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A VARIANT OF THE PRIME NUMBER THEOREM AND SOME RELATED PROBLEMS

B. CHEN, J.-Y. HU AND J. WU

ABSTRACT. Let $\Lambda(n)$ and $\mu(n)$ be the von Mangoldt function and the Möbius function. Let $\mathbb{1}_p(n)$ be the characteristic function of prime numbers and let $[t]$ be the integral part of real number t . In this paper, we prove that asymptotic formulas

$$\sum_{n \leq x} \Lambda\left(\left[\frac{x}{n}\right]\right) = \left(\sum_{d=1}^{\infty} \frac{\Lambda(d)}{d(d+1)}\right)x + O_{\varepsilon}(x^{7/15+\varepsilon})$$

$$\sum_{n \leq x} \mu(n)^2 \Lambda\left(\left[\frac{x}{n}\right]\right) = \left(\frac{6}{\pi^2} \sum_{d=1}^{\infty} \frac{\Lambda(d)}{d(d+1)}\right)x + O(x^{2/3} \mathcal{L}(x)^{-c})$$

hold as $x \rightarrow \infty$, where $\mathcal{L}(x) := \exp((\log x)^{3/5}(\log \log x)^{-1/5})$, $\varepsilon > 0$ is an arbitrarily small positive number and $c > 0$ is a positive constant. The first asymptotic formula improves a recent result of Zhang, which requires $\frac{7}{15} + \frac{1}{195}$ in place of $\frac{7}{15}$. We also improve some results of Ma-Chen-Wu [13] and of Zhou-Feng [20].

1. INTRODUCTION

Let $\Lambda(n)$ be the von Mangoldt function. The prime number theorem states, in strong form, as follows: there is a constant $c > 0$ such that for $x \rightarrow \infty$ the asymptotic formula

$$(1.1) \quad \sum_{n \leq x} \Lambda(n) = x + O(x \mathcal{L}(x)^{-c})$$

holds, where \log_k is the k -fold iterated logarithm and $\mathcal{L}(x) := \exp((\log x)^{3/5}(\log_2 x)^{-1/5})$. Furthermore, for any $\varepsilon > 0$,

$$\sum_{n \leq x} \Lambda(n) = x + O_{\varepsilon}(x^{1/2+\varepsilon}) \Leftrightarrow \text{Riemann Hypothesis,}$$

Clearly it is also interesting to study the distribution of prime numbers in different sets of integers. Recently many authors are interested by the sparse set

$$\mathfrak{S}(x) := \left\{ \left[\frac{x}{n}\right] : n \in \mathbb{N} \cap [1, x] \right\},$$

which is sparse in $\mathbb{N} \cap [1, x]$:

$$|\mathfrak{S}(x)| \underset{x \rightarrow \infty}{\sim} 2\sqrt{x}.$$

Under some simple assumptions of arithmetic function f , Bordellès-Dai-Heyman-Pan-Shparlinski [3, Theorem 2.6] established an asymptotic formula of

$$(1.2) \quad S_f(x) := \sum_{n \leq x} f\left(\left[\frac{x}{n}\right]\right).$$

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Subsequently, Wu [17, Theorem 1.2(i)] and Zhai [18, Theorem 1] improved their results independently, by showing the following result: If $f(n) \ll n^\theta$ with some $\theta \in [0, 1[$, then for any $\varepsilon > 0$ and $x \rightarrow \infty$,

$$S_f(x) = c_f x + O_\varepsilon(x^{(1+\theta)/2+\varepsilon}),$$

where

$$c_f := \sum_{d=1}^{\infty} \frac{f(d)}{d(d+1)}.$$

In particular, this implies that

$$S_\Lambda(x) = c_\Lambda x + O_\varepsilon(x^{1/2+\varepsilon}) \quad (x \rightarrow \infty).$$

In order to break the $\frac{1}{2}$ -barrier, after application of the Vaughan identity, a key point is to give non trivial bound of the following triple exponential sums

$$(1.3) \quad S_\delta := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_m c_n e\left(\frac{hx}{mn + \delta}\right),$$

where $e(t) := e^{2\pi it}$, $\delta \in \mathbb{R} \setminus (-\mathbb{N}^*)$, $|a_h| \leq 1$, $|b_m| \leq 1$, $|c_n| \leq 1$ and the symbol $h \sim H$ means that $H \leq h < 2H$. By the Cauchy-Schwarz inequality and the method of one-dimensional exponential sum, Ma and Wu [11] succeeded to prove

$$S_\Lambda(x) = c_\Lambda x + O_\varepsilon(x^{35/71+\varepsilon}) \quad (x \rightarrow \infty).$$

Subsequently the exponent $\frac{35}{71}$ has been improved to $\frac{97}{203}$ by Bordellès [2, Corollary 1.3] using a result of Baker on 2-dimensional exponential sums [1, Theorem 6]. When $\delta = 0$, this triple exponential sum (or more general multiple exponential sums with monomials) has been studied by Heath-Brown [8], Fouvry-Iwaniec [6], Sargos-Wu [15] and Robert-Sargos [14]. In [10], Liu, Wu and Yang adapted and refined Heath-Brown's approach to the case of $\delta \neq 0$ (see Proposition 2.1(ii) below) and improved $\frac{97}{203}$ to $\frac{9}{19}$. It seems interesting to generalize Fouvry-Iwaniec's method (or Robert-Sargos' improved version) to the case of $\delta \neq 0$ for improving the exponent $\frac{9}{19}$. As Zhang observed in [19], this seems not easy to do. In order to eliminate the influence of δ he proposed a simple argument of partial integration such that he could apply directly Robert-Sargos' result (see Lemma 2.2 below) to improve $\frac{9}{19}$ to $\frac{7}{15} + \frac{1}{195}$. By a rather complicated processus of parameter optimization combining Bourgain's new exponent pair (see [5]), he can obtain a slightly better exponent 0.47118 in place of $\frac{7}{15} + \frac{1}{195} \approx 0.47179$. He also wrote [19, page 166]: "It is worth emphasizing that only by involving the ideas in this paper, the best possible result should be $\frac{7}{15}$."

In this paper, by combining Zhang's idea with Liu-Wu-Yang's estimate of S_δ (see also (2.2) below with simple exponent pair $(\frac{1}{2}, \frac{1}{2})$), we shall establish the best exponent $\frac{7}{15}$ which can be obtained by the improved version of Robert-Sargos on Fouvry-Iwaniec's method.

Theorem 1. *For any $\varepsilon > 0$, we have*

$$(1.4) \quad S_\Lambda(x) = c_\Lambda x + O_\varepsilon(x^{7/15+\varepsilon}),$$

$$(1.5) \quad S_{\mathbb{1}_\mathbb{P}}(x) = c_{\mathbb{1}_\mathbb{P}} x + O_\varepsilon(x^{7/15+\varepsilon}),$$

as $x \rightarrow \infty$, where $\mathbb{1}_\mathbb{P}$ is the characteristic function of the set of prime numbers \mathbb{P} .

The asymptotic formula (1.5) improves Theorem 2 of Ma and Wu [12], which requires $\frac{9}{19}$ in place of $\frac{7}{15}$. For comparison, we have

$$\frac{9}{19} \approx 0.47368, \quad \frac{7}{15} + \frac{1}{195} \approx 0.47179, \quad \frac{7}{15} \approx 0.46666.$$

Recently, motivated by a conjecture of Kellner concerning denominators of Bernoulli polynomials [9] *, Ma, Chen and Wu [13] proposed and investigated a more general problem: Let $\theta \in]0, 1]$ be a fixed constant. Define

$$(1.6) \quad S_{\mathbb{1}_P}(x, \theta) := \sum_{\substack{n \leq x^\theta \\ [\frac{x}{n}] \text{ is prime}}} 1, \quad \pi_S(x, \theta) := \sum_{\substack{p \leq x \\ \exists n \leq x^\theta \text{ such that } [\frac{x}{n}] = p}} 1$$

such that $S_{\mathbb{1}_P}(x, 1) = S_{\mathbb{1}_P}(x)$ and $\pi_S(x, 1) = \pi_S(x)$ (In the notation of [13], we have $S_{\mathbb{1}_P}(x, \theta) = \pi_\theta^{(3)}(x)$ and $\pi_S(x, \theta) = \pi_\theta^{(4)}(x)$). They proved that for any positive integer $L \geq 1$, the asymptotic formulas

$$(1.7) \quad S_{\mathbb{1}_P}(x, \theta) = \begin{cases} \frac{x^\theta}{\log x^{1-\theta}} \left\{ \sum_{\ell=0}^{L-1} \frac{(-1)^\ell \ell!}{(\log x^{1-\theta})^\ell} + O\left(\frac{1}{(\log x)^L}\right) \right\} & \text{if } \frac{23}{47} < \theta < 1 \\ c_{\mathbb{1}_P} x + O(x^{26/53} (\log x)^{119/53}) & \text{if } \theta = 1 \end{cases}$$

and

$$(1.8) \quad \pi_S(x, \theta) = \begin{cases} \frac{x^\theta}{\log x^{1-\theta}} \left\{ \sum_{\ell=0}^{L-1} \frac{(-1)^\ell \ell!}{(\log x^{1-\theta})^\ell} + O\left(\frac{1}{(\log x)^L}\right) \right\} & \text{if } \frac{23}{47} < \theta \leq \frac{1}{2} \\ \frac{4\sqrt{x}}{\log x} \left\{ \sum_{\ell=0}^{L-1} \frac{2^{2\ell} (2\ell)!}{(\log x)^{2\ell}} + O\left(\frac{1}{(\log x)^{2L-1}}\right) \right\} & \text{if } \frac{1}{2} < \theta \leq 1 \end{cases}$$

hold as $x \rightarrow \infty$ (see [13, Theorems 4.2, 4.4 and Corollary 4.3]). They conjectured that the first asymptotic formula in (1.7) also holds for $0 < \theta \leq \frac{23}{47}$.

Obviously (1.5) of Theorem 1 sharpens the case of $\theta = 1$ of (1.7) by reducing the error term $O(x^{26/53} (\log x)^{119/53})$ to $O(x^{7/15+\epsilon})$. Very recently with the help of Liu-Wu-Yang's work [10, Proposition 4.1] (see also (4.2) below), Zhou and Feng [20] improved Ma-Chen-Wu's (1.7) by extending $\frac{23}{47} < \theta < 1$ to $\frac{9}{19} < \theta < 1$. Here we propose better results.

Theorem 2. (i) Let $\frac{7}{15} < \theta < 1$, then there is a positive constant $c_\theta > 0$ such that asymptotic formula

$$(1.9) \quad S_{\mathbb{1}_P}(x, \theta) = x \int_0^{x^{\theta-1}} \frac{dt}{\log(1/t)} + O(x^\theta \mathcal{L}(x)^{-c_\theta})$$

holds as $x \rightarrow \infty$. Furthermore the first asymptotic formula of (1.7) holds for $\frac{7}{15} < \theta < 1$.

(ii) Let $\frac{7}{15} < \theta < 1$, then there is a positive constant $c_\theta > 0$ such that asymptotic formulas

$$(1.10) \quad \pi_S(x, \theta) = \begin{cases} x \int_0^{x^{\theta-1}} \frac{dt}{\log(1/t)} + O(x^\theta \mathcal{L}(x)^{-c_\theta}) & \text{if } \frac{7}{15} < \theta \leq \frac{1}{2} \\ \int_2^{\sqrt{x}} \left(\frac{1}{\log t} + \frac{1}{\log(x/t)} \right) dt + O(\sqrt{x} \mathcal{L}(x)^{-c_\theta}) & \text{if } \frac{1}{2} < \theta \leq 1 \end{cases}$$

hold for $x \rightarrow \infty$. Furthermore (1.8) holds with $\frac{7}{15}$ in place of $\frac{23}{47}$.

*About this conjecture and related works, we refer reader to Bordellès, Luca, Moree and Shparlinski's interesting paper [4].

For comparison, we have

$$\frac{23}{47} \approx 0.489361, \quad \frac{9}{19} \approx 0.473684 \quad \text{and} \quad \frac{7}{15} \approx 0.46666.$$

Let $\mu(n)$ be the Möbius function, then $\mu(n)^2$ is the characteristic function of square-free integers. Define

$$S_{\Lambda}^{\sharp}(x) := \sum_{n \leq x} \mu(n)^2 \Lambda\left(\left[\frac{x}{n}\right]\right), \quad S_{\mathbb{1}_{\mathbb{P}}}^{\sharp}(x) := \sum_{n \leq x} \mu(n)^2 \mathbb{1}_{\mathbb{P}}\left(\left[\frac{x}{n}\right]\right).$$

Theorem 3. *There is a positive constant $c > 0$ such that*

$$(1.11) \quad S_{\Lambda}^{\sharp}(x) = (6\pi^{-2})c_{\Lambda}x + O(x^{2/3}\mathcal{L}(x)^{-c}),$$

$$(1.12) \quad S_{\mathbb{1}_{\mathbb{P}}}^{\sharp}(x) = (6\pi^{-2})c_{\mathbb{1}_{\mathbb{P}}}x + O(x^{2/3}\mathcal{L}(x)^{-c}),$$

as $x \rightarrow \infty$.

Theorem 3 shows that events “ n is a square-free integer” and “ $\lceil \frac{x}{n} \rceil$ is a prime number” are independent.

2. TRIPLE EXPONENTIAL SUMS

As in [6], we write S_{δ} in the following form

$$S_{\delta} := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_m c_n e\left(X \frac{MN}{H} \frac{h}{mn + \delta}\right),$$

where $X := xH/(MN)$, $|a_h| \leq 1$, $|b_m| \leq 1$, $|c_n| \leq 1$ and the symbol $h \sim H$ means that $H \leq h < 2H$. The aim of this section is to give some bounds for this triple exponential sums, which will play a key role in the proof of Theorem 1.

Proposition 2.1. (i) *Let $\delta \in \mathbb{R} \setminus (-\mathbb{N}^*)$ be a fixed constant. For any $\varepsilon > 0$, we have*

$$(2.1) \quad S_{\delta} \ll_{\varepsilon} (1 + X(MN)^{-1})((XH^3M^3N^2)^{1/4} + (HM)^{3/4}N + HMN^{1/2} + X^{-1/2}HMN)Y^{\varepsilon}$$

uniformly for $X > 0$, $H \geq 1$, $M \geq 1$ and $N \geq 1$, where $Y := 1 + XHMN$ and the implied constant depends on (δ, ε) at most.

(ii) *Let (κ, λ) be an exponent pair and let $\delta \in \mathbb{R} \setminus (-\mathbb{N}^*)$ be a fixed constant. For any $\varepsilon > 0$, we have*

$$(2.2) \quad S_{\delta} \ll ((X^{\kappa}H^{2+\kappa}M^{2+\kappa}N^{1+\kappa+\lambda})^{1/(2+2\kappa)} + (HM)^{1/2}N + HMN^{1/2} + X^{-1/2}HMN)Y^{\varepsilon}$$

uniformly for $X > 0$, $H \geq 1$, $M \geq 1$ and $N \geq H$, where $Y := 1 + XHMN$ and the implied constant depends on $(\kappa, \lambda, \delta, \varepsilon)$.

The second assertion of Proposition 2.1 is a special case of [10, Proposition 3.1]. In order to prove the first assertion, we need a result of Robert-Sargos (see [14, Theorem 1]), which is an improvement of [6, Theorem 3] and [15, Theorem 7].

Lemma 2.2. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha\beta\gamma(\gamma - 1) \neq 0$. Define*

$$S := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_m c_n e\left(X \frac{h^{\alpha} m^{\beta} n^{\gamma}}{H^{\alpha} M^{\beta} N^{\gamma}}\right).$$

Then for any $\varepsilon > 0$ we have

$$(2.3) \quad S \ll ((XH^3M^3N^2)^{1/4} + (HM)^{3/4}N + HMN^{1/2} + X^{-1/2}HMN)Y^{\varepsilon}$$

uniformly for $X > 0$, $H \geq 1$, $M \geq 1$ and $N \geq 1$, where $Y := HMN$ and the implied constant depends on $(\alpha, \beta, \gamma, \varepsilon)$.

Proof of (2.1). Noticing that

$$X \frac{MN}{H} \frac{h}{mn + \delta} = X \frac{MN}{H} \frac{h}{mn} - X \frac{MN}{H} \frac{\delta h}{mn(mn + \delta)},$$

we have

$$S_\delta = \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_m c_n e\left(-X \frac{MN}{H} \frac{\delta h}{mn(mn + \delta)}\right) e\left(X \frac{MN}{H} \frac{h}{mn}\right).$$

By a partial integration, we can write

$$\begin{aligned} S_\delta &= \int_N^{2N} e\left(-X \frac{MN}{H} \frac{\delta h}{mt(mt + \delta)}\right) d\left(\sum_{h \sim H} \sum_{m \sim M} \sum_{N \leq n < t} a_h b_m c_n e\left(X \frac{MN}{H} \frac{h}{mn}\right)\right) \\ &\ll (1 + X(MN)^{-1}) \max_{N \leq N_1 \leq 2N} \left| \sum_{h \sim H} \sum_{m \sim M} \sum_{N \leq n < N_1} a_h b_m c_n e\left(X \frac{MN}{H} \frac{h}{mn}\right) \right|. \end{aligned}$$

Applying (2.3) to bound the last triple exponential sums, we get (2.1). □

3. PRELIMINARY LEMMAS

In this section, we shall cite two lemmas, which will be needed in the next section. The first one is the Vaughan identity [16, formula (3)].

Lemma 3.1. *Let $\tau(n)$ be the classical divisor function. There are six real arithmetical functions $\alpha_k(n)$ verifying $|\alpha_k(n)| \ll \tau(n)$ for $(n \geq 1, 1 \leq k \leq 6)$ such that, for all $D \geq 100$ and any arithmetical function g , we have*

$$(3.1) \quad \sum_{D < d \leq 2D} \Lambda(d)g(d) = S_1 + S_2 + S_3 + S_4,$$

where

$$\begin{aligned} S_1 &:= \sum_{m \leq D^{1/3}} \alpha_1(m) \sum_{D \leq mn < 2D} g(mn), \\ S_2 &:= \sum_{m \leq D^{1/3}} \alpha_2(m) \sum_{D \leq mn < 2D} g(mn) \log n, \\ S_3 &:= \sum_{\substack{D^{1/3} < m, n \leq D^{2/3} \\ D \leq mn < 2D}} \alpha_3(m) \alpha_4(n) g(mn), \\ S_4 &:= \sum_{\substack{D^{1/3} < m, n \leq D^{2/3} \\ D \leq mn < 2D}} \alpha_5(m) \alpha_6(n) g(mn). \end{aligned}$$

The sums S_1 and S_2 are called as type I, S_3 and S_4 are called as type II.

The second one is due to Vaaler (see [7, Theorem A.6]).

Lemma 3.2. Let $\psi(t) := \{t\} - \frac{1}{2}$, where $\{t\}$ means the fractional part of real number t . For $x \geq 1$ and $H \geq 1$, we have

$$(3.2) \quad \psi(x) = - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hx)}{2\pi i h} + R_H(x),$$

where $e(t) := e^{2\pi i t}$, $\Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|$ and the error term $R_H(x)$ satisfies

$$(3.3) \quad |R_H(x)| \leq \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx).$$

4. A KEY INEQUALITY

The aim of this section is to prove the following proposition, which will play a key role for the proof of Theorem 1. Define

$$(4.1) \quad \mathfrak{S}_\delta(x, D) := \sum_{d \sim D} \Lambda(d) \psi\left(\frac{x}{d+\delta}\right),$$

where $\Lambda(n)$ is the von Mangoldt function, $\psi(t) := t - [t] - \frac{1}{2}$ and the symbol $d \sim D$ means that $D \leq d < 2D$.

Proposition 4.1. Let $\delta \in \mathbb{R} \setminus (-\mathbb{N}^*)$ be a fixed constant.

(i) For any $\varepsilon > 0$, we have

$$(4.2) \quad \mathfrak{S}_\delta(x, D) \ll_\varepsilon (x^2 D^7)^{1/12} x^\varepsilon$$

uniformly for $x^{6/13} \leq D \leq x^{2/3}$.

(ii) For any $\varepsilon > 0$, we have

$$(4.3) \quad \mathfrak{S}_\delta(x, D) \ll_\varepsilon ((x^6 D)^{1/14} + (x^8 D^{-3})^{1/14}) x^\varepsilon$$

uniformly for $x^{22/45} \leq D \leq x^{8/15}$.

Proof. The first assertion is [10, Proposition 4.1]. Next we shall prove the second one.

Using (3.2) of Lemma 3.2, for any $H \geq 1$ we can write

$$\mathfrak{S}_\delta(x, D) = - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{1}{2\pi i h} \sum_{d \sim D} \Lambda(d) e\left(\frac{hx}{d+\delta}\right) + \sum_{d \sim D} \Lambda(d) R\left(\frac{x}{d+\delta}\right).$$

In view of (3.3) of Lemma 3.2, we can derive that

$$\begin{aligned} |\mathfrak{S}_\delta(x, D)| &\leq \left| \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(-hx)}{2\pi i h} \sum_{d \sim D} \Lambda(d) e\left(\frac{hx}{d+\delta}\right) \right| \\ &\quad + \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(-hx) \sum_{d \sim D} \Lambda(d) e\left(\frac{hx}{d+\delta}\right). \end{aligned}$$

Noticing the fact that $0 < \Phi(t) < 1$ ($0 < |t| < 1$) and using the prime number theorem, for any $H \geq 1$ we have

$$(4.4) \quad \mathfrak{S}_\delta(x, D) \ll \frac{D}{H} + \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{d \sim D} \Lambda(d) e\left(\frac{hx}{d+\delta}\right) \right|.$$

For the last inner sum, we apply the Vaughan identity (3.1) to write

$$(4.5) \quad \mathfrak{S}_\delta(x, D) \ll DH^{-1} + \mathfrak{S}_{\delta,1} + \mathfrak{S}_{\delta,2} + \mathfrak{S}_{\delta,3} + \mathfrak{S}_{\delta,4},$$

where $\mathfrak{S}_{\delta,j} := \max_{1 \leq H_1 \leq H} H_1^{-1} S_j(H_1, D)$ and

$$\begin{aligned} S_1(H_1, D) &:= \sum_{h \sim H_1} \left| \sum_{m \leq D^{1/3}} \alpha_1(m) \sum_{D \leq mn < 2D} e\left(\frac{hx}{mn + \delta}\right) \right|, \\ S_2(H_1, D) &:= \sum_{h \sim H_1} \left| \sum_{m \leq D^{1/3}} \alpha_2(m) \sum_{D \leq mn < 2D} (\log n) e\left(\frac{hx}{mn + \delta}\right) \right|, \\ S_3(H_1, D) &:= \sum_{h \sim H_1} \left| \sum_{\substack{D^{1/3} < m, n \leq D^{2/3} \\ D \leq mn < 2D}} \alpha_3(m) \alpha_4(n) e\left(\frac{hx}{mn + \delta}\right) \right|, \\ S_4(H_1, D) &:= \sum_{h \sim H_1} \left| \sum_{\substack{D^{1/3} < m, n \leq D^{2/3} \\ D \leq mn < 2D}} \alpha_5(m) \alpha_6(n) e\left(\frac{hx}{mn + \delta}\right) \right|. \end{aligned}$$

The sums S_1 and S_2 are called as type I, S_3 and S_4 are called as type II. In view of symmetricity of m and n in $S_3(H_1, D)$ and $S_4(H_1, D)$, we can suppose that $m \leq n$ in these two sums.

Firstly we bound $\mathfrak{S}_{\delta,1}$ and $\mathfrak{S}_{\delta,2}$.

Applying the exponent pair (κ', λ') to the sum over n , we can derive that

$$\begin{aligned} S_1(H_1, D) &\ll x^\varepsilon \sum_{m \leq D^{1/3}} \sum_{h \sim H_1} \left\{ \left(\frac{hx}{D^2/m}\right)^{\kappa'} \left(\frac{D}{m}\right)^{\lambda'} + \frac{(D^2/m)}{hx} \right\} \\ &\ll (x^{\kappa'} D^{(1-5\kappa'+2\lambda')/3} H_1^{1+\kappa'} + x^{-1} D^2) x^\varepsilon \end{aligned}$$

for all $H \geq 1$. The same bound also holds for $S_2(H_1, D)$ since the factor $\log n$ can be removed by a simple partial integration. Thus these imply that

$$(4.6) \quad \mathfrak{S}_{\delta,j} \ll (x^{\kappa'} D^{(1-5\kappa'+2\lambda')/3} H^{\kappa'} + x^{-1} D^2) x^\varepsilon$$

for all $H \geq 1$ and $j = 1, 2$.

Next we shall bound $\mathfrak{S}_{\delta,3}$ and $\mathfrak{S}_{\delta,4}$.

Define

$$\begin{aligned} S &:= \sum_{h \sim H_1} \left| \sum_{m \sim M} \sum_{n \sim N} \alpha_3(m) \alpha_4(n) e\left(\frac{hx}{mn + \delta}\right) \right| \\ &= \sum_{h \sim H_1} \left| \sum_{m \sim M} \sum_{n \sim N} \alpha_3(m) \alpha_4(n) e\left(X \frac{MN}{H_1} \frac{h}{mn + \delta}\right) \right|, \end{aligned}$$

where $X = xH_1(MN)^{-1}$, $D^{1/3} \leq M \leq D^{1/2}$, $D \leq mn < 2D$ and $H_1 \leq H \leq N$. Applying (2.1) of Proposition 2.1 with $(H, M, N) = (H_1, N, M)$, it follows that

$$(4.7) \quad \begin{aligned} S &\ll ((XH_1^3 M^2 N^3)^{1/4} + (H_1^3 M^4 N^3)^{1/4} + (H_1^2 M N^2)^{1/2} + X^{-1/2} H_1 M N \\ &\quad + (X^5 H_1^3 M^{-2} N^{-1})^{1/4} + (X^4 H_1^3 N^{-1})^{1/4} + (X^2 H_1^2 M^{-1})^{1/2} + X^{1/2} H_1) Y^\varepsilon. \end{aligned}$$

By (2.2) of Proposition 2.1 with $(H, M, N) = (H_1, M, N)$, it follows that

$$(4.8) \quad \begin{aligned} S &\ll ((X^\kappa H_1^{2+\kappa} M^{2+\kappa} N^{1+\kappa+\lambda})^{1/(2+2\kappa)} \\ &\quad + (H_1 M)^{1/2} N + H_1 M N^{1/2} + X^{-1/2} H_1 M N) Y^\varepsilon. \end{aligned}$$

Write

$$\begin{aligned}\varsigma_1 &:= (XH_1^3M^2N^3)^{1/4}, \\ \varsigma_2 &:= (X^5H_1^3M^{-2}N^{-1})^{1/4}, \\ \psi &:= (X^\kappa H_1^{2+\kappa}M^{2+\kappa}N^{1+\kappa+\lambda})^{1/(2+2\kappa)}.\end{aligned}$$

From (4.7) and (4.8), for $H_1 \leq H \leq N$ we can deduce that

$$(4.9) \quad \begin{aligned}S &\ll ((X^{1+\kappa-\lambda}H_1^{5+\kappa-3\lambda}M^{4+\kappa-2\lambda}N^{4+\kappa-2\lambda})^{1/(6+2\kappa-4\lambda)} \\ &\quad + (H_1^3M^4N^3)^{1/4} + (H_1^2MN^2)^{1/2} + X^{-1/2}H_1MN \\ &\quad + (X^{5+\kappa-5\lambda}H_1^{5+\kappa-3\lambda}M^{\kappa+2\lambda}N^{\kappa+2\lambda})^{1/(6+2\kappa-4\lambda)} \\ &\quad + (X^4H_1^3N^{-1})^{1/4} + (X^2H_1^2M^{-1})^{1/2} + X^{1/2}H_1 \\ &\quad + (H_1M)^{1/2}N + H_1MN^{1/2})Y^\varepsilon,\end{aligned}$$

where we have used the following inequality

$$\begin{aligned}\min(\psi, \varsigma_1) &\leq (\psi^{1+\kappa}\varsigma_1^{2-2\lambda})^{1/(3+\kappa-2\lambda)} \\ &= (X^{1+\kappa-\lambda}H_1^{5+\kappa-3\lambda}M^{4+\kappa-2\lambda}N^{4+\kappa-2\lambda})^{1/(6+2\kappa-4\lambda)}, \\ \min(\psi, \varsigma_2) &\leq (\psi^{1+\kappa}\varsigma_2^{2-2\lambda})^{1/(3+\kappa-2\lambda)} \\ &= (X^{5+\kappa-5\lambda}H_1^{5+\kappa-3\lambda}M^{\kappa+2\lambda}N^{\kappa+2\lambda})^{1/(6+2\kappa-4\lambda)}.\end{aligned}$$

Noticing that $H_1 \geq 1$ and $M \leq D^{1/2} \leq N$ imply that

$$(H_1M)^{1/2}N \leq (H_1^2MN^2)^{1/2}, \quad H_1MN^{1/2} \leq (H_1^2MN^2)^{1/2}$$

and

$$((X^2H_1^2M^{-1})^{1/2})^{1/2}((H_1^2MN^2)^{1/2})^{1/2} = (XH_1^2N)^{1/2} \geq X^{1/2}H_1,$$

the inequality (4.9) can be simplified as

$$\begin{aligned}S &\ll ((X^{1+\kappa-\lambda}H_1^{5+\kappa-3\lambda}M^{4+\kappa-2\lambda}N^{4+\kappa-2\lambda})^{1/(6+2\kappa-4\lambda)} \\ &\quad + (H_1^3M^4N^3)^{1/4} + (H_1^2MN^2)^{1/2} + X^{-1/2}H_1MN \\ &\quad + (X^{5+\kappa-5\lambda}H_1^{5+\kappa-3\lambda}M^{\kappa+2\lambda}N^{\kappa+2\lambda})^{1/(6+2\kappa-4\lambda)} \\ &\quad + (X^4H_1^3N^{-1})^{1/4} + (X^2H_1^2M^{-1})^{1/2})Y^\varepsilon\end{aligned}$$

for $H_1 \leq H \leq N$. Using $D^{1/3} \leq M \leq D^{1/2}$ and $D \leq mn < 2D$, we deduce that

$$\begin{aligned}S &\ll ((X^{1+\kappa-\lambda}H_1^{5+\kappa-3\lambda}D^{4+\kappa-2\lambda})^{1/(6+2\kappa-4\lambda)} + (H_1^6D^7)^{1/8} + (H_1^6D^5)^{1/6} + X^{-1/2}H_1D \\ &\quad + (X^{5+\kappa-5\lambda}H_1^{5+\kappa-3\lambda}D^{\kappa+2\lambda})^{1/(6+2\kappa-4\lambda)} + (X^8H_1^6D^{-1})^{1/8} + (X^6H_1^6D^{-1})^{1/6})Y^\varepsilon\end{aligned}$$

for $H_1 \leq H \leq N$, where we have removed terms $(H_1^2D^3)^{1/4}$ and $(H_1^3D^5)^{1/6}$ since

$$(H_1^2D^3)^{1/4} \leq (H_1^3D^5)^{1/6} \leq (H_1^6D^7)^{1/8}.$$

Replacing X by xH_1D^{-1} , it follows that

$$\begin{aligned}S &\ll ((x^{1+\kappa-\lambda}H_1^{6+2\kappa-4\lambda}D^{3-\lambda})^{1/(6+2\kappa-4\lambda)} + (H_1^6D^7)^{1/8} + (H_1^6D^5)^{1/6} + (x^{-1}H_1D^3)^{1/2} \\ &\quad + (x^{5+\kappa-5\lambda}H_1^{10+2\kappa-8\lambda}D^{-5+7\lambda})^{1/(6+2\kappa-4\lambda)} + (x^8H_1^{14}D^{-9})^{1/8} + (x^6H_1^{12}D^{-7})^{1/6})x^\varepsilon.\end{aligned}$$

for $H_1 \leq H \leq D^{1/2} (\leq N)$. This implies that for $j = 3, 4$,

$$(4.10) \quad \begin{aligned}\mathfrak{S}_{\delta,j} &\ll ((x^{5+\kappa-5\lambda}H_1^{4-4\lambda}D^{-5+7\lambda})^{1/(6+2\kappa-4\lambda)} + (x^8H_1^6D^{-9})^{1/8} + (x^6H_1^6D^{-7})^{1/6} \\ &\quad + (x^{1+\kappa-\lambda}D^{3-\lambda})^{1/(6+2\kappa-4\lambda)} + D^{7/8})x^\varepsilon\end{aligned}$$

for $H \leq D^{1/2}$ ($\leq N$), where we have removed terms $D^{5/6}$ and $(x^{-1}D^3)^{1/2}$ since $D^{5/6} \leq D^{7/8}$ and $(x^{-1}D^3)^{1/2} \leq D^{7/8}$ provided $D \leq x^{4/5}$.

Inserting (4.6) and (4.10) into (4.5), it follows that

$$\begin{aligned} \mathfrak{S}_\delta(x, D) &\ll (DH^{-1} + (x^{5+\kappa-5\lambda}H^{4-4\lambda}D^{-5+7\lambda})^{1/(6+2\kappa-4\lambda)}) \\ &\quad + (x^8H^6D^{-9})^{1/8} + (x^6H^6D^{-7})^{1/6} + x^{\kappa'}D^{(-5\kappa'+2\lambda'+1)/3}H^{\kappa'} \\ &\quad + (x^{1+\kappa-\lambda}D^{3-\lambda})^{1/(6+2\kappa-4\lambda)} + D^{7/8}x^\varepsilon \end{aligned}$$

for all $H \leq D^{1/2}$, where we have removed term $x^{-1}D^2$ since $x^{-1}D^2 \leq D^{7/8}$ (provided $D \leq x^{8/9}$). Optimizing H on $]0, D^{1/2}]$, we obtain that

$$(4.11) \quad \begin{aligned} \mathfrak{S}_\delta(x, D) &\ll ((x^{5+\kappa-5\lambda}D^{-1+3\lambda})^{1/(10+2\kappa-8\lambda)} + (x^8D^{-3})^{1/14} + (x^6D^{-1})^{1/12} \\ &\quad + (x^{3\kappa'}D^{-2\kappa'+2\lambda'+1})^{1/(3\kappa'+3)} + (x^{1+\kappa-\lambda}D^{3-\lambda})^{1/(6+2\kappa-4\lambda)} + D^{7/8})x^\varepsilon. \end{aligned}$$

Taking $(\kappa, \lambda) = (\kappa', \lambda') = (\frac{1}{2}, \frac{1}{2})$ in (4.11), we find that

$$\begin{aligned} \mathfrak{S}_\delta(x, D) &\ll ((x^6D)^{1/14} + (x^8D^{-3})^{1/14} + (x^6D^{-1})^{1/12} \\ &\quad + (x^3D^2)^{1/9} + (x^2D^5)^{1/10} + D^{7/8})x^\varepsilon. \end{aligned}$$

Noticing that

$$\begin{aligned} (x^3D^2)^{1/9} &\leq (x^6D)^{1/14} \text{ (provided } D \leq x^{12/19}), \\ (x^2D^5)^{1/10} &\leq (x^6D)^{1/14} \text{ (provided } D \leq x^{8/15}), \\ D^{7/8} &\leq (x^6D)^{1/14} \text{ (provided } D \leq x^{8/15}), \\ (x^6D^{-1})^{1/12} &\leq (x^8D^{-3})^{1/14} \text{ (provided } D \leq x^{6/11}), \end{aligned}$$

we can obtain the required inequality. \square

5. PROOF OF THEOREM 1

5.1. Proof of (1.4).

First we write

$$(5.1) \quad S_\Lambda(x) = \sum_{n \leq x} \Lambda\left(\left[\frac{x}{n}\right]\right) = \tilde{S}_\Lambda(x) + O(x^{7/15} \log x)$$

with

$$\tilde{S}_\Lambda(x) := \sum_{x^{7/15} < n \leq x} \Lambda\left(\left[\frac{x}{n}\right]\right).$$

Putting $d = [x/n]$, then

$$x/n - 1 < d \leq x/n \Leftrightarrow x/(d+1) < n \leq x/d.$$

Thus we can write

$$(5.2) \quad \begin{aligned} \tilde{S}_\Lambda(x) &= \sum_{d \leq x^{8/15}} \Lambda(d) \sum_{x/(d+1) < n \leq x/d} 1 \\ &= \sum_{d \leq x^{8/15}} \Lambda(d) \left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{x}{d+1} + \psi\left(\frac{x}{d+1}\right) \right) \\ &= c_\Lambda x + \mathcal{R}_1^\Lambda(x, x^{7/15}) - \mathcal{R}_0^\Lambda(x, x^{7/15}) + O(x^{7/15}), \end{aligned}$$

where

$$\mathcal{R}_\delta^\Lambda(x, x^{7/15}) = \sum_{x^{7/15} < d \leq x^{8/15}} \Lambda(d) \psi\left(\frac{x}{d+\delta}\right)$$

and we have used the following bounds

$$x \sum_{d > x^{8/15}} \frac{\Lambda(d)}{d(d+1)} \ll x^{7/15}, \quad \sum_{d \leq x^{7/15}} \Lambda(d) \left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right) \right) \ll x^{7/15}.$$

Noticing that

$$|\mathcal{R}_\delta^\Lambda(x, x^{7/15})| \ll (\log x) \max_{x^{7/15} \leq D \leq x^{8/15}} |\mathfrak{S}_\delta(x, D)|,$$

we can apply Proposition 4.1(i) (for $x^{7/15} \leq D \leq x^{34/67}$) and Proposition 4.1(ii) (for $x^{34/67} \leq D \leq x^{8/15}$) to get

$$|\mathcal{R}_\delta^\Lambda(x, x^{7/15})| \ll_\varepsilon x^{7/15+\varepsilon}.$$

Inserting this into (5.2), we find that

$$(5.3) \quad \tilde{S}_\Lambda(x) = c_\Lambda x + O_\varepsilon(x^{7/15+\varepsilon}).$$

Now the required result follows from (5.1) and (5.3).

5.2. Proof of (1.5).

As before, we have

$$(5.4) \quad S_{\mathbb{1}_\mathbb{P}}(x) = \tilde{S}_{\mathbb{1}_\mathbb{P}}(x) + O(x^{7/15}),$$

where

$$(5.5) \quad \begin{aligned} \tilde{S}_{\mathbb{1}_\mathbb{P}}(x) &:= \sum_{x^{7/15} < n \leq x} \mathbb{1}_\mathbb{P}\left(\left[\frac{x}{n}\right]\right) = \sum_{p \leq x^{8/15}} \sum_{x/(p+1) < n \leq x/p} 1 \\ &= \sum_{p \leq x^{8/15}} \left(\frac{x}{p} - \psi\left(\frac{x}{p}\right) - \frac{x}{p+1} + \psi\left(\frac{x}{p+1}\right) \right) \\ &= c_{\mathbb{1}_\mathbb{P}} x + \mathcal{R}_1^{\mathbb{1}_\mathbb{P}}(x, x^{7/15}) - \mathcal{R}_0^{\mathbb{1}_\mathbb{P}}(x, x^{7/15}) + O(x^{7/15}) \end{aligned}$$

and

$$\mathcal{R}_\delta^{\mathbb{1}_\mathbb{P}}(x, x^{7/15}) = \sum_{x^{7/15} < p \leq x^{8/15}} \psi\left(\frac{x}{p+\delta}\right).$$

We write

$$\begin{aligned} \mathcal{R}_\delta^{\mathbb{1}_\mathbb{P}}(x, x^{7/15}) &\ll x^\varepsilon \max_{x^{7/15} < D \leq x^{8/15}} \left| \sum_{D < p \leq 2D} \psi\left(\frac{x}{p+\delta}\right) \right| \\ &\ll x^\varepsilon \max_{x^{7/15} < D \leq x^{8/15}} \int_D^{2D} \left| \frac{1}{\log t} d\left(\sum_{D < p \leq t} (\log p) \psi\left(\frac{x}{p+\delta}\right) \right) \right|. \end{aligned}$$

Noticing that

$$\sum_{D < p \leq t} (\log p) \psi\left(\frac{x}{p+\delta}\right) = \sum_{D < n \leq t} \Lambda(n) \psi\left(\frac{x}{n+\delta}\right) + O(t^{1/2}),$$

we can apply Proposition 4.1(i) (for $x^{7/15} \leq D \leq x^{34/67}$) and Proposition 4.1(ii) (for $x^{34/67} \leq D \leq x^{8/15}$) to get

$$(5.6) \quad \mathcal{R}_\delta^{\mathbb{1}_\mathbb{P}}(x, x^{7/15}) \ll_\varepsilon x^{7/15+\varepsilon}.$$

Now (1.5) follows from (5.4), (5.5) and (5.6). \square

6. PROOF OF THEOREM 2

6.1. Proof of (i).

Let $\frac{7}{15} < \theta < 1$. Similar to (5.4) and (5.5), we have

$$(6.1) \quad S_{\mathbb{1}_{\mathbb{P}}}(x, \theta) = \tilde{S}_{\mathbb{1}_{\mathbb{P}}}(x, \theta) + O(x^{7/15}),$$

where

$$(6.2) \quad \begin{aligned} \tilde{S}_{\mathbb{1}_{\mathbb{P}}}(x, \theta) &= \sum_{x^{1-\theta} < p \leq x^{8/15}} \left(\frac{x}{p} - \psi\left(\frac{x}{p}\right) - \frac{x}{p+1} + \psi\left(\frac{x}{p+1}\right) \right) \\ &= x \sum_{p \geq x^{1-\theta}} \frac{1}{p(p+1)} + \mathcal{R}_1^{\mathbb{1}_{\mathbb{P}}}(x, x^{1-\theta}) - \mathcal{R}_0^{\mathbb{1}_{\mathbb{P}}}(x, x^{1-\theta}) + O(x^{7/15}), \end{aligned}$$

where

$$\mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}}}(x, x^{1-\theta}) = \sum_{x^{1-\theta} < p \leq x^{8/15}} \psi\left(\frac{x}{p+\delta}\right).$$

Trivially we have

$$\mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}}}(x, x^{1-\theta}) \ll \sum_{x^{\max(7/15, 1-\theta)} < p \leq x^{8/15}} \psi\left(\frac{x}{p+\delta}\right) + x^{7/15}.$$

The same argument for proving (5.6) allows us to derive that for any $\varepsilon > 0$, the last sum is $O(x^{7/15+\varepsilon})$. Thus

$$\mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}}}(x, x^{1-\theta}) \ll_\varepsilon x^{7/15+\varepsilon}.$$

Inserting this into (6.2) and noticing that

$$x \sum_{p \geq x^{1-\theta}} \frac{1}{p^3} \ll x^{1-2(1-\theta)} \ll x^{2\theta-1},$$

we obtain that

$$(6.3) \quad \tilde{S}_{\mathbb{1}_{\mathbb{P}}}(x, \theta) = x \sum_{p \geq x^{1-\theta}} \frac{1}{p^2} + O_\varepsilon(x^{\max(7/15+\varepsilon, 2\theta-1)}).$$

With the help of the prime number theorem

$$(6.4) \quad \pi(t) := \sum_{p \leq t} 1 = \int_2^t \frac{du}{\log u} + O(t\mathcal{L}(t)^{-c})$$

where $c > 0$ is a positive constant, we easily derive that

$$\begin{aligned} x \sum_{p > x^{1-\theta}} \frac{1}{p^2} &= x \int_{x^{1-\theta}}^\infty \frac{du}{u^2 \log u} + O(x^\theta \mathcal{L}(x)^{-c_\theta}) \\ &= x \int_{x^{\theta-1}}^0 \frac{dt}{\log t} + O(x^\theta \mathcal{L}(x)^{-c_\theta}) \quad (u = 1/t). \end{aligned}$$

Inserting this into (6.3) and noticing that $\max(7/15 + \varepsilon, 2\theta - 1) < \theta$ (provided that $0 < \varepsilon \leq \frac{1}{2}(\theta - \frac{7}{15})$), it follows that

$$(6.5) \quad \tilde{S}_{\mathbb{1}_{\mathbb{P}}}(x, \theta) = x \int_0^{x^{\theta-1}} \frac{dt}{\log(1/t)} + O(x^\theta \mathcal{L}(x)^{-c_\theta}).$$

Now (1.9) follows from (6.1) and (6.5).

By integration by parts, it follows that

$$\begin{aligned} x \int_{x^{\theta-1}}^0 \frac{dt}{(\log t)^\ell} &= x \left(\left[\frac{t}{(\log t)^\ell} \right]_{x^{\theta-1}}^0 + \ell \int_{x^{\theta-1}}^0 \frac{dt}{(\log t)^{\ell+1}} \right) \\ &= \frac{(-1)^{\ell-1} x^\theta}{(1-\theta)^\ell (\log x)^\ell} + x \ell \int_{x^{\theta-1}}^0 \frac{dt}{(\log t)^{\ell+1}}. \end{aligned}$$

After a simple argument of recurrence, we can find that

$$x \int_0^{x^{\theta-1}} \frac{dt}{\log(1/t)} = \sum_{\ell=1}^L \frac{(-1)^{\ell-1} (\ell-1)!}{(1-\theta)^\ell} \frac{x^\theta}{(\log x)^\ell} + O\left(\frac{x^\theta}{(\log x)^{L+1}}\right) \quad (\ell \in \mathbb{N}^*).$$

Thus this and (1.9) imply the second assertion of (i).

6.2. Proof of (ii).

When $\frac{7}{15} < \theta \leq \frac{1}{2}$, we have $S_{\mathbb{1}_\mathbb{P}}(x, \theta) = \pi_{\mathbb{S}}(x, \theta)$. Thus the first assertion gives us immediately the required result.

Next we suppose that $\frac{1}{2} < \theta \leq 1$. In this case, the required result is essentially [12, Theorem 1]. In fact, noticing that $\left[\frac{x}{n}\right] = p \Leftrightarrow x/(p+1) < n \leq x/p$ and $n \leq x^\theta \Leftrightarrow x^{1-\theta} - 1 < p \leq x$, we can write

$$(6.6) \quad \pi_{\mathbb{S}}(x, \theta) = \sum_{x^{1-\theta} - 1 < p \leq x} \mathbb{1}\left(\left[\frac{x}{p}\right] - \left[\frac{x}{p+1}\right] > 0\right) = G_1(x, \theta) + G_2(x),$$

where $\mathbb{1}(Q) = 1$ if the statement Q is true and 0 otherwise, and

$$\begin{aligned} G_1(x, \theta) &:= \sum_{x^{1-\theta} - 1 < p \leq \sqrt{x}} \mathbb{1}\left(\left[\frac{x}{p}\right] - \left[\frac{x}{p+1}\right] > 0\right), \\ G_2(x) &:= \sum_{\sqrt{x} < p \leq x} \mathbb{1}\left(\left[\frac{x}{p}\right] - \left[\frac{x}{p+1}\right] > 0\right). \end{aligned}$$

For $p \leq \sqrt{x} - 1$, we have

$$\left[\frac{x}{p}\right] - \left[\frac{x}{p+1}\right] > \frac{x}{p(p+1)} - 1 > 0.$$

Thus the prime number theorem (6.4) gives us

$$(6.7) \quad G_1(x) = \pi(\sqrt{x}) + O(\pi(x^{1-\theta})) = \text{Li}(\sqrt{x}) + O(\sqrt{x} \mathcal{L}(x)^{-c_\theta})$$

for $x \geq 3$, where $c_\theta > 0$ is a positive constant depending on θ .

On the other hand, according to [12, (2.8)], we have

$$(6.8) \quad G_2(x) = \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)} + O(\sqrt{x} \mathcal{L}(x)^{-c}).$$

Now the required result (1.10) follows from (6.6), (6.7) and (6.8).

The second assertion is an immediate consequence of (1.10) thanks to a simple partial integration. \square

7. PROOF OF THEOREM 3

7.1. Proof of (1.11).

Let $N(x) \in [1, x^{2/3}]$ be a function to be chosen later. Write

$$(7.1) \quad S_{\Lambda}^{\sharp}(x) = \tilde{S}_{\Lambda}^{\sharp}(x) + O(N(x) \log x),$$

where

$$\tilde{S}_{\Lambda}^{\sharp}(x) := \sum_{N(x) < n \leq x} \mu(n)^2 \Lambda\left(\left[\frac{x}{n}\right]\right).$$

Noticing that $d = [x/n] \Leftrightarrow x/(d+1) < n \leq x/d$, we can write

$$\begin{aligned} \tilde{S}_{\Lambda}^{\sharp}(x) &= \sum_{d \leq x/N(x)} \Lambda(d) \sum_{\max(N(x), x/(d+1)) < n \leq x/d} \mu(n)^2 \\ &= \sum_{d \leq x/N(x)} \Lambda(d) \sum_{x/(d+1) < n \leq x/d} \mu(n)^2 + O(N(x) \log x), \end{aligned}$$

where we have used the following bound

$$\begin{aligned} \sum_{x/N(x)-1 < d \leq x/N(x)} \Lambda(d) \sum_{N(x) < n \leq x/d} \mu(n)^2 &\ll \sum_{n \leq 2N(x)} \mu(n)^2 \sum_{x/N(x)-1 < d \leq x/N(x)} \Lambda(d) \\ &\ll N(x) \log x. \end{aligned}$$

With the help of classic result

$$(7.2) \quad \sum_{n \leq x} \mu(n)^2 = (6/\pi^2)x + O(x^{1/2} \mathcal{L}(x)^{-3c}),$$

we can derive that

$$\tilde{S}_{\Lambda}^{\sharp}(x) = \sum_{d \leq x/N(x)} \Lambda(d) \left\{ \frac{6}{\pi^2} \cdot \frac{x}{d(d+1)} + O\left(\left(\frac{x}{d}\right)^{1/2} \mathcal{L}(x)^{-3c}\right) \right\} + O(N(x) \log x).$$

We have

$$\sum_{d \leq x/N(x)} \frac{\Lambda(d)}{d^{1/2}} \ll (x/N(x))^{1/2} \quad \text{and} \quad x \sum_{d \leq x/N(x)} \frac{\Lambda(d)}{d(d+1)} = c_{\Lambda} x + O(N(x)).$$

Inserting these into the preceding formula, it follows that

$$(7.3) \quad \tilde{S}_{\Lambda}^{\sharp}(x) = (6/\pi^2)c_{\Lambda} x + O(N(x) + xN(x)^{-1/2} \mathcal{L}(x)^{-3c}).$$

Now the required result follows from (7.1) and (7.3) thanks to the choice of $N(x) = x^{2/3} \mathcal{L}(x)^{-2c}$.

7.2. Proof of (1.12).

As before, we have trivially

$$(7.4) \quad S_{\mathbb{1}_{\mathbb{P}}}^{\sharp}(x) = \tilde{S}_{\mathbb{P}}^{\sharp}(x) + O(N(x)),$$

where

$$\tilde{S}_{\mathbb{1}_{\mathbb{P}}}^{\sharp}(x) := \sum_{N(x) < n \leq x} \mu(n)^2 \mathbb{1}_{\mathbb{P}}\left(\left[\frac{x}{n}\right]\right).$$

Using (7.2), we have

$$\begin{aligned} \tilde{S}_{\mathbb{1}_p}^{\sharp}(x) &= \sum_{p \leq x/N(x)} \sum_{x/(p+1) < n \leq x/p} \mu(n)^2 + O(N(x)) \\ &= \sum_{p \leq x/N(x)} \left\{ \frac{6}{\pi^2} \frac{x}{p(p+1)} + O\left(\left(\frac{x}{p}\right)^{1/2} \mathcal{L}(x)^{-3c}\right) \right\} + O(N(x)) \\ &= (6/\pi^2)c_{\mathbb{1}_p}x + O(N(x) + xN(x)^{-1/2}\mathcal{L}(x)^{-3c}). \end{aligned}$$

Inserting this into (7.4) and taking $N(x) = x^{2/3}\mathcal{L}(x)^{-2c}$, we get (1.12). \square

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