LOCAL ZETA FUNCTIONS FOR A CLASS OF P-ADIC SYMMETRIC SPACES (II)

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Part II:
Explicit functional equation for zeta functions attached to the minimal spherical principal series.

Abstract: *In this paper we study the zeta functions associated to the minimal spherical principal series of representations for a class of reductive p-adic symmetric spaces, which are realized as open orbits of some prehomogeneous spaces. These symmetric spaces have been studied in the paper arXiv: 2003.05764. We prove that the zeta functions satisfy a functional equation which is given explicitly (see Theorem 4.3.9 and Theorem 4.4.5). Moreover, for a subclass of these spaces, we define L-functions and ε-factors associated to the representations.*

AMS classification: 22E50, 11S40, 43A85, 22E35.

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INTRODUCTION

In this second part we define and study the local zeta functions associated to spherical representations for the class of symmetric spaces introduced in the first part (see [8]).

Let us first describe, formally, the general setting where this paper takes place. Let G be a connected reductive algebraic group over a p-adic local field F of characteristic zero. Suppose we are given an irreducible regular prehomogeneous vector space (G, V) defined over F (see [15] for example) and denote by Δ_0 the fundamental relative invariant. Then the dual representation (G, V^*) is still a prehomogeneous vector space of the same type. We denote by Δ_0^* its fundamental relative invariant.

Suppose, for sake of simplicity, only in this introduction, that (G, V) has only one open orbit Ω. Let H be the isotropy subgroup of an element $I \in \Omega$. Moreover, suppose that $G/H \simeq \Omega$ is a symmetric space corresponding to an involution σ of G.

Then the dual space (G, V^*) has the same property. More precisely the open orbit $\Omega^* \subset V^*$ contains the element $I^* = \frac{d\Delta_0}{\Delta_0}(I)$ (see [15]) and the isotropy subgroup of I^* is still H.

Consider now a minimal σ-split parabolic subgroup P of G (this means that P and $\sigma(P)$ are opposite and that P is minimal for this property). Again for sake of simplicity, we suppose that P also has a unique open orbit in V. It is well known (see [1]) that if χ is a character of L which is trivial on $L \cap H$, the induced representation $\pi_\chi = \mathrm{Ind}_P^G(\chi)$ is generically H- distinguished (or H-spherical). This means that for "almost all" characters χ the dual space I_χ^* of the space I_χ of π_χ contains a nonzero H-invariant vector ξ. Therefore if $w \in I_\chi$ the coefficient $\langle \pi_\chi^*(g)\xi, w\rangle$ is right H-invariant and hence can be considered as a function on Ω (or Ω^*). The minimal spherical series for G/H is the set of the representations π_χ.

Let $\mathcal{S}(V)$ (respectively $\mathcal{S}(V^*)$) be the spaces of locally constant functions with compact support on V (respectively V^*). Let also d^*X (resp. d^*Y) be the G-invariant measure on V (resp. V^*). For $\Phi \in \mathcal{S}(V)$ and $\Psi \in \mathcal{S}(V^*)$, and $s \in \mathbb{C}$, let us define (formally!) the following local zeta functions:

$$Z(\Phi, s, \xi, w) = \int_\Omega \Phi(X)|\Delta_0(X)|^s \langle \pi_\chi^*(X)\xi, w\rangle d^*X,$$

$$Z^*(\Psi, s, \xi, w) = \int_{\Omega^*} \Psi(Y)|\Delta_0^*(Y)|^s \langle \pi_\chi^*(Y)\xi, w\rangle d^*Y.$$

It is expected that these zeta functions can be correctly defined (via absolute convergence and meromorphic continuation) and that they should verify a functional of the type:

$$Z^*(\mathcal{F}(\Phi), m - s, \xi, w) = \gamma(s, \chi) Z(\Phi, s, \xi, w),$$

where $\mathcal{F} : \mathcal{S}(V) \longrightarrow \mathcal{S}(V^*)$ is the Fourier transform, m is a suitable "shift", and $\gamma(s, \chi)$ a meromorphic function.

The aim of this paper is to perform this program, including an explicit form of $\gamma(s, \chi)$ in terms of local Tate factors and Weil constants related to the Fourier transform of some quadratic characters. This is done even in the case where G has several open orbits, each of it being a symmetric space, for a class of p-adic prehomogeneous vector spaces (which is essentially the class described in the first part ([8])). See Theorem 4.3.9 and 4.4.5 below.

Our results contain, as a particular case, the case of $GL_n(F) \times GL_n(F)$ acting on the space $M_n(F)$ of n by n matrices, which gives rise to the Godement-Jacquet zeta function for the principal minimal series for $GL_n(F)$.
Moreover, in the case where G and P have a unique open orbit in V^+, the space of H-invariant linear form on I_χ is 1-dimensional. In that case we define and prove the existence of L-functions which describe the poles of the zeta functions $Z(\Phi, s, \xi, w)$ and $Z^*(\Psi, s, \xi, w)$ for all $\Phi \in \mathcal{S}(V^+)$, $\Psi \in \mathcal{S}(V^-)$ and $w \in I_\chi$. We define also the corresponding ε-factors. This generalizes the results of Godement-Jacquet for the principal minimal series for $GL_n(F)$. See Proposition 4.5.3 and Theorem 4.5.5 below.

Let us now describe briefly the content of the paper.
• In section 1 we re-define briefly the class of commutative prehomogeneous vector spaces we are interested in. These spaces are associated to 3-gradings of a reductive Lie algebra $\tilde{\mathfrak{g}}$ of the form

$$\tilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$$

which satisfy some regularity and irreducibility conditions. The facts about this objects which are needed in the sequel are recalled from [8] . For technical reasons we also introduce a subgroup $\tilde{P}$ of the σ-split parabolic P which will play an important role. The representations $(P, V^\pm)$ and $(\tilde{P}, V^\pm)$ are prehomogeneous and the open orbits are described.

• Section 2 is devoted to fix some imported tools. First of all we define the Fourier transform $\mathcal{F} : \mathcal{S}(V^+) \longrightarrow \mathcal{S}(V^-)$ and prove that there exists a unique pair of measures on $V^+ \times V^-$ which are dual for $\mathcal{F}$ and verify some additional compatibility condition. We also define and study the so-called *mean functions* (see definition 2.3.4) which correspond, roughly speaking, to integration on an $\mathcal{N}$ orbit where $\mathcal{N}$ is the nilradical of a σ-split parabolic, non necessarily minimal. We also normalize the measures on various subspaces of $\mathfrak{g}$. These normalizations are necessary to compute precisely the factors in the final functional equations. In this section these normalizations are also needed to compute precisely the Weil constants corresponding to some quadratic forms occurring in the classification of the orbits (see Proposition 2.5.5).

• As said before the representations $(\tilde{P}, V^\pm)$ are prehomogeneous and regular. But there are several fundamental relative invariants named $\Delta_0, \Delta_1, \ldots, \Delta_k$. In this situation one can classically define zeta functions associated to this prehomogeneous space. Roughly speaking they are of the form $\mathcal{K}(f, s) = \int_{V^+} f(X)|\Delta(X)|^s dX$ where $f \in \mathcal{S}(V^+)$, $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$ and where $|\Delta(X)|^s = |\Delta_0(X)|^{s_0} \ldots |\Delta_k(X)|^{s_k}$. See Definition 3.2.2. In such a situation it is known from the work of F. Sato ([15]) that there exists a functional equation if the prehomogeneous spaces satisfies a certain condition $(A2')$. In section 3 we prove that the spaces $(\tilde{P}, V^\pm)$ satisfy this condition (Theorem 3.1.3) and compute explicitly the constants in the functional equation (Theorem 3.3.3).

• Section 4 contains the main results. The symmetric spaces G/H_i (i=1,...,p) we consider are the open G-orbits in V^+. Let σ_i be the corresponding involution of G. The key point here is the fact that the parabolic subgroup P defined previously is minimal σ_i-split for all σ_i. Therefore these symmetric spaces have the same minimal spherical series.
The zeta functions associated to such representations are defined in 4.3.1 as integrals depending on several complex parameters μ_j and z. We prove that they are rational functions in the variables

$q^{\pm\mu_j}, q^{\pm z}$ (q is the residual characteristic), that they satisfy a functional equation which is explicitly computed (see Theorem 4.3.9). The main ingredient of the proofs is the work of P. Blanc and P. Delorme ([1]) and the close relation, via the Poisson kernel, between these zeta functions associated to representations and the zeta functions of the prehomogeneous space (P, V^+) studied in section 3. We also give a second version of the main Theorem, which modulo the introduction of a operator valued "gamma" factor, has a very simple form (see Theorem 4.4.5). Finally, in the case where G and P have a unique open orbit si V^+, we define and prove the existence of L-functions ε-factors associated to these representations (Proposition 4.5.3 and Theorem 4.5.5).

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1. A CLASS OF COMMUTATIVE PREHOMOGENEOUS VECTOR SPACES

1.1. Notations and preliminaries.

Let F be a p-adic field of characteristic 0, i.e. a finite extension of $\mathbb{Q}_p$. Moreover we will always suppose that the residue class field $\mathbf{k}$ has characteristic $\neq 2$ (non dyadic case) and we let q denote the cardinal of $\mathbf{k}$.

Let $\mathscr{O}_F$ be the ring of integers of F and denote by $\mathscr{O}_F^*$ the subgroup of units of F^*.

We fix a set of representatives $\{1, \varepsilon, \pi, \varepsilon\pi\}$ of $\mathscr{C} = F^*/F^{*2}$, where ε is a unit which is not a square in F^* and π is a uniformizer of F.

We will denote by $\overline{F}$ an algebraic closure of F. In the sequel, if U is a F-vector space, we will set

$$\overline{U} = U \otimes_F \overline{F}.$$

Definition 1.1.1. *Throughout this paper, a reductive Lie algebra $\widetilde{\mathfrak{g}}$ over F satisfying the following three hypothesis will be called a regular graded Lie algebra:*

($\mathbf{H_1}$) *There exists an element $H_0 \in \widetilde{\mathfrak{g}}$ such that $\mathrm{ad}H_0$ defines a $\mathbb{Z}$-grading of the form*

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+ \quad (V^+ \neq \{0\}) \,,$$

$$\text{where } [H_0, X] = \begin{cases} 0 & \text{for } X \in \mathfrak{g} \,; \\ 2X & \text{for } X \in V^+ \,; \\ -2X & \text{for } X \in V^- \,. \end{cases}$$

(Therefore, in fact, $H_0 \in \mathfrak{g}$)

($\mathbf{H_2}$) *The (bracket) representation of $\mathfrak{g}$ on $\overline{V^+}$ is irreducible. (In other words, the representation $(\mathfrak{g}, V^+)$ is absolutely irreducible)*

($\mathbf{H_3}$) *There exist $I^+ \in V^+$ and $I^- \in V^-$ such that $\{I^-, H_0, I^+\}$ is an $\mathfrak{sl}_2$-triple.*

Let us first recall the structure results of such algebras which were obtained in [8] and used in the rest of the paper.

We fix a maximal split abelian Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ containing H_0. Then $\mathfrak{a}$ is also maximal split abelian in $\widetilde{\mathfrak{g}}$. We denote by $\widetilde{\Sigma}$ and Σ the roots system of $(\widetilde{\mathfrak{g}}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{a})$ respectively.

Let H_λ be the coroot of $\lambda \in \widetilde{\Sigma}$. For $\mu \in \mathfrak{a}^*$, we denote by $\widetilde{\mathfrak{g}}^\mu$ the subspace of weight μ of $\widetilde{\mathfrak{g}}$.

From Theorem 1.6.1, Proposition 1.7.7 and Theorem 1.8.1 in [8], we know that there exists a unique $k \in \mathbb{N}$ and a family of 2 by 2 strongly orthogonal roots $\lambda_0, \ldots, \lambda_k$ in $\widetilde{\Sigma}^+ \setminus \Sigma$, unique modulo the action of the Weyl group of Σ, such that:

(**S1**) $H_0 = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}$,

(**S2**) If we set $\mathfrak{a}^0 = \oplus_{j=0}^k F H_{\lambda_j}$ and if for $p, q \in \mathbb{Z}$, we define $E_{i,j}(p,q)$ to be the space of $X \in \widetilde{\mathfrak{g}}$ such that $[H_{\lambda_s}, X] = \begin{cases} pX & \text{if } s = i\ ; \\ qX & \text{if } s = j\ ; \\ 0 & \text{if } s \notin \{i,j\}\ . \end{cases}$.

Then we have the following decompositions

$$(a) \qquad \mathfrak{g} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \left(\oplus_{i \neq j} E_{i,j}(1,-1)\right)\ ;$$
$$(b) \qquad V^+ = \left(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}\right) \oplus \left(\oplus_{i<j} E_{i,j}(1,1)\right)\ ;$$
$$(c) \qquad V^- = \left(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{-\lambda_j}\right) \oplus \left(\oplus_{i<j} E_{i,j}(-1,-1)\right)\ .$$

In the rest of the paper, we fix such a family $(\lambda_0, \ldots, \lambda_k)$ of roots. The integer $k+1$ is called the *rank* of the graded Lie algebra $\widetilde{\mathfrak{g}}$. From Proposition 1.9.3 in [8], we know that for $j = 0, \ldots, k$, the spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ have the same dimension and also that for $i \neq j$, the spaces $E_{i,j}(\pm 1, \pm 1)$ have the same dimension.

Notation 1.1.2. In the rest of the paper we will use the following notations:

$$\ell = \dim \widetilde{\mathfrak{g}}^{\lambda_j} \text{ for } j = 0, \ldots, k$$
$$d = \dim E_{i,j}(\pm 1, \pm 1) \text{ for } i \neq j \in \{0, \ldots, k\}$$
$$e = \dim \tilde{\mathfrak{g}}^{(\lambda_i + \lambda_j)/2} \text{ for } i \neq j \in \{0, \ldots, k\} \ (e \text{ may be equal to } 0).$$

From the classification of the simple graded Lie algebras ([8] §2.2), the integer ℓ is either the square of an integer, or equal to 3 and $e \in \{0,1,2,3,4\}$. Moreover $d - e$ is even.

Let $\mathfrak{k}$ be a reductive Lie algebra. Denote by $\mathrm{Aut}(\mathfrak{k})$ the group of automorphisms of $\mathfrak{k}$ and by $\mathrm{Aut}_e(\mathfrak{k})$ the subgroup of elementary automorphisms, that is the automorphisms which are finite products of $e^{\mathrm{ad}x}$, where $\mathrm{ad}(x)$ is nilpotent on $\mathfrak{k}$. Define

$$\mathrm{Aut}_0(\mathfrak{k}) = \mathrm{Aut}(\mathfrak{k}) \cap \mathrm{Aut}_e(\widetilde{\mathfrak{k}} \otimes \overline{F}).$$

Let $G = \mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0) = \{g \in \mathrm{Aut}_0(\widetilde{\mathfrak{g}}),\ g.H_0 = H_0\}$ be the centralizer of H_0 in $\mathrm{Aut}_0(\widetilde{\mathfrak{g}})$.

From loc.cit., the group G is the group of F-point of an algebraic group defined over F. Moreover, the Lie algebra of the group G is $\mathfrak{g} \cap [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] + [V^-, V^+]$.

We recall also that (G, V^+) is an absolutely irreducible prehomogeneous vector space, of commutative type, and regular. By ([8] Theorem 1.11.4), there exists on V^+ a unique (up to scalar multiplication) relative invariant polynomial Δ_0 which is absolutely irreducible (i.e. irreducible as a polynomial on $\overline{V^+}$). We denote by χ_0 the character of G associated to Δ_0.

Similarly , (G, V^-) is a prehomogeneous vector space of the same type, whose absolutely irreducible polynomial relative invariant is denoted by ∇_0. Its associated character is denoted by χ_0^-.

We set

$$m = \frac{\dim V^+}{\deg(\Delta_0)}. \tag{1.1}$$

Let dX (resp. dY) be arbitrary Haar measures on V^+ (resp. V^-). As we know from ([8], proof of Lemma 4.5.4) that $\det g_{|V^+} = \chi_0(g)^m$ and $\det g_{|V^-} = \chi_0(g)^{-m}$, it is worth noticing that

$$d^*X = \frac{dX}{|\Delta_0(X)|^m} \text{ and } d^*Y = \frac{dY}{|\nabla_0(Y)|^m} \tag{1.2}$$

are G-invariant measures on V^+ and V^- respectively.

In this article, we will always suppose that $\ell = 1$, except in section 2 in which the results of §2.1, §2.2, §2.3 and §2.4 are valid without any condition on ℓ. For $\ell = 1$, we have $m = 1 + \frac{kd}{2}$.

According to ([8] §1.10), we fix a non degenerate extension $\widetilde{B}$ of the Killing form of $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ to $\widetilde{\mathfrak{g}}$. We define the *normalized Killing form* by setting:

$$b(X, Y) = -\frac{k+1}{4 \dim V^+}\widetilde{B}(X, Y) \quad X \in, Y \in \tilde{\mathfrak{g}}. \tag{1.3}$$

Let $X_s \in \tilde{\mathfrak{g}}^{\lambda_s}$ for $s = 0, \dots k$. For $i \neq j$ let us consider the quadratic form q_{X_i,X_j} on $E_{i,j}(-1,-1)$ defined by

$$q_{X_i,X_j}(Y) = -\frac{1}{2}b([X_i, Y], [X_j, Y]). \tag{1.4}$$

Definition 1.1.3. *We fix $\mathfrak{sl}_2$-triples $\{Y_s, H_{\lambda_s}, X_s\}$, $s \in \{0, \dots, k\}$, with $Y_s \in \tilde{\mathfrak{g}}^{-\lambda_s}$ and $X_s \in \tilde{\mathfrak{g}}^{\lambda_s}$ such that, for $i \neq j$, the quadratic forms q_{X_i,X_j} are all G-equivalent (this means that there exists $g \in G$ such that $q_{X_0,X_1} = q_{X_i,X_j} \circ g$), and such that each of the forms q_{X_i,X_j} represents 1 (i.e. there exists $u \in E_{i,j}(-1,-1)$ such that $q_{X_i,X_j}(u) = 1$). The existence of such triples is given in ([8] Proposition 3.5.2). We set*

$$q_e := q_{X_0,X_1}, \quad I^+ = X_0 + \ldots + X_k, \quad \textit{et} \quad I^- = Y_0 + \ldots Y_k.$$

We normalize the polynomials Δ_0 and ∇_0 by setting $\Delta_0(I^+) = \nabla_0(I^-) = 1$.

From Proposition 3.5.2 of [8] (and its proof) we know the following facts concerning q_e:
- The quadratic form q_0 is hyperbolic of rank $d = \dim E_{0,1}(-1,-1)$ (sum of $d/2$ hyperbolic planes).
- If $e \neq 0$, the quadratic form q_e is the sum of an anisotropic form $q_{an,e}$ of rank e, which represents 1, and of an hyperbolic form $q_{hyp,d-e}$ of rank $d - e$.
Setting $q_{an,0} = 0$, we can write $q_e = q_{an,e} + q_{hyp,d-e}$.

From Lemma 3.8.6 of [8], the form $q_{an,2}$ represents exactly two classes of squares. As 1 is represented, there exists another class of squares, say $-\alpha.(F^*)^2$ which is represented (here $-\alpha \notin (F^*)^2$ of course).
- Suppose first that $\alpha \notin (F^*)^2$. Then the quadratic extension $E = F[\sqrt{\alpha}]$ of F is such that $N_{E/F}(E^*) = \mathrm{Im}(q_{an,2})^*$, where $N_{E/F}$ is the norm associated to E.
- If $\alpha \in (F^*)^2$. Then we are in the case where $-1 \notin (F^*)^2$ and $q_{an,2}$ represents the two classes 1 and -1. In that case $q_{an,2} \simeq x^2 + y^2$ because two anisotropic forms of rank 2 are equivalent if and only if they represent the same two classes of squares ([8] Lemma 3.8.6) and because -1 is the sum

of two squares ([11] Chapter VI, Corollary 2.6. p.154). In that case if we set $E = F[\sqrt{-1}]$, we have $N_{E/F}(E^*) = \mathrm{Im}(q_{an,2})^*$.
Hence we have shown that there exists a quadratic extension $E = F[\xi]$ of F (unique up to isomorphism) such that $N_{E/F}(E^*) = \mathrm{Im}(q_{an,2})^*$.

Definition 1.1.4. .
Define $S_e = \{a \in F^*, aq_e \simeq q_e\}$. *Then*

$$S_e = \begin{cases} F^* & \text{for} \quad e = 0 \text{ or } 4 \\ F^{*2} & \text{for} \quad e = 1 \text{ or } 3 \\ N_{E/F}(E^*) = \mathrm{Im}(q_{an,e})^* & \text{for} \quad e = 2. \end{cases}$$

We denote by $\mathscr{S}_e$ *a set of representatives of* F^*/S_e *in* F^*/F^{*2}.

As two hyperbolic forms of the same rank are always equivalent, we also have $S_e = \{a \in F^*, aq_{an,e} \simeq q_{an,e}\}$.
The precise description of the G-orbits in V^+ has been given in ([8] §3). The number of orbits depends on e and, in some cases, on k (cf. [8] Theorem 3.6.3). This is summarized as follows:

(1) If $e = 0$ or 4, the group G has a unique open orbit in V^+,
(2) if $e \in \{1, 2, 3\}$, the number of open G-orbits in V^+ depends on e and on the parity of k:
   (a) if $e = 2$ then G has a unique open orbit in V^+ if k is even and 2 open orbits if k is odd,
   (b) if $e = 1$ then G has a unique open orbit in V^+ if $k = 0$, it has 4 open orbits if $k = 1$, it has 2 open orbits if $k \geq 2$ is even , and 5 open orbits if $k \geq 2$ is odd .
   (c) if $e = 3$, then G has 4 open orbits.

Let us denote by $\Omega^{\pm}$ the union of the open G-orbits $V^{\pm}$. Then one has:

$$\Omega^+ = \{X \in V^+; \Delta_0(X) \neq 0\} \quad \text{and} \quad \Omega^- = \{Y \in V^-; \nabla_0(Y) \neq 0\}.$$

If $X \in \Omega^+$, there exists a unique element $\iota(X) \in \Omega^-$ such that $\{\iota(X), H_0, X\}$ is an $\mathfrak{s}l_2$-triple.
From ([8] Remark after Definition 4.5.5), we know that the mapping $\iota : \Omega^+ \to \Omega^-$ is a G-equivariant isomorphism from Ω^+ on Ω^-.

Definition 1.1.5. *Let* $\{y, h, x\}$ *be a* $\mathfrak{s}l_2$*-triple with* $y \in V^-, h \in \mathfrak{a}$ *and* $x \in V^+$. *For* $t \in F^*$, *we define the following elements of* G*:*

$$\theta_x(t) = e^{t\ \mathrm{ad}_{\mathfrak{g}} x} e^{t^{-1}\ \mathrm{ad}_{\mathfrak{g}} y} e^{t\ \mathrm{ad}_{\mathfrak{g}} x} \quad \text{and} \quad h_x(t) = \theta_x(t)\theta_x(-1).$$

By ([4] Chap. VIII, §1, Proposition 6), $h_x(t)$ *acts by* t^n *on the weight space of weight* n *under the action of* h.
By ([8] Lemma 1.11.3), the element $h_{I^+}(\sqrt{t})$ *belongs to* G *(and even to the Levi subgroup* L *defined below). It stabilizes* $\mathfrak{g}$ *and acts by* $t.Id_{V^+}$ *on* V^+. *We set*

$$m_t = h_{I^+}(\sqrt{t}).$$

1.2. **The subgroups P and $\tilde{P}$ and their open orbits in V^+.**
Set $\mathfrak{a}^0 = \oplus_{j=0}^k FH_{\lambda_j}$. Consider the parabolic subgroup P of G defined as follows (see [8] §4.2):

$$P = LN, \text{ where } L = Z_G(\mathfrak{a}^0) \text{ and } N = \exp \operatorname{ad} \mathfrak{n}, \ \mathfrak{n} = \oplus_{0\leq i<j\leq k} E_{i,j}(1,-1).$$

The representations (P, V^+) and (P, V^-) are prehomogeneous (see [8] Proposition 4.3.1). The corresponding sets of fundamental relative invariants are respectively denoted by $\Delta_0, \ldots, \Delta_k$ and $\nabla_0, \ldots, \nabla_k$. These polynomials are normalized by the conditions $\Delta_j(I^+) = \nabla_j(I^-) = 1$ for $j = 0, \ldots, k$. Let χ_j and χ_j^- be the characters of P associated respectively to Δ_j and ∇_j pour $j = 0, \ldots, k$.

From ([8] Theorem 4.3.6 and Theorem 4.5.3), we know that the union $\mathcal{O}^\pm$ of the open P-orbits in $V^\pm$ is given by

$$\mathcal{O}^+ = \{X \in V^+; \Delta_j(X) \neq 0, j = 0, \ldots, k\} \quad \text{and} \quad \mathcal{O}^- = \{Y \in V^+; \nabla_j(Y) \neq 0, j = 0, \ldots, k\}.$$

Definition 1.2.1.

(1) *As $\ell = 1$ and as L centralizes $\mathfrak{a}^0 = \oplus_{j=0}^k FH_{\lambda_j}$, each element $l \in L$ acts by scalar multiplication on X_j for $j = 0, \ldots, k$. Define the character x_j of L by*

$$l.X_j = x_j(l)X_j, \quad l \in L.$$

(2) *The subgroups $\tilde{L}$ and $\tilde{P}$ are defined by*

$$\tilde{L} = \{l \in L; x_j(l) \in S_e, \ j = 0, \ldots, k\}, \quad \tilde{P} = \tilde{L}N.$$

Remark 1.2.2.
Define the map $\mathcal{T}$ by

$$\begin{array}{rcl} \mathcal{T} : L & \longrightarrow & (F^*)^{k+1} \\ l & \longmapsto & (x_0(l), \ldots, x_k(l)) \end{array}$$

From Theorem 3.8.8, Theorem 3.8.9 (1) and (2)(d), and Theorem 3.8.10 (1) and (2)(d) in [8] we can see that

$$\mathrm{Im}\mathcal{T} = \cup_{x\in\mathscr{S}_e} x S_e^{k+1}.$$

In particular we have $\mathcal{T}(\tilde{L}) = S_e^{k+1}$.

Definition 1.2.3. *For $a = (a_0, \ldots, a_k) \in \mathscr{S}_e^{k+1}$, define the sets*

$$\mathcal{O}^+(a) = \{X \in V^+; \Delta_j(X)a_j \ldots a_k \in S_e, \text{ for } \ j = 0, \ldots, k\},$$

and

$$\mathcal{O}^-(a) = \{Y \in V^-; \nabla_j(Y)a_0 \ldots a_{k-j} \in S_e, \text{ for } \ j = 0, \ldots, k\}.$$

Lemma 1.2.4.
Remember that $m_x \in L$, $x \in F^*$ is the element of L whose action on V^+ is the multiplication by x (cf. Definition 1.1.5).
1) *One has*

$$L = \cup_{x\in\mathscr{S}_e} m_x \tilde{L}$$

2) *The representations $(\tilde{P}, V^\pm)$ are prehomogeneous vector spaces which have the same sets of fundamental relative invariants as $(P, V^\pm)$,*

3) *The open orbits of $\tilde{P}$ in V^+ (resp. V^-) are the sets*

$$\mathcal{O}^+(a) = \tilde{P}.I^+(a), \text{ where } I^+(a) = a_0X_0 + \ldots + a_kX_k,$$

$$\text{(respectively } \mathcal{O}^-(a) = \tilde{P}.I^-(a), \text{ where } I^-(a) = a_0^{-1}Y_0 + \ldots + a_k^{-1}Y_k),$$

for $a = (a_0, \ldots, a_k) \in \mathscr{S}_e^{k+1}$.
Moreover, any open P-orbit in $V^\pm$ is given by $\cup_{x\in\mathscr{S}_e} m_x\mathcal{O}^\pm(a)$ for $a = (a_0, \ldots, a_{k-1}, 1) \in \mathscr{S}_e^{k+1}$.

Proof.
If $e = 0$ or 4, we have $S_e = F^*$ and hence $\mathscr{S}_e = \{1\}$ and $P = \tilde{P}$. Then Theorem 4.3.6 and Theorem 4.5.3 in [8] imply the result.

Suppose now that $e \in \{1, 2, 3\}$. This implies $k \geq 1$.

Using Theorem 3.8.8, Theorem 3.8.9 (1) et (2)(d), and Theorem 3.8.10 (1) et (2)(d) in [8] , we know that two diagonal elements $a_0X_0 + \ldots + a_kX_k$ and $b_0X_0 + \ldots + b_kX_k$ are L-conjugated if and only if there exists $x \in \mathscr{S}_e$ such that $xa_jb_j \in S_e$ for all $j = 0, \ldots, k$.
If $l \in L$, then $l.I^+ = x_0(l)X_0 + \ldots + x_k(l)X_k$. Hence there exists $x \in \mathscr{S}_e$ such that $xx_j(l) \in S_e$ for all $j = 0, \ldots, k$. Therefore $m_xl \in \tilde{L}$, this proves assertion 1).

As $P/\tilde{P}$ is a finite group, assertion 2) is true.

Let us now show assertion 3). Let $a = (a_0, \ldots, a_k) \in \mathscr{S}_e^{k+1}$. From Theorem 4.3.2 in [8], we have for $l \in \tilde{L}$ and $n \in N$:

$$\Delta_j(ln.(a_0X_0 + \ldots + a_kX_k)) = \prod_{s=j}^{k} x_s(l)a_s.$$

As $x_s(l) \in S_e$ for $s = 0, \ldots, k$, we obtain from the definition of $\mathcal{O}^+(a_0, \ldots, a_k)$ that $\tilde{P}.(a_0X_0 + \ldots + a_kX_k) \subset \mathcal{O}^+(a_0, \ldots, a_k)$.

Conversely, let $Z \in \mathcal{O}^+(a_0, \ldots, a_k)$. From Lemma 4.3.3 in [8], there exists $n \in N$ et $z_0, \ldots z_k$ in F^* such that $n.Z = z_0X_0 + \cdots + z_kX_k$. From the definition of $\mathcal{O}^+(a_0, \ldots, a_k)$ (and as $F^{*2} \subset S_e$), there exist $\mu_j \in S_e$ such that $z_j = a_j\mu_j$ for $j = 0, \ldots, k$.
From Remark 1.2.2 there exists $l \in \tilde{L}$ such that $x_j(l) = \mu_j^{-1}$ for $j = 0, \ldots, k$. Then we have $ln.Z = a_0X_0 + \ldots + a_kX_k$ and hence

$$\mathcal{O}^+(a_0, \ldots, a_k) = \tilde{P}.I^+(a).$$

The proof concerning the open $\tilde{P}$-orbits in V^- is similar. And the last assertion is a consequence of the description of the P-orbits in $V^\pm$ given in ([8] Theorem 4.3.6 and Theorem 4.5.3). □

2. Mean functions and Weil formula

Except for the last section §2.5 which is only valid for $\ell = 1$, the results of this paragraph are valid for any regular graded algebra (see Definition 1.1.1). Hence ℓ is either the square of an integer, or $\ell = 3$ (Notation 1.1.2).

2.1. **A class of parabolic subgroups.**

Let us fix $p \in \{0, \dots, k-1\}$ and define the following elements:

$$I_{p,1}^+ = \sum_{i=0}^{p} X_i, \qquad H_0^{p,1} = \sum_{i=0}^{p} H_{\lambda_i}, \quad \text{and} \quad I_{p,1}^- = \sum_{i=0}^{p} Y_i,$$

and

$$I_{p,2}^+ = \sum_{i=p+1}^{k} X_i, \qquad H_0^{p,2} = \sum_{i=p+1}^{k} H_{\lambda_i}, \quad \text{and} \quad I_{p,2}^- = \sum_{i=p+1}^{k} Y_i.$$

Hence we have

$$I^+ = I_{p,1}^+ + I_{p,2}^+, \quad H_0 = H_0^{p,1} + H_0^{p,2}, \quad \text{and} \quad I^- = I_{p,1}^- + I_{p,2}^-,$$

and $\{I_{p,j}^-, H_0^{p,j}, I_{p,j}^+\}$ is an $\mathfrak{sl}_2$-triple, for $j = 1, 2$.
For a fixed $p \in \{0, 1, \dots, k\}$ and $i, j \in \mathbb{Z}$ we define the following spaces:

$$K_p(i,j) = \{X \in \mathfrak{g},\ [H_{0,p}^1, X] = iX,\ [H_{0,p}^2, X] = jX\}.$$

The same way as in ([8] Theorem 1.8.1) one shows that $K_p(i,j) \neq \{0\} \Longrightarrow |i| + |j| \leq 2$.
More precisely one obtains the following decompositions:

$$\begin{aligned} V^- &= K_p(-2,0) \oplus K_p(-1,-1) \oplus K_p(0,-2), \\ \mathfrak{g} &= K_p(1,-1) \oplus K_p(0,0) \oplus K_p(-1,1), \\ V^+ &= K_p(2,0) \oplus K_p(1,1) \oplus K_p(0,2). \end{aligned}$$

Hence there are four new regular prehomogeneous vector spaces (corresponding to graded algebras) at hand. Namely $(K_p(0,0), K_p(2,0))$, $(K_p(0,0), K_p(0,2))$, $(K_p(0,0), K_p(-2,0))$, $(K_p(0,0), K_p(0,-2))$.
For further use we need to make explicit the fundamental relative invariants corresponding to these spaces.
The graded Lie algebra $\tilde{\mathfrak{l}}_j = \tilde{\mathfrak{g}}^{-\lambda_j} \oplus [\tilde{\mathfrak{g}}^{-\lambda_j}, \tilde{\mathfrak{g}}^{\lambda_j}] \oplus \tilde{\mathfrak{g}}^{\lambda_j}$ satisfies also the conditions of Definition 1.1.1.
We denote by κ the common degree of the corresponding relative invariants (see [8], section 1.13). More precisely $\kappa = \delta$ if $\ell = \delta^2$ and $\kappa = 2$ if $\ell = 3$.

Definition 2.1.1.
Let $p = 0, \dots, k-1$. From the definition in ([8] §1.14 and §4.5) the fundamental relative invariant of $(K_p(0,0), K_p(0,2))$ is Δ_{p+1} whose degree is $\kappa(k+1-(p+1)) = \kappa(k-p)$. Similarly the fundamental relative invariant of $(K_p(0,0), K_p(-2,0)$ is ∇_{k-p} whose degree is $\kappa(k+1-(k-p)) = \kappa(p+1)$.
We denote by $\widetilde{\Delta}_{k-p}$ the fundamental relative invariant of $(K_p(0,0), K_p(2,0))$ whose degree is $\kappa(k+1-(k-p)) = \kappa(p+1)$.
We denote by $\widetilde{\nabla}_{p+1}$ the fundamental relative invariant of $(K_p(0,0), K_p(0,-2))$ whose degree is $\kappa(k+1-(p+1)) = \kappa(k-p)$.
It is easy to see from the definitions that the polynomials $\widetilde{\Delta}_p$ $(p = 0, \dots, k-1)$ are the fundamental relative invariants of the prehomogeneous vector space (P^-, V^+) where P^- is the parabolic subgroup opposite to P. Also the polynomials $\widetilde{\nabla}_p$ $(p = 0, \dots, k-1)$ are the fundamental relative invariants of the prehomogeneous vector space (P, V^-).

Lemma 2.1.2. *The representations $(K_p(0,0), K_p(2,0))$ and $(K_p(0,0), K_p(0,2))$ are absolutely irreducible.*

Proof. Recall that we denote by $\overline{F}$ the algebraic closure of $F$and for any F-vector space U, we define

$$\overline{U} = U \otimes_F \overline{F}$$

The space $\overline{K_p(0,2)}$ contains the highest weight vector of $\overline{V^+}$ (for the action of $\overline{\mathfrak{g}}$). Let v_1 be such a non zero vector. Of course v_1 is also a highest weight vector for the action of $\overline{K_p(0,0)}$. Suppose that $\overline{K_p(0,2)}$ is reducible under the action of $\overline{K_p(0,0)}$. Then it would exist a second highest weight vector v_2 for this action. Let us examine how these vectors move under the action of $\overline{n}^-$ (= the sum of the negative root spaces in $\overline{\mathfrak{g}}$). One has

$$\overline{n}^- = \overline{K_p(1,-1)} \oplus \overline{K_p(0,0)}^-.$$

The action of $\overline{K_p(1,-1)}$ on $\overline{K_p(0,2)}$ is as follows:

$$v_1, v_2 \in \overline{K_p(0,2)} \overset{\mathrm{ad}\overline{K_p(1,-1)}}{\longrightarrow} \overline{K_p(1,1)} \overset{\mathrm{ad}\overline{K_p(1,-1)}}{\longrightarrow} \overline{K_p(2,0)} \overset{\mathrm{ad}\overline{K_p(1,-1)}}{\longrightarrow} \{0\}.$$

On the other hand the action of $\overline{K_p(0,0)}^-$ stabilizes $\overline{K_p(0,2)}$

$$v_1, v_2 \in \overline{K_p(0,2)} \overset{\mathrm{ad}\overline{K_p(0,0)}^-}{\longrightarrow} \overline{K_p(0,2)}.$$

Let $\mathcal{U}(\overline{K_p(0,0)})$ be the enveloping algebra of $\overline{K_p(0,0)}$. As $\overline{K_p(0,2)}$ is supposed to be reducible under $\overline{K_p(0,0)}$ we obtain

$$\overline{V_1} = \mathcal{U}(\overline{K_p(0,0)})v_1 \subsetneq \overline{K_p(0,2)}.$$

As $\overline{V^+}$ is irreducible under $\overline{\mathfrak{g}}$ we should have:

$$\begin{aligned} \overline{V^+} &= \mathcal{U}(\overline{K_p(1,-1)})\mathcal{U}(\overline{K_p(0,0)})v_1 \\ &= \mathcal{U}(\overline{K_p(1,-1)})\overline{V_1} \subset \overline{V_1} \oplus \overline{K_p(1,1)} \oplus \overline{K_p(2,0)} \subset \overline{K_p(0,2)} \oplus \overline{K_p(1,1)} \oplus \overline{K_p(2,0)} = \overline{V^+}. \end{aligned}$$

But this is not possible as $\overline{V_1} \subsetneq \overline{K_p(0,2)}$.
As $\overline{K_p(2,0)}$ contains a lowest weight vector of the representation $(\overline{\mathfrak{g}}, \overline{V^+})$, the proof for $\overline{K_p(2,0)}$ is analogue. □

It is worth noticing that from the preceding Lemma 2.1.2 and from Proposition 1.5.2 in [8] the Lie algebras $K_p(-2,0) \oplus [K_p(2,0), K_p(-2,0)] \oplus K_p(2,0)$ and $K_p(0,-2) \oplus [K_p(0,2), K_p(0,-2)] \oplus K_p(0,2)$ are absolutely simple.

Let us fix $p \in \{0,1,\ldots,k\}$. Consider now the maximal parabolic subalgebra of $\mathfrak{g}$ given by

$$\begin{aligned} \mathfrak{p}_p &= \mathcal{Z}_{\mathfrak{g}}(H^1_{0,p}) \oplus K_p(1,-1) = \mathcal{Z}_{\mathfrak{g}}(H^2_{0,p}) \oplus K_p(1,-1) = \mathcal{Z}_{\mathfrak{g}}(F.H^1_{0,p} \oplus F.H^2_{0,p}) \oplus K_p(1,-1) \\ &= K_p(0,0) \oplus K_p(1,-1) \end{aligned}$$

(As $H_0 = H^1_{0,p} + H^2_{0,p}$ and as $\mathfrak{g} = \mathcal{Z}_{\widetilde{\mathfrak{g}}}(H_0)$, we get $\mathcal{Z}_{\mathfrak{g}}(H^1_{0,p}) = \mathcal{Z}_{\mathfrak{g}}(H^2_{0,p}) = \mathcal{Z}_{\mathfrak{g}}(F.H^1_{0,p} \oplus F.H^2_{0,p})$.)

The maximal parabolic subgroup P_p corresponding to $\mathfrak{p}_p$ is now defined as follows. Define first:

$$N_p = \exp\ \mathrm{ad}K_p(1,-1)$$

$$L_p = \mathcal{Z}_G(F.H^1_{0,p} \oplus F.H^2_{0,p}) = \mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(F.H^1_{0,p} \oplus F.H^2_{0,p}) = \mathcal{Z}_G(F.H^1_{0,p}) = \mathcal{Z}_G(F.H^2_{0,p}).$$

Then

$$P_p = L_p N_p$$

The Lie algebra of L_p is $K_p(0,0)$, the Lie algebra of N_p is $K_p(1,-1)$ and hence the Lie algebra of P_p is $K_p(0,0) \oplus K_p(1,-1)$.

2.2. **Fourier transform and normalization of measures.**

We will, in the sequel of this paragraph, normalize the Haar measures such that there is no constant appearing in some integral formulas occurring later, in particular in Theorem 2.3.3 and Theorem 2.4.2. We use here the same method as in ([2] chap. 4) and in ([12] §3.2). For the convenience of the reader we give some details.

Remember that we use the normalized Fourier Killing form $b(\cdot,\cdot)$ defined by

$$b(X,Y) = -\frac{k+1}{4 \dim V^+}\widetilde{B}(X,Y) \quad X \in, Y \in \tilde{\mathfrak{g}}.$$

Using b the space V^- is identified with the dual space V^{+*}, and the natural action of $g \in G$ on V^- corresponds to the dual action of g on V^{+*}.

We fix once and for all a non trivial additive character ψ of F. Consider Haar measures dX and dY on V^+ and V^- respectively.

Definition 2.2.1. *The Fourier transform $f \in \mathcal{S}(V^+)$ is the function $\mathcal{F}(f) \in \mathcal{S}(V^-)$ defined by*

$$\mathcal{F}(f)(Y) = \int_{V^+} f(X)\psi(b(X,Y))dX, \quad Y \in V^-,$$

and the inverse Fourier transform of $g \in \mathcal{S}(V^-)$ is the function $\overline{\mathcal{F}}(g) \in \mathcal{S}(V^+)$ defined by

$$\overline{\mathcal{F}}(g)(X) = \int_{V^-} g(Y)\overline{\psi(b(X,Y))}dY, \quad X \in V^+.$$

Using the same character ψ and the same form b, we define similarly:
- Fourier transforms $\mathcal{F}_{2,0}$ anf $\overline{\mathcal{F}}_{2,0}$ between $\mathcal{S}(K_p(2,0))$ and $\mathcal{S}(K_p(-2,0))$,
- Fourier transforms $\mathcal{F}_{0,2}$ anf $\overline{\mathcal{F}}_{0,2}$ between $\mathcal{S}(K_p(0,2))$ and $\mathcal{S}(K_p(0,-2))$,
- Fourier transforms $\mathcal{F}_{1,1}$ anf $\overline{\mathcal{F}}_{1,1}$ between $\mathcal{S}(K_p(1,1))$ and $\mathcal{S}(K_p(-1,-1))$,
- Fourier transforms $\mathcal{F}_{-1,1}$ anf $\overline{\mathcal{F}}_{-1,1}$ between $\mathcal{S}(K_p(-1,1))$ and $\mathcal{S}(K_p(1,-1))$.

It is well known that there exists a pair of Haar measures (dX, dY) on V^+ and V^- respectively wich are so-called *dual* for $\mathcal{F}$. This means that $\overline{\mathcal{F}} \circ \mathcal{F} = Id_{\mathcal{S}(V^+)}$. Moreover if such a pair (dX, dY) of dual measures is given, any other pair of dual measures is of the form $(\alpha dX, \frac{1}{\alpha}dY)$, with $\alpha \in \mathbb{R}^{+*}$.

Let us indicate one way to construct such dual measures. Define the Fourier transform on F by

$$\mathcal{F}_1 h(y) = \int_F h(x)\psi(xy)dx, \quad h \in \mathcal{S}(F),\ \mathcal{F}_1 h \in \mathcal{S}(F),$$

and fix once and for all a self-dual measure (for $\mathcal{F}_1$) dx on F.

Let us choose a basis $(e_1, \dots, e_n)$ of V^+ which contains a base of each root space contained in V^+. Consider the dual base $(e_1^*, \dots, e_n^*)$ of V^- (*i.e* $b(e_i, e_j^*) = \delta_{ij}$). Denote by $X = \sum_{i=1}^n x_i e_i$ (resp. $Y = \sum_{j=1}^n y_j e_j^*$) a general element in V^+ (resp. in V^-) and define the measures

$$d_1X = dx_1 \dots dx_n \quad d_1Y = dy_1 \dots dy_n,$$

where dx_i, dy_j are copies of dx.

Then the measures d_1X, d_1Y are dual for $\mathcal{F}$. Let us denote by ι the bijective map from Ω^+ to Ω^-, defined for $X \in \Omega^+$ by the fact that $(\iota(X), H_0, X)$ is an $\mathfrak{sl}_2$-triple. This map is clearly G-equivariant:

$\iota(g.X) = g.\iota(X)$ for $X \in \Omega^+$ and $g \in G$. Moreover this map can also be defined as a map from $\overline{\Omega^+}$ onto $\overline{\Omega^-}$, which is $\overline{G}$-equivariant, where $\overline{G}$ is the group which is defined the same way as G, but for the extended grading of $\overline{\overline{\mathfrak{g}}}$.

It is worth noticing that similarly one can consider maps still denoted by ι from $K_p(2,0)'$ onto $K_p(-2,0)'$ and from $K_p(0,2)'$ onto $K_p(0,-2)'$ defined by the conditions that $(\iota(u), H_0^{p,1}, u)$ and $(\iota(v), H_0^{p,2}, v)$ are $\mathfrak{sl}_2$-triples ($u \in K_p(2,0)'$, $v \in K_p(0,2)'$).

We then normalize the various relative invariants by the conditions:

$$\nabla_0(\iota(X)) = \frac{1}{\Delta_0(X)} \quad \text{for all } X \in \Omega^+,$$

$$\nabla_{k-p}(\iota(u)) = \frac{1}{\widetilde{\Delta}_{k-p}(u)} \quad \text{for all } u \in K_p(2,0)',$$

$$\widetilde{\nabla}_{p+1}(\iota(v)) = \frac{1}{\Delta_{p+1}(v)} \quad \text{for all } v \in K_p(0,2)'.$$

It is easy to see, due to the $\overline{G}$-equivariance and the $\mathcal{Z}_G(FH_0^{p,1} + FH_0^{p,2})$-equivariance, that these conditions are equivalent to the same equalities holding only for fixed elements X, u, v.

We will also impose the conditions:

$$\Delta_0(u+v) = \widetilde{\Delta}_{k-p}(u)\Delta_{p+1}(v) \quad \text{for all } u \in K_p(2,0),\ v \in K_P(0,2),$$

$$\nabla_0(u'+v') = \nabla_{k-p}(u')\widetilde{\nabla}_{p+1}(v') \quad \text{for all } u' \in K_p(-2,0),\ v' \in K_P(0,-2).$$

Let (Y, H_0, X) be an $\mathfrak{sl}_2$-triple with $X \in V^+$, $Y \in V^-$. Recall from Definition 1.1.5 that we have set for $t \in F^*$

$$\theta_X(t) = \exp(t\mathrm{ad}X)\exp(\frac{1}{t}\mathrm{ad}Y)\exp(t\mathrm{ad}X) = \exp(\frac{1}{t}\mathrm{ad}Y)\exp(t\mathrm{ad}X)\exp(\frac{1}{t}\mathrm{ad}Y),$$

and $\theta_X = \theta_X(-1)$. Then

$$\theta_X(X) = (-1)^2 Y = Y,\ \theta_X(Y) = (-1)^2 X,\ \theta_X(H_0) = -H_0.$$

Moreover θ_X^2 acts by $(-1)^r$ on the weight r space for any finite dimensional module. Hence θ_X is an involution of $\overline{\overline{\mathfrak{g}}}$. Is is also easy to see that for $g \in G$ and $X \in \Omega^+$ we have $\theta_{gX} = g\theta_X g^{-1}$.

The restriction $\theta_{X|_{V^+}}$ is a linear map from V^+ to V^-. From the preceding identity one obtains

$$\det(\theta_{gX|_{V^+}}) = (\det g_{|_{V^-}})\det(\theta_{X|_{V^+}})(\det g^{-1}{}_{|_{V^+}}) = (\det g_{|_{V^-}})^2\det(\theta_{X|_{V^+}}).$$

Hence $\det(\theta_{X|_{V^+}})$ is a relative invariant whose character is $(\det g_{|_{V^-}})^2$. As we know ([2], p.95) that $\det g_{|_{V^-}} = \chi_0(g)^{-m}$, where $m = \frac{\dim V^+}{\deg(\Delta_0)} = \frac{\dim V^+}{\kappa(k+1)}$, we get that there exists $c \in F^*$ such that

$$\det(\theta_{X|_{V^+}}) = c\Delta_0(X)^{-2m}. \tag{2.1}$$

Fix $X_0 \in \Omega^+$, and remark that $\theta_{X_0} g \theta_{X_0}^{-1} \in G$. Then

$$\nabla_0(\theta_{X_0}(gX)) = \nabla_0(\theta_{X_0}g\theta_{X_0}^{-1}\theta_{X_0}(X)) = \chi_0^{-1}(\theta_{X_0}g\theta_{X_0}^{-1})\nabla_0(\theta_{X_0}(X)).$$

Therefore the map $X \longmapsto \nabla_0(\theta_{X_0}(X))$ is a polynomial relative invariant with the same degree as ∇_0 (hence the same degree as Δ_0). Thus $\nabla_0(\theta_{X_0}(X)) = \alpha\Delta_0(X)$ with $\alpha \in F^*$. Replacing X by X_0, and using the normalization above we obtain

$$\alpha\Delta_0(X_0) = \nabla_0(\theta_{X_0}(X_0)) = \nabla_0(\iota(X_0)) = \frac{1}{\Delta_0(X_0)}.$$

Hence $\alpha = \dfrac{1}{\Delta_0(X_0)^2}$ and

$$\nabla_0(\theta_{X_0}(X)) = \frac{\Delta_0(X)}{\Delta_0(X_0)^2} \ \ (X \in \Omega^+). \tag{2.2}$$

Proposition 2.2.2.

There exists a unique pair of Haar measures (dX, dY) on $V^+ \times V^-$ such that
1) $\overline{\mathcal{F}} \circ \mathcal{F} = Id_{\mathcal{S}(V^+)}$ *(in other words the measures are dual for $\mathcal{F}$)*
2) $\forall f \in L^1(V^-)$ *and* $\forall X_0 \in \Omega^+$

$$\int_{V^+} f(\theta_{X_0}(X))dX = |\Delta_0(X_0)|^{2m} \int_{V^-} f(Y)dY. \tag{2.3}$$

In fact if the identity (2.3) is true for one fixed X_0, then it holds for all.
Moreover if $d^*X = \frac{dX}{|\Delta_0(X)|^m}$ and $d^*Y = \frac{dY}{|\nabla_0(Y)|^m}$ denote the corresponding G-invariant measures on V^+ and V^- respectively, we have for $\Psi \in L^1(\Omega^-, d^*Y)$ and $\Phi \in L^1(\Omega^+, d^*X)$:

$$\int_{\Omega^+} \Psi(\theta_{X_0}(X))d^*X = \int_{\Omega^-} \Psi(Y)d^*Y \ \text{ and } \ \int_{\Omega^-} \Phi(\theta_{X_0}(Y))d^*Y = \int_{\Omega^+} \Phi(X)d^*X.$$

Proof.

Let us start with the pair of dual Haar measures (d_1X, d_1Y) described before. Suppose that (dX, dY) is a couple of Haar measures satisfying the condition 1). Then there exists $\lambda \in \mathbb{R}^{*+}$ such that $dX = \lambda d_1X$ and $dY = \frac{1}{\lambda}d_1Y$. We make the change of variable $Y = \theta_{X_0}(X)$ in the left hand side of 2). From the definition of d_1X and d_1Y we get $d_1Y = |\det(\theta_{X_0|_{V^+}})|d_1X$. Hence

$$\begin{aligned}\int_{V^+} f(\theta_{X_0}(X))dX &= \int_{V^+} f(\theta_{X_0}(X))\lambda d_1X = \int_{V^-} f(Y)\frac{1}{|\det(\theta_{X_0|_{V^+}})|}\lambda d_1Y \\ &= \int_{V^-} f(Y)\frac{1}{|\det(\theta_{X_0|_{V^+}})|}\lambda^2 dY = \frac{\lambda^2}{|c|}|\Delta_0(X_0)|^{2m}\int_{V^-} f(Y)dY),\end{aligned}$$

where the last equality comes from (2.1).
Therefore if we take $\lambda = \sqrt{|c|} = |\det(\theta_{X_0|_{V^+}})|^{\frac{1}{2}}|\Delta_0(X_0)|^{2m}$ we have proved the first part of the statement. And the fact that c does not depend on X_0 implies that if (2.3) holds for one X_0 then it holds for all.
Using (2.2) and 2) we obtain:

$$\begin{aligned}\int_{\Omega^+} \Psi(\theta_{X_0}(X))d^*X &= \int_{\Omega^+} \Psi(\theta_{X_0}(X))\frac{dX}{|\Delta_0(X)|^m} \\ &= |\Delta_0(X_0)|^{-2m}\int_{V^+} \Psi(\theta_{X_0}(X))\frac{dX}{|\nabla_0(\theta_{X_0}(X))|^m} \\ &= \int_{V^-} \Psi(Y)d^*Y.\end{aligned}$$

Setting $\Phi = \Psi \circ \theta_{X_0}$ in this identity, we obtain the last assertion. □

Proposition 2.2.3.

Fix $X_0 \in \Omega^+$ and let $Y_0 = \iota(X_0)$. Define $H = \mathcal{Z}_G(X_0) = \mathcal{Z}_G(Y_0)$. One defines two G-invariant measures $d_1\dot{g}$ and $d_2\dot{g}$ on G/H by setting for $\Phi \in \mathcal{S}(G.X_0)$ and $\Psi \in \mathcal{S}(G.Y_0)$

$$\int_{G.X_0} \Phi(X)d^*X = \int_{G/H} \Phi(g.X_0)d_1\dot{g} \ \text{ and } \ \int_{G.Y_0} \Psi(Y)d^*Y = \int_{G/H} \Psi(g.Y_0)d_2\dot{g}.$$

Then $d_1\dot{g} = d_2\dot{g}$.

Proof.
From the last assertion of Proposition 2.2.2 we obtain
$\int_{G/H} \Phi(g.X_0)d_1\dot{g} = \int_{\Omega^+} \Phi(X)d^*X = \int_{\Omega^-} \Phi(\theta_{X_0}(Y))d^*Y = \int_{G/H} \Phi(\theta_{X_0}(g.Y_0))d_2\dot{g}$
$= \int_{G/H} \Phi(\theta_{X_0}g\theta_{X_0}\theta_{X_0}(Y_0))d_2\dot{g} = \int_{G/H} \Phi(g.X_0)d_2\dot{g}$
(The last equality is obtained by taking into account that the conjugation $g \longmapsto \theta_{X_0}g\theta_{X_0}$ is an involution of G which stabilizes H, so we make the change of variable $g = \theta_{X_0}g\theta_{X_0}$). □

Definition 2.2.4. *We will denote by* $d\dot{g}$ *the* G*-invariant measure appearing in Proposition 2.2.3.*

Proposition 2.2.5.
Let σ *be an involution of* $\tilde{\mathfrak{g}}$ *such that* $\sigma(H_0) = -H_0$. *We also suppose that there exists* $X_0 \in V^+$ *such that* $(\sigma(X_0), H_0, X_0)$ *is an* $\mathfrak{sl}_2$*-triple. Then for* $\Psi \in L^1(\Omega^-, d^*Y)$ *we have*

$$\int_{\Omega^+} \Psi(\sigma(X))d^*X = \int_{\Omega^-} \Psi(Y)d^*Y.$$

Proof.
One should note that if σ is such an involution, then $\sigma(\mathfrak{g}) = \mathfrak{g}$ and $\sigma(V^+) = V^-$. From Proposition 2.2.2 we obtain

$$\int_{\Omega^+} \Psi(\sigma(X))d^*X = \int_{\Omega^+} \Psi \circ \sigma\theta_{X_0}(\theta_{X_0}(X))d^*X = \int_{\Omega^-} \Psi(\sigma\theta_{X_0}(Y))d^*Y.$$

It is easy to see that if we set $Y_0 = \sigma(X_0)$ we have

$$\theta_{X_0} \circ \sigma = \sigma \circ \exp(-\mathrm{ad}(Y_0)) \exp(-\mathrm{ad}(X_0)) \exp(-\mathrm{ad}(Y_0)) = \sigma \circ \theta_{X_0}.$$

Therefore $\sigma \circ \theta_{X_0}$ is an involution preserving V^-. Hence if we make the change of variable $Y = \sigma\theta_{X_0}(Y)$ we obtain the assertion. □

Proposition 2.2.6.
Remember that for $X \in \Omega^+$, *the element* $\iota(X) \in \Omega^-$ *is defined by the fact that* $(\iota(X), H_0, X)$ *is an* $\mathfrak{sl}_2$*-triple. For* $\Psi \in L^1(\Omega^-, d^*Y)$ *we have*

$$\int_{\Omega^+} \Psi(\iota(X))d^*X = \int_{V^-} \Psi(Y)d^*Y.$$

Proof.
Let $\Omega^+ = \cup_{i=1}^{r_0}\Omega_i^+$ be the decomposition of Ω^+ into open G-orbits. Let $\Omega_i^- = \iota(\Omega_i^+)$. Take $X_i \in \Omega_i^+$ and define $Y_i = \iota(X_i) \in \Omega_i^-$. Define also $\mathcal{H}_i = \mathcal{Z}_G(X_i) = \mathcal{Z}_G(Y_i)$. It is enough to prove the assertion for $\Psi \in \mathcal{S}(\Omega_i^-)$. Using Proposition 2.2.3 we obtain:
$\int_{\Omega_i^+} \Psi(\iota(X))d^*X = \int_{G/\mathcal{H}_i} \Psi(\iota(gX_i))d\dot{g} = \int_{G/\mathcal{H}_i} \Psi(g\iota(X_i))d\dot{g} = \int_{G/\mathcal{H}_i} \Psi(gY_i)d\dot{g} = \int_{\Omega_i^-} \Psi(Y)d^*Y.$
□

We now consider the subset $(e_{i_1}, \ldots, e_{i_r})$ of $(e_1, \ldots, e_n)$ which is a basis of $K_p(1,1)$ (see the discussion after definition 2.2.1). Then $(e_{i_1}^*, \ldots, e_{i_r}^*)$ is the dual basis of $K_p(-1,-1)$ with respect to the form b. If d_1w and d_1w' are the Haar measures on $K_p(1,1)$ and $K_p(-1,-1)$ respectively, defined

through the preceding basis and the choice of a self dual measure on F, then these measures are dual for $\mathcal{F}_{1,1}$.

Proposition 2.2.7.
There exists a unique pair of Haar measures (dw, dw') on $K_p(1,1) \times K_p(-1,-1)$ such that
1) $\overline{\mathcal{F}}_{1,1} \circ \mathcal{F}_{1,1} = Id_{\mathcal{S}(K_p(1,1))}$ *(in other words the measures are dual for $\mathcal{F}_{1,1}$)*
2) $\forall g \in L^1(K_p(1,1))$ *and* $\forall u \in K_p(2,0)'$ *and* $\forall v \in K_p(0,2)'$

$$\int_{K_p(1,1)} g(\theta_{u+v}(w))dw = |\Delta_{p+1}(v)|^{\frac{\dim(K_p(1,1))}{\deg(\Delta_{p+1})}} |\widetilde{\Delta}_{k-p}(u)|^{\frac{\dim(K_p(1,1))}{\deg(\widetilde{\Delta}_{k-p})}} \int_{K_p(-1,-1)} g(w')dw'.$$

As $\theta_{u+v}{}^2_{|_{K_p(1,1)}} = Id_{K_p(-1,-1)}$ it is easy to see that condition 2) is equivalent to the following condition 2').
2') $\forall f \in L^1(K_p(-1,-1))$ *and* $\forall u \in K_p(2,0)'$ *and* $\forall v \in K_p(0,2)'$

$$\int_{K_p(1,1)} f(w)dw = |\Delta_{p+1}(v)|^{\frac{\dim(K_p(1,1))}{\deg(\Delta_{p+1})}} |\widetilde{\Delta}_{k-p}(u)|^{\frac{\dim(K_p(1,1))}{\deg(\widetilde{\Delta}_{k-p})}} \int_{K_p(-1,-1)} f(\theta_{u+v}(w'))dw'.$$

Proof.
For u, v as in 2), the map θ_{u+v} is a linear isomorphism from $K_p(1,1)$ onto $K_p(-1,-1)$ and for $g \in \mathcal{Z}_G(FH_0^{p,1} + FH_0^{p,2})$ we have:

$$\det(\theta_{g(u+v)_{|_{K_p(1,1)}}}) = \det(g_{|_{K_p(-1,-1)}})^2 \det(\theta_{u+v|_{K_p(1,1)}}).$$

Therefore $\det(\theta_{u+v|_{K_p(1,1)}})$ is a relative invariant of the prehomogeneous vector space $(K_p(0,0), K_p(2,0) \oplus K_p(0,2)$. Hence there exist $c \in F^*$ and $\alpha, \beta \in \mathbb{Z}$ such that

$$\det(\theta_{u+v|_{K_p(1,1)}}) = c\widetilde{\Delta}_{k-p}(u)^\alpha \Delta_{p+1}(v)^\beta. \tag{2.4}$$

For $t_1, t_2 \in F$, let us consider the element $g_0 = h_u(t_1)h_v(t_2) \in \mathcal{Z}_G(FH_0^{p,1} + FH_0^{p,2})$ (see Definition 1.1.5). This element acts by $t_2^{-1}t_1^{-1}$ on $K_p(-1,-1)$, by t_1^2 on $K_p(2,0)$ and by t_2^2 on $K_p(0,2)$ Hence

$$\det(g_{0|_{K_p(-1,-1)}})^2 = t_2^{-2\dim K_p(-1,-1)} t_1^{-2\dim K_p(-1,-1)}$$

But from (2.4) above we also have:

$$\det(\theta_{g_0(u+v)_{|_{K_p(1,1)}}}) = ct_1^{2\alpha \deg \widetilde{\Delta}_{k-p}} t_2^{2\beta \deg \Delta_{p+1}} \det(\theta_{(u+v)_{|_{K_p(1,1)}}}).$$

Therefore $\alpha = -\frac{\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}$ and $\beta = -\frac{\dim K_p(1,1)}{\deg \Delta_{p+1}}$. Hence

$$\det(\theta_{u+v|_{K_p(1,1)}}) = c\widetilde{\Delta}_{k-p}(u)^{-\frac{\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}} \Delta_{p+1}(v)^{-\frac{\dim K_p(1,1)}{\deg \Delta_{p+1}}}. \tag{2.5}$$

Let us start with the pair of dual Haar measures (d_1w, d_1w') described before. Suppose that (dw, dw') is a couple of Haar measures satisfying the condition 1). Then there exists $\lambda \in \mathbb{R}^{*+}$ such that $dw = \lambda d_1w$ and $dw' = \frac{1}{\lambda} d_1w'$. We make the change of variable $w' = \theta_{u+v}(w)$ in the left hand side of 2). From the definition of d_1w and d_1w' we get $d_1w' = |\det(\theta_{u+v|_{K_p(1,1)}})| d_1w$. Hence

$$\begin{aligned}\int_{K_p(1,1)} g(\theta_{u+v}(w))dw &= \int_{K_p(1,1)} g(\theta_{u+v}(w))\lambda d_1 w = \int_{K_p(-1,-1)} g(w')\frac{1}{|\det(\theta_{u+v}|_{K_p(1,1)}|}\lambda d_1 w' \\ &= \int_{K_p(-1,-1)} g(w')\frac{1}{|\det(\theta_{u+v}|_{K_p(1,1)})|}\lambda^2 dw' \\ &= \frac{\lambda^2}{|c|}|\widetilde{\Delta}_{k-p}(u)|^{\frac{\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}}|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,1)}{\deg \Delta_{p+1}}}\int_{K_p(-1,-1)} g(w')dw',\end{aligned}$$

where the last equality comes from (2.5). Therefore if we take $\lambda = \sqrt{|c|}$ the result is proved.

□

As above we consider Haar measures d_1A and d_1A' on $K_p(-1,1)$ and $K_p(1,-1)$ respectively, which are dual for $\mathcal{F}_{-1,1}$ (see Definition 2.2.1).These measures are defined through the choice of a basis of $K_p(-1,1) \subset V^+$, the dual basis of $K_p(1,-1) \subset V^-$ (via the form b) and the choice of a self dual measure on F.

Proposition 2.2.8.
There exists a unique pair of Haar measures (dA, dA') on $K_p(-1,1) \times K_p(1,-1)$ such that
1) $\overline{\mathcal{F}}_{-1,1} \circ \mathcal{F}_{-1,1} = Id_{\mathcal{S}(K_p(-1,1))}$ *(in other words the measures are dual for $\mathcal{F}_{-1,1}$)*
2) $\forall g \in L^1(K_p(1,-1))$ *and* $\forall u \in K_p(2,0)'$ *and* $\forall v \in K_p(0,2)'$

$$\int_{K_p(-1,1)} g(\theta_{u+v}(A))dA = |\Delta_{p+1}(v)|^{\frac{\dim(K_p(1,-1))}{\deg(\Delta_{p+1})}}|\widetilde{\Delta}_{k-p}(u)|^{-\frac{\dim(K_p(1,-1))}{\deg(\widetilde{\Delta}_{k-p})}}\int_{K_p(1,-1)} g(A')dA'.$$

As $\theta_{u+v}{}^2_{|K_p(-1,1)} = Id_{K_p(-1,1)}$ it is easy to see that condition is equivalent to the following condition $2'$).
$2'$) $\forall f \in L^1(K_p(-1,1))$ *and* $\forall u \in K_p(2,0)'$ *and* $\forall v \in K_p(0,2)'$

$$\int_{K_p(-1,1)} f(A)dA = |\Delta_{p+1}(v)|^{\frac{\dim(K_p(1,-1))}{\deg(\Delta_{p+1})}}|\widetilde{\Delta}_{k-p}(u)|^{-\frac{\dim(K_p(1,-1))}{\deg(\widetilde{\Delta}_{k-p})}}\int_{K_p(1,-1)} f(\theta_{u+v}(A'))dA'.$$

Proof.
The proof is almost the same as for the preceding Proposition. The only change comes from the fact that the element $g_0 = h_u(t_1)h_v(t_2)$ acts by $t_1t_2^{-1}$ on $K_p(1,-1)$. □

Remark 2.2.9. It is clear that Proposition 2.2.2 applies to the graded algebras $K_p(-2,0) \oplus K_p(0,0) \oplus K_p(2,0)$ and $K_p(0,-2) \oplus K_p(0,0) \oplus K_p(0,2)$.
Therefore there exists a unique pair of measures (du, du') on $K_p(2,0) \times K_p(-2,0)$ such that
1) These measures are dual for the Fourier transform $\mathcal{F}_{2,0}$ between $\mathcal{S}(K_p(2,0))$ and $\mathcal{S}(K_p(-2,0))$,
2) $\forall f \in L^1(K_p(-2,0)$ and $\forall u_0 \in K_p(2,0)'$

$$\int_{K_p(2,0)} f(\theta_{u_0}(u))du = |\widetilde{\Delta}_{k-p}(u_0)|^{\frac{2\dim K_p(2,0)}{\deg \widetilde{\Delta}_{k-p}}}\int_{K_p(-2,0)} f(u')du'.$$

Similarly there exists a unique pair of measures (dv, dv') on $K_p(0,2) \times K_p(0,-2)$ such that
1) These measures are dual for the Fourier transform $\mathcal{F}_{0,2}$ between $\mathcal{S}(K_p(0,2))$ and $\mathcal{S}(K_p(0,-2))$

2) $\forall f \in L^1(K_p(0,-2)$ and $\forall v_0 \in K_p(0,2)'$

$$\int_{K_p(0,2)} f(\theta_{v_0}(v))dv = |\Delta_{p+1}(v_0)|^{\frac{2\dim K_p(0,2)}{\deg \Delta_{p+1}}} \int_{K_p(0,-2)} f(v')dv'.$$

Definition 2.2.10. Normalization of the measures

From now on we will always make the following choice of measures:

a) On $V^+ \times V^-$ we take the measures (dX, dY) which were defined in Proposition 2.2.2.

b) On $K_p(2,0) \times K_p(-2,0)$ and on $K_p(0,2) \times K_p(0,-2)$ respectively, we take the pairs measures (du, du') and (dv, dv') which were defined in Remark 2.2.9.

c) On $K_p(1,1) \times K_p(-1,-1)$ we take the pair of measures (dw, dw') which were defined in Proposition 2.2.7.

d) On $K_p(-1,1) \times K_p(1,-1)$ we take the pair of measures (dA, dA') which were defined in Proposition 2.2.8.

Proposition 2.2.11.

a) $dX = dudvdw$

b) $dY = du'dv'dw'$

Proof.

Define $dX := dudvdw$ and $dY := du'dv'dw'$. It is now enough to prove that the pair (dX, dY) satisfies the two conditions of Proposition 2.2.2.

1) It is clear from the definitions that the measures dX, dY are dual for $\mathcal{F}$.

2) Let $f \in \mathcal{S}(V^+)$, $u_0 \in K_p(2,0)'$, $v_0 \in K_p(0,2)'$. Then we have, using Remark 2.2.9 and Proposition 2.2.7:

$$\int_{V^+} f(\theta_{u_0+v_0}(u+v+w))dudvdw = \int_{V^+} f(\theta_{u_0}(u)+\theta_{v_0}(v)+\theta_{u_0+v_0}(w))dudvdw$$
$$= |\widetilde{\Delta}_{k-p}(u_0)|^{\frac{2\dim K_p(2,0)+\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}} |\Delta_{p+1}(v_0)|^{\frac{2\dim K_p(0,2)+\dim K_p(1,1)}{\deg \Delta_{p+1}}} \int_{V^-} f(u'+v'+w')du'dv'dw'.$$

From Proposition 1.9.5 p.26 in [8] we know that $\dim V^+ = (k+1)(\ell + \frac{kd}{2})$, $\dim K_p(2,0) = (p+1)(\ell + \frac{pd}{2})$, $\dim K_p(0,2) = (k-p)(\ell + \frac{(k-p-1)d}{2})$. Moreover as $K_p(1,1) = \oplus_{0\le i\le p; p+1\le j\le k} E_{i,j}(1,1)$, we obtain that $\dim K_p(1,1) = (p+1)(k-p)d$. We also have $\deg \widetilde{\Delta}_{k-p} = \kappa(p+1)$ and $\deg \Delta_{p+1} = \kappa(k-p)$. A simple calculation shows then that

$$\frac{2\dim K_p(2,0)+\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}} = \frac{2\dim K_p(0,2)+\dim K_p(1,1)}{\deg \Delta_{p+1}} = \frac{2}{\kappa}(\ell + \frac{kd}{2}) = 2m.$$

Finally we get

$$\int_{V^+} f(\theta_{u_0+v_0}(u+v+w))dudvdw = |\Delta_0(u_0+v_0)|^{2m} \int_{V^-} f(u'+v'+w')du'dv'dw'$$

□

2.3. **Mean functions.**

We will first define the mean functions $T_f^+(u,v)$ ($u \in K_p(2,0), v \in K_p(0,2)$) and $T_f^-(u',v')$ ($u' \in K_p(-2,0), v \in K_p(0,-2)$) which were introduced by I. Muller ([12] Definition 4.3.1 p. 83) and used in the real case by N. Bopp and H. Rubenthaler ([2] p.104).

Let us denote $K_p(\pm 2,0)'$ and $K_p(0,\pm 2)'$ the sets of generic elements in $K_p(\pm 2,0)$ and $K_p(0,\pm 2)$, respectively (as $(L_p, K_p(\pm 2,0))$ and $(L_p, K_p(0,\pm 2))$ are prehomogeneous, the notion of generic element is clear).

In the rest of the section we adopt the following convention concerning the variables:
$u \in K_p(2,0)$, $v \in K_p(0,2)$, $u' \in K_p(-2,0)$, $v' \in K_p(0,-2)$, $A \in K_p(-1,1)$, $A' \in K_p(1,-1)$, $w \in K_p(1,1)$, $w' \in K_p(-1,-1)$, $X \in V^+$, $Y \in V^-$.

Lemma 2.3.1.
There exist positive constants α, β, γ and δ which depend on the choice of Lebesgue measures on the different spaces such that the following assertions hold.
1) If $v \in K_p(0,2)'$, then for all $f \in L^1(K_p(1,1))$ one has :

$$\int_{K_p(1,1)} f(w)dw = \alpha |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(1,-1)} f([A',v])dA'.$$

2) If $v \in K_p(0,2)'$, then for all $g \in L^1(K_p(-1,1))$ one has :

$$\int_{K_p(-1,1)} g(A)dA = \beta |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(-1,-1)} g([w',v])dw'.$$

3) If $u' \in K_p(-2,0)'$, then for all $f \in L^1(K_p(-1,-1))$ one has :

$$\int_{K_p(-1,-1)} f(w')dw' = \gamma |\nabla_{k-p}(u')|^{\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}} \int_{K_p(1,-1)} f([A',u'])dA'.$$

4) If $u' \in K_p(-2,0)'$, then for all $g \in L^1(K_p(-1,1))$ one has :

$$\int_{K_p(-1,1)} g(A)dA = \delta |\nabla_{k-p}(u')|^{\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}} \int_{K_p(1,1)} g([w,u'])dw.$$

(Remember that $\dim K_p(1,-1) = \dim K_p(-1,1) = \dim K_p(1,1) = \dim K_p(-1,-1)$).

Proof. :
We only prove assertion 1). The proofs of 2), 3) and 4) are similar.
If $v \in K_p(0,2)'$ the map :

$$\mathrm{ad}(v) : K_p(1,-1) \longrightarrow K_p(1,1)$$

is an isomorphism (classical result for $\mathfrak{sl}_2$-modules). Therefore there exists a non zero positive constant $\alpha(v)$ such that

$$\int_{K_p(1,1)} f(w)dw = \alpha(v) \int_{K_p(1,-1)} f([A',v])dA'$$

On the other hand the subspace $K_p(1,-1)$ is stable under L_p. This implies that there exists a character χ_1 of L_p such that for $F \in L^1(K_p(1,-1))$ and $g \in L_p$ one has:

$$\int_{K_p(1,-1)} F(gA')dA' = \chi_1(g) \int_{K_p(1,-1)} F(A')dA'.$$

Moreover, the set $K_p(0,2)'$ is also L_p-invariant. Hence:

$$\begin{aligned}\int_{K_p(1,1)} f(w)dw &= \alpha(gv)\int_{K_p(1,-1)} f([A',gv])dA' \\ &= \alpha(gv)\int_{K_p(1,-1)} f(g[g^{-1}A',v])dA' \\ &= \alpha(gv)\chi_1^{-1}(g)\int_{K_p(1,-1)} f(g[A',v])dA' \\ &= \alpha(gv)\chi_1^{-1}(g)\frac{1}{\alpha(v)}\int_{K_p(1,1)} f(g.w)dw.\end{aligned}$$

But there is also a character χ_2 of L_p such that

$$\int_{K_p(1,1)} f(g.w)dw = \chi_2(g)\int_{K_p(1,1)} f(w)dw.$$

Hence

$$\alpha(gv) = \chi_1(g)\chi_2(g)^{-1}\alpha(v).$$

This means that $\alpha(v)$ is a positive relative invariant on $K_p(0,2)$ under the action of L_p, hence there exist $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}^{*+}$ such that

$$\alpha(v) = \alpha|\Delta_{p+1}(v)|^n.$$

To compute n, let us take $g = h_{I_{p,2}^+}(t)$ as defined in Definition 1.1.5. Then $gv = t^2v$.
Thus, we have

$$\begin{aligned}\int_{K_p(1,1)} f(w)dw &= \alpha(v)\int_{K_p(1,-1)} f([A',v])dA' \\ &= \alpha(t^2v)\int_{K_p(1,-1)} f([A',t^2v])dA' \\ &= \alpha(t^2v)\int_{K_p(1,-1)} f([t^2A',v])dA' \\ &= \alpha(t^2v)t^{-2\dim(K_p(1,-1))}\int_{K_p(1,-1)} f([A',v])dA'\end{aligned}$$

Hence $\alpha(t^2v) = t^{2\dim(K_p(1,-1))}\alpha(v)$. We obtain then:

$$\alpha|\Delta_{p+1}(t^2v)|^n = \alpha t^{2n\deg\Delta_{p+1}}|\Delta_{p+1}(v)|^n = \alpha t^{2\dim K_p(1,-1)}|\Delta_{p+1}(v)|^n.$$

Therefore $n = \frac{\dim K_p(1,-1)}{\deg\Delta_{p+1}}$ and $\alpha(v) = \alpha|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg\Delta_{p+1}}}$
This ends the proof of assertion 1).

□

Proposition 2.3.2.

If the measures are normalized as in Definition 2.2.10, then the four constants $\alpha, \beta, \delta, \gamma$ occuring in Lemma 2.3.1 are equal to 1.

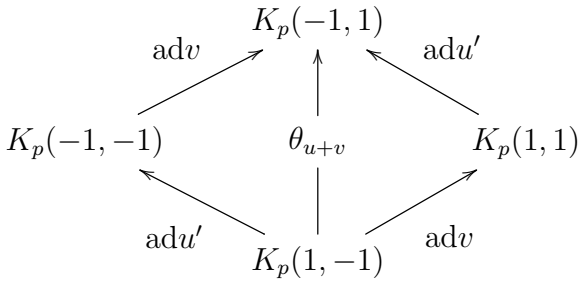
Proof.

Let $u \in K_P(2,0)'$, $v \in K_p(0,2)'$, and define $u' = \iota(u)$, $v' = \iota(v)$. If $A \in K_p(1,-1)$, then $\mathrm{ad}uA = \mathrm{ad}v'A = 0$ and hence $\mathrm{ad}v\,\mathrm{ad}u'A = \mathrm{ad}u'\,\mathrm{ad}vA = \mathrm{ad}u'\mathrm{ad}(u+v)A = \mathrm{ad}[u',u+v]A +$

$\mathrm{ad}(u+v)\mathrm{ad}u'A = A + \mathrm{ad}(u+v)\mathrm{ad}u'A = A + \mathrm{ad}(u+v)\mathrm{ad}(u'+v')A = \theta_{u+v}(A)$ (for the last equality see Theorem 4.1.1 in [8]). Hence we have

$$\forall A \in K_P(1,-1),\ \mathrm{ad}v\,\mathrm{ad}u'A = \mathrm{ad}u'\,\mathrm{ad}vA = \theta_{u+v}(A). \tag{2.6}$$

It is equivalent to say that the following diagram is commutative:



Take now $w' \in K_p(-1,-1)$. Then (recall that $\theta_{u+v}(v) = v'$)
$\mathrm{ad}\theta_{u+v}(v)\mathrm{ad}\,v\,w' = \mathrm{ad}[\theta_{u+v}(v), v]\,w' + \mathrm{ad}\,v\,\mathrm{ad}\theta_{u+v}(v)w' = -w' + \mathrm{ad}\,v\,\mathrm{ad}\theta_{u+v}(v)w' = -w'$.
Hence

$$\mathrm{ad}v\circ\theta_{u+v}\circ\mathrm{ad}v\,w' = \theta_{u+v}\circ\theta_{u+v}\circ\mathrm{ad}v\circ\theta_{u+v}\circ\mathrm{ad}v\,w' = \theta_{u+v}\circ\mathrm{ad}\theta_{u+v}(v)\circ\mathrm{ad}v\,w' = -\theta_{u+v}(w'). \tag{2.7}$$

Similarly one can also prove that for all $w \in K_p(1,1)$:

$$\mathrm{ad}u' \circ \theta_{u+v} \circ \mathrm{ad}u'w = -\theta_{u+v}(w) \tag{2.8}$$

– **Step 1**. We will first prove that $\alpha\delta = \alpha\beta = \beta\gamma = \gamma\delta = 1$.
Let $u \in K_p(2,0)'$ and $v \in K_p(0,2)'$.
Using successively relation 4) and 1) in Lemma 2.3.1, the relation $\nabla_{k-p}(u') = \widetilde{\Delta}_{k-p}(u)^{-1}$ and equation (2.6) we obtain:

$$\begin{aligned}
\int_{K_p(-1,1)} g(A)dA &= \delta|\nabla_{k-p}(u')|^{\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}} \int_{K_p(1,1)} g(-\mathrm{ad}u'w)dw \\
&= \alpha\delta|\nabla_{k-p}(u')|^{\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}}|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(1,-1)} g(\mathrm{ad}u'\mathrm{ad}vA')dA' \\
&= \alpha\delta|\widetilde{\Delta}_{k-p}(u)|^{-\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}}|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(1,-1)} g(\mathrm{ad}u'\mathrm{ad}vA')dA' \\
&= \alpha\delta|\widetilde{\Delta}_{k-p}(u)|^{-\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}}|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(1,-1)} g(\theta_{u+v}(A'))dA'
\end{aligned}$$

Then Proposition 2.2.8 implies that $\alpha\delta = 1$.
Using successively relation 1) in Lemma 2.3.1, Proposition 2.2.8, equation 2) in Lemma 2.3.1, and finally equation (2.7) above we obtain:

$$\begin{aligned}
\int_{K_p(1,1)} f(w)dw &= \alpha|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\Delta_{p+1}}} \int_{K_p(1,-1)} f([A',v])dA' \\
&= \alpha|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,1)}{\Delta_{p+1}}}|\Delta_{p+1}(v)|^{-\frac{\dim K_p(1,1)}{\Delta_{p+1}}}|\widetilde{\Delta}_{k-p}(u)|^{\frac{\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}} \int_{K_p(-1,1)} f([\theta_{u+v}(A),v])dA \\
&= \alpha\beta|\widetilde{\Delta}_{k-p}(u)|^{\frac{\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}}|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(-1,-1)} f([\theta_{u+v}([w',v]),v])dw' \\
&= \alpha\beta|\widetilde{\Delta}_{k-p}(u)|^{\frac{\dim K_p(1,1)}{\deg \widetilde{\Delta}_{k-p}}}|\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(-1,-1)} f(-\theta_{u+v}(w'))dw'
\end{aligned}$$

Then Proposition 2.2.7 implies that $\alpha\beta = 1$.

The proofs that $\beta\gamma = 1$ and $\gamma\delta = 1$ are similar.

- Step 2 We will now prove that $\alpha\gamma = 1$. From Step 1, this will imply that $\alpha = \beta = \gamma = \delta = 1$.

Let $f \in \mathcal{S}(K_p(1,1)$. Then by Lemma 2.3.1, 3), for $w \in K_p(1,1)$

$$\begin{aligned} f(w) &= (\overline{\mathcal{F}}_{1,1} \circ \mathcal{F}_{1,1} f)(w) = \int_{K_p(-1,-1)} \mathcal{F}_{1,1} f(w') \overline{\psi(b(w',w))} dw' \\ &= \gamma |\nabla_{k-p}(u')|^{\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}} \int_{K_p(1,-1)} \mathcal{F}_{1,1} f([A',u']) \overline{\psi(b([A',u'],w))} dA'. \end{aligned} \tag{2.9}$$

On the other hand, using Lemma 2.3.1, 1) we have

$$\begin{aligned} \mathcal{F}_{1,1}(w') &= \int_{K_p(1,1)} f(w)\psi(b(w,w'))dw \\ &= \alpha |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(1,-1)} f([A',v])\psi(b(A',[v,w']))dA'. \end{aligned}$$

If we define $\varphi(A') = f([A',v])$ the preceding equation becomes:

$$\mathcal{F}_{1,1}(w') = \alpha |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \overline{\mathcal{F}}_{1,-1}\varphi(-[v,w']) \quad \text{(see Definition 2.2.1)}.$$

Then, from equation (2.6) we get

$$\mathcal{F}_{1,1}([A',u']) = \alpha |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \overline{\mathcal{F}}_{1,-1}\varphi(\mathrm{ad}v\mathrm{ad}u'A') = \alpha |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \overline{\mathcal{F}}_{1,-1}\varphi(\theta_{u+v}(A')).$$

Therefore equation (2.9) above can be written as

$$f(w) = \gamma\alpha |\nabla_{k-p}(u')|^{\frac{\dim K_p(1,-1)}{\deg \nabla_{k-p}}} |\Delta_{p+1}(v)|^{\frac{\dim K_p(1,-1)}{\deg \Delta_{p+1}}} \int_{K_p(1,-1)} \overline{\mathcal{F}}_{1,-1}\varphi(\theta_{u+v}(A'))\overline{\psi(b(A',\mathrm{ad}u'w))}dA'.$$

Using Proposion 2.2.8, 2') (and the relation $\nabla_{k-p}(u') = \widetilde{\Delta}_{k-p}(u)^{-1}$) we obtain:

$$\begin{aligned} f(w) &= \gamma\alpha \int_{K_p(-1,1)} \overline{\mathcal{F}}_{1,-1}\varphi(A)\overline{\psi(b(\theta_{u+v}A, \mathrm{ad}u'w))}dA \\ &= \gamma\alpha \int_{K_p(-1,1)} \overline{\mathcal{F}}_{1,-1}\varphi(A)\overline{\psi(b(A, \theta_{u+v} \circ \mathrm{ad}u'w))}dA \\ &= \gamma\alpha \int_{K_p(-1,1)} \overline{\mathcal{F}}_{1,-1}\varphi(A)\psi(b(A, -\theta_{u+v} \circ \mathrm{ad}u'w))dA \\ &= \gamma\alpha\varphi(-\theta_{u+v} \circ \mathrm{ad}u'w) = \gamma\alpha f([-\theta_{u+v} \circ \mathrm{ad}u'w, v]) \\ &= \gamma\alpha f(\mathrm{ad}v \circ \theta_{u+v} \circ \mathrm{ad}u'w). \end{aligned}$$

Define $A' \in K_p(1,-1)$ such that $w = \mathrm{ad}vA'$ (always possible). Then (using (2.6))

$$f(w) = \gamma\alpha f(\mathrm{ad}v \circ \theta_{u+v} \circ \mathrm{ad}u' \circ \mathrm{ad}vA') = \gamma\alpha f(\mathrm{ad}v \circ \theta_{u+v}^2 A') = \gamma\alpha f(\mathrm{ad}vA') = \gamma\alpha f(w).$$

Hence $\gamma\alpha = 1$.

□

Theorem 2.3.3.

1) For $f \in L^1(V^+)$ we have

$$\int_{V^+} f(X)dX = \int_{u\in K_p(2,0)} \int_{A\in K_p(1,-1)} \int_{v\in K_p(0,2)} f(e^{\mathrm{ad}A}(u+v))|\Delta_{p+1}(v)|^{\frac{(p+1)d}{\kappa}} du\, dA\, dv,$$

2) For $g \in L^1(V^-)$ we have

$$\int_{V^-} g(Y)dY = \int_{u' \in K_p(-2,0)} \int_{A \in K_p(1,-1)} \int_{v' \in K_p(0,-2)} g(e^{\mathrm{ad}A}(u'+v'))|\nabla_{p+1}(u')|^{\frac{(k-p)d}{\kappa}} du'\, dA\, dv'.$$

Proof. We only prove the first formula. The proof of the second is similar. The right hand side is equal to

$$\int_{u \in K_p(2,0)} \int_{A \in K_p(1,-1)} \int_{v \in K_p(0,2)} f(u + \frac{1}{2}(\mathrm{ad}A)^2 v + (\mathrm{ad}A)v + v)|\Delta_{p+1}(v)|^{\frac{(p+1)d}{\kappa}} du\, dA\, dv.$$

We make first the change of variable $u \longrightarrow (u + \frac{1}{2}(\mathrm{ad}A)^2 v)$ (which is possible as $\frac{1}{2}(\mathrm{ad}A)^2 v \in K_p(2,0)$) and we get

$$\int_{u \in K_p(2,0)} \int_{A \in K_p(1,-1)} \int_{v \in K_p(0,2)} f(u + [A,v] + v)|\Delta_{p+1}(v)|^{\frac{(p+1)d}{\kappa}} du\, dA\, dv,$$

and then, using Lemma 2.3.1, Proposition 2.3.2 and Proposition 2.2.11, we obtain

$$\int_{K_p(2,0)} \int_{K_p(1,1)} \int_{K_p(0,2)} f(u+w+v)du\, dw\, dv = \int_{V^+} f(X)dX.$$

□

Definition 2.3.4. *(Introduced in ([12] Definition 4.3.1) for a special kind of maximal parabolic, see also ([2] Definition 4.20).)*
Let $p \in \{0, \ldots, k\}$ 1) For $f \in L^1(V^+)$ and $(u,v) \in K_p(2,0) \times K_p(0,2)$, we define

$$T_f^{p,+}(u,v) = |\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} \int_{K_p(1,-1)} f(e^{\mathrm{ad}A}(u+v))dA.$$

2) For $g \in L^1(V^-)$ and $(u',v') \in K_p(-2,0) \times K_p(0,-2)$, we define

$$T_g^{p,-}(u',v') = |\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} \int_{K_p(1,-1)} f(e^{\mathrm{ad}A}(u'+v'))dA.$$

Then Theorem 2.3.3 can be re-formulated in the following manner:

Theorem 2.3.5.
1) Let $f \in L^1(V^+)$, then the function $(u,v) \longmapsto T_f^{p,+}(u,v)|\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}}$ belongs to $L^1(K_p(2,0) \times K_p(0,2))$ and

$$\int_{V^+} f(X)dX = \int_{u \in K_p(2,0)} \int_{v \in K_p(0,2)} T_f^{p,+}(u,v)|\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} du\, dv.$$

2) Let $g \in L^1(V^-)$, then then the function $(u',v') \longmapsto T_g^{p,-}(u',v')|\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}}$ belongs to $L^1(K_p(-2,0) \times K_p(0,-2))$ and

$$\int_{V^-} g(Y)dY = \int_{u' \in K_p(-2,0)} \int_{v' \in K_p(0,-2)} T_g^{p,-}(u',v')|\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} du'\, dv'.$$

We investigate now the smoothness of $T_f^{p,+}$ and $T_g^{p,-}$.

Theorem 2.3.6.

(1) (a) *For $f \in \mathcal{S}(V^+)$, $u \in K_p(2,0)$, $v \in K_p(0,2)'$, $A \in K_p(1,-1)$, the function*

$$A \longmapsto f(e^{\mathrm{ad}A}(u+v))$$

belongs to $\mathcal{S}(K_p(1,-1))$ and $T_f^{p,+}(u,v)$ is everywhere defined on $K_p(2,0) \times K_p(0,2)'$. Moreover $T_f^{p,+}(u,v)$ is locally constant on $K_p(2,0) \times K_p(0,2)'$.

(b) *If $f \in \mathcal{S}(V^+)$ with compact support in $\mathcal{O}_{p+1}^+ = \{X \in V^+,\ \Delta_{p+1}(X) \neq 0\}$, then $T_f^{p,+}$ is everywhere defined on $K_p(2,0) \times K_p(0,2)$ and is locally constant with compact support in $K_p(2,0) \times K_p(0,2)'$.*

(2) (a) *For $g \in \mathcal{S}(V^-)$, $u' \in K_p(-2,0)'$, $v' \in K_p(0,-2)$, $A \in K_p(1,-1)$, the function*

$$A \longmapsto g(e^{\mathrm{ad}A}(u'+v'))$$

belongs to $\mathcal{S}(K_p(1,-1))$ and $T_g^{p,-}(u',v')$ is everywhere defined on $K_p(-2,0)' \times K_p(0,-2)$. Moreover $T_g^{p,-}(u',v')$ is locally constant on $K_p(-2,0)' \times K_p(0,-2)$.

(b) *If $g \in \mathcal{S}(V^-)$ with compact support in $\mathcal{O}_{k-p}^- = \{Y \in V^-,\ \nabla_{k-p} \neq 0\}(Y)$, then $T_g^{p,-}$ is everywhere defined on $K_p(-2,0) \times K_p(0,-2)$ and is locally constant with compact support in $K_p(-2,0)' \times K_p(0,-2)$.*

Proof.

We only prove the first assertion. The proof of the second one will be similar.

We fix $f \in \mathcal{S}(V^+)$. By ([14], II.1.3), we can suppose that there exist $f_1 \in \mathcal{S}(K_p(2,0))$, $f_2 \in \mathcal{S}(K_p(1,1))$ and $f_3 \in \mathcal{S}(K_p(0,2))$ such that

$$f(u+x+v) = f_1(u)f_2(x)f_3(v), \quad u \in K_p(2,0), x \in K_p(1,1) \text{ and } v \in K_p(0,2).$$

Let $u \in K_p(2,0)$ and $v \in K_p(0,2)'$. We consider the map

$$\alpha : \begin{cases} K_p(1,-1) & \longrightarrow \mathbb{C} \\ A & \longmapsto f(e^{\mathrm{ad}A}(u+v)) = f_1(u+\frac{1}{2}(\mathrm{ad}A)^2 v) f_2([A,v]) f_3(v). \end{cases}$$

Denote by K_2 the (compact) support of f_2. As the map $A \longmapsto [A,v] = \alpha_v(A)$ is a linear isomorphism from $K_p(1,-1)$ onto $K_p(1,1)$ we see that there exists a compact subset $K_2' \subset K_p(1,-1)$ such that $f_2([A,v]) = 0$ if $A \notin K_2'$.

This shows that α has compact support.

In the rest of the proof we will denote by $||\ ||$ a fixed norm on the various involved vector spaces.

We will now show that α is locally constant. This means that for $A_0 \in K_p(1,-1)$, there exists $\varepsilon > 0$ such that

$$h \in K_P(1,-1),\ \|h\| < \varepsilon \Longrightarrow \alpha(A_0+h) = \alpha(A_0).$$

We consider first the function $A \longmapsto f_1(u+\frac{1}{2}(\mathrm{ad}A)^2 v)$. As the map $A \mapsto (\mathrm{ad}A)^2 v$ is continuous, for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$||h|| \leq \eta \Rightarrow \|\mathrm{ad}(A_0+h)^2 v - \mathrm{ad}(A_0)^2 v\| < \epsilon.$$

As f_1 is locally constant, this implies that the function $A \longmapsto f_1(u+\frac{1}{2}(\mathrm{ad}A)^2 v)$ is locally constant. A similar argument shows that the function $A \longmapsto f_2([A,v])$ is locally constant. Hence we have proved that $\alpha \in \mathcal{S}(K_p(1,-1))$.

But now, the fact that $T_f^{p,+}(u,v)$ is everywhere defined on $K_p(2,0)\times K_p(0,2)'$ is clear as the integral which defines $T_f^{p,+}(u,v)$ is the integral of α over a compact set.
The fact that $T_f^{p,+}$ is locally constant on $K_p(2,0)\times K_p(0,2)'$ will be a consequence on the proof of 1)b). See below.

Let us now prove 1) b). Here we suppose that $f\in\mathcal{S}(\mathcal{O}_{p+1})$. Denote by K_i the support of f_i $(i=1,2,3)$.
If $\Delta_{p+1}(u+v)=\Delta_{p+1}(v)=0$ (in other words $v\notin K_p(0,2)'$), then $f(e^{\mathrm{ad}A}(u+v))=f(u+\frac{1}{2}(\mathrm{ad}A)^2v+(\mathrm{ad}A)v+v)=0$ because Supp$(f)\subset\mathcal{O}_{p+1}$. Then $K_3\subset K_p(0,2)'$ and $T_f^{p,+}$ is now defined everywhere with support in $K_p(2,0)\times K_p(0,2)'$.

Let us now prove that $T_f^{p,+}$ has compact support.
Remember from above that the map α_v defined by $\alpha_v(A)=[A,v]$ is an isomorphism form $K_p(1,-1)$ onto $K_p(1,1)$. Consider now the map

$$\begin{array}{rcl} K_3\times K_2 & \stackrel{\varphi}{\longrightarrow} & K_p(1,-1)\\ (v,x) & \longmapsto & \varphi(v,x)=\alpha_v^{-1}(x).\end{array}$$

As it is continuous, its image $K=\varphi(K_3\times K_2)$ is compact. Moreover the support of the map

$$A\longmapsto f_2([A,v])$$

is contained in $\alpha_v^{-1}(K_2)$, so that the support of all the maps $A\longmapsto f_2([A,v])$ when $v\in K_3$ is contained in $K=\varphi(K_3\times K_2)\subset K_p(1,-1)$. In other words, the support of all the maps

$$A\longmapsto f(e^{\mathrm{ad}A}(u+v))$$

when $v\in K_3$, is contained in K.
Since the map $\phi:(A,v)\mapsto\frac{1}{2}(\mathrm{ad}A)^2v$ is continuous from $K_p(1,-1)\times K_p(0,2)$ to $K_p(2,0)$, the set $K_0=\phi(K\times K_3)$ is a compact subset of $K_p(2,0)$. Thus there exists $C\in\mathbb{R}^{*+}$ such that for $v\in K_3$ and $A\in K$, we have

$$||\frac{1}{2}(\mathrm{ad}A)^2v||\leq C.$$

Then $||u+\frac{1}{2}(\mathrm{ad}A)^2v||\geq\,|\,||u||-||\frac{1}{2}(\mathrm{ad}A)^2v||\,|\geq||u||-||\frac{1}{2}(\mathrm{ad}A)^2v||\geq||u||-C$. As f_1 has compact support, there exists $C'>0$ such that for $v\in K_3$ and $A\in K$,

$$||u||>C'\Longrightarrow f_1(u+\frac{1}{2}(\mathrm{ad}A)^2v)=0.$$

Hence

$$T_f^{p,+}(u,v)=|\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}}\int_{K_p(1,-1)}f(e^{\mathrm{ad}A}(u+v))dA$$

has compact support for $f\in\mathcal{S}(\mathcal{O}_{p+1})$.

It remains to prove that $T_f^{p,+}$ is locally constant.
As the integral in the definition of $T_f^{p,+}$ is equal to zero for $v\notin K_3$ and as $|\Delta_{p+1}(v)|$ is locally constant on $K_p(0,2)'$, it is enough to prouve that

$$\Psi(u,v)=\int_{K_p(1,-1)}f_1(u+\frac{1}{2}(\mathrm{ad}A)^2v)f_2([A,v])f_3(v)dA$$

is locally constant.

This means that, for $u \in K_p(2,0)$ and $v \in K_p(0,2)$, there exists $\epsilon > 0$ such that

$$(h,k) \in K_p(2,0) \times K_p(0,2),\ ||h|| \leq \epsilon, ||k|| \leq \epsilon \Longrightarrow \Psi(u+h, v+k) = \Psi(u,v).$$

As f_i is locally constant with compact support, there exists $\epsilon_i > 0$ such that f_i is constant on each ball of radius ϵ_i $(i = 1, 2)$. Also there exists $\delta_2 > 0$, such that for $||k|| \leq \delta_2$, and $A \in K$, we have $||[A,k]|| \leq c||A||\,||k|| \leq \epsilon_2$ (for a $c \in \mathbb{R}^*$) , and hence

$$f_2([A, v+k]) = f_2([A,v] + [A,k]) = f_2([A,v]). \tag{2.10}$$

Similarly, there exists $\delta_1 > 0$ and $c > 0$ such that

$$||h + \frac{1}{2}(\mathrm{ad}A)^2 k|| \leq ||h|| + c||A||^2||k|| \leq \epsilon_1 \text{ if } ||h|| \leq \delta_1,\ ||k|| \leq \delta_1 \text{ and } A \in K.$$

Hence

$$f_1(u+h+\frac{1}{2}(\mathrm{ad}A)^2(v+k)) = f_1(u+\frac{1}{2}(\mathrm{ad}A)^2(v)+h+\frac{1}{2}(\mathrm{ad}A)^2(k)) = f_1(u+\frac{1}{2}(\mathrm{ad}A)^2(v)). \tag{2.11}$$

Finally (2.10) and (2.11) imply that $\Psi(u+h, v+k) = \Psi(u,v)$ if $||h||,\ ||k|| \leq Min(\delta_1, \delta_2, \epsilon_3)$. This ends the proof of 1)b).

Lets us return to the end of the proof of 1)a). Taking now K_3 to be an open compact subset of $K_p(0,2)'$ which contains a fixed element v (instead of the support of f_3), and taking into account that $|\Delta_{p+1}(.)|^{\frac{(p+1)d}{2\kappa}}$ is locally constant on K_3, the same proof as before shows that $T_f^{p,+}$ is locally constant on $K_p(0,2) \times K_p(0,2)'$. □

In the rest of the paper all the involved measures are normalized as in Definition 2.2.10.

2.4. Weil formula and the computation of $T^{p,-}_{\mathcal{F}f}$.

Notation 2.4.1. From now on, for sake of simplicity, and as the context will be clear, we will always denote by the same letter $\mathcal{F}$, the Fourier transforms between various subspaces of V^+, V^- and $\mathfrak{g}$.

In this section we will make a connection, roughly speaking, between $T^{p,-}_{\mathcal{F}f}$ and $\mathcal{F}T^{p,+}_f$ (for the precise statement see Theorem 2.4.4 below).

Define $E = K_p(1,-1)$. Then E^* is identified to $K_p(-1,1)$ through the normalized Killing form b (see (1.3)). More precisely an element $B \in K_p(-1,1)$ is identified with the form $y^* \in E^*$ defined by

$$\langle y^*, A\rangle = b(B,A) \text{ for } A \in K_p(1,-1).$$

As usual the Fourier transform of a function $f \in \mathcal{S}(K_p(1,-1))$ by

$$\mathcal{F}f(B) = \hat{f}(y^*) = \int_{K_p(1,-1)} f(A)\psi(\langle y^*, A\rangle)dA = \int_{K_p(1,-1)} f(A)\psi(b(B,A))dA.$$

(Remember that we have chosen, once and for all, a non trivial unitary additive character ψ of F).

Suppose now that a non degenerate quadratic form Q is given on E. Denote by

$$\beta(A,A') = Q(A+A') - Q(A) - Q(A') \qquad \text{for } A, A' \in E$$

the corresponding bilinear form. This bilinear form induces a linear isomorphism α_β between E and E^* given by

$$b(\alpha_\beta(A), A') = \beta(A,A') \quad \text{for } A, A' \in E.$$

Then there exists a constant $C > 0$ such that for any $f^* \in \mathcal{S}(E^*)$

$$\int_{E^*} f^*(B)dB = C \int_E f^*(\alpha_\beta(A))dA. \tag{2.12}$$

Theorem 2.4.2. (A. Weil)
There exists a constant $\gamma_\psi(Q)$ of module 1 such that for $f \in \mathcal{S}(E)$

$$\int_E \mathcal{F}f(\alpha_\beta(A))\psi(Q(A))dA = \frac{1}{\sqrt{C}}\gamma_\psi(Q)\int_E f(A)\psi(-Q(A))dA.$$

Proof.
For another Haar measure $d\mu(A)$ on E, we consider for $f \in \mathcal{S}(E)$, a "new" Fourier transform by

$$\mathcal{F}'f(A') = \int_E f(A)\psi(b(\alpha_\beta(A'), A)d\mu(A) \quad (A, A' \in E).$$

First of all we determine $\lambda > 0$ such the measure $d\mu(A) = \lambda dA$ is self dual for the Fourier transform $\mathcal{F}'$. From the definitions we see that $\mathcal{F}'f(A') = \lambda\mathcal{F}f(\alpha_\beta(A'))$.
Then, using (2.12) above, the measure λdA is self dual for $\mathcal{F}'$ if and only if

$$\int_E \mathcal{F}'f(A')\lambda dA' = \lambda^2 \int_E \mathcal{F}f(\alpha_\beta(A'))dA' = \frac{\lambda^2}{C}\int_{E^*} \mathcal{F}f(B)dB = \frac{\lambda^2}{C}f(0) = f(0).$$

Hence the self dual measure for $\mathcal{F}'$ is $\sqrt{C}dA$. Once this self dual measure is known, the formula in the Theorem is juste a rewriting of formula (1-4) p. 500 in [13] (where self-dual measures are needed). And this is a particular case of a more general result of A. Weil concerning the Fourier transform of quadratic characters of abelian groups ([17]).

□

We will now make a particular choice of Q. Consider two generic elements $u' \in K_p(-2, 0)'$ and $v \in K_p(0, 2)'$. We define the quadratic form $Q_{u',v}$ on $K_p(1, -1)$ as in ([8], §3.5) by

$$Q_{u',v}(A) = \frac{1}{2}b((\mathrm{ad}A)^2u', v).$$

We compute now the bilinear form $\beta_{u',v}$ corresponding to this form (the computation is the same as in [2], p.106, we give it here for the convenience of the reader):

$$\begin{aligned}
\beta_{u',v}(A, A') &= Q_{u',v}(A + A') - Q_{u',v}(A) - Q_{u',v}(A') \\
&= \tfrac{1}{2}b((\mathrm{ad}A + \mathrm{ad}A')^2u', v) - \tfrac{1}{2}b((\mathrm{ad}A)^2u', v) - \tfrac{1}{2}b((\mathrm{ad}A')^2u', v) \\
&= \tfrac{1}{2}b([A, [A', u']], v) + \tfrac{1}{2}b([A', [A, u']], v) \\
&= -\tfrac{1}{2}b([A', u'], [A, v]) + \tfrac{1}{2}b(A', [[A, u'], v]) \\
&= \tfrac{1}{2}b(A', [[A, v], u']) + \tfrac{1}{2}b(A', [[A, u'], v]) \\
&= b(A', [[A, v], u']) \ \text{(since } \mathrm{ad}u'\ \mathrm{ad}v = \mathrm{ad}v\ \mathrm{ad}u') \\
&= b(A', [[A, u'], v])
\end{aligned}$$

Hence if the map $\alpha_{\beta_{u',v}} : E \longrightarrow E^*$ is defined by

$$\langle \alpha_{\beta_{u',v}}(A), A' \rangle = \beta_{u',v}(A, A'),$$

we obtain (modulo the identification $E* = K_p(-1,1)$):

$$\alpha_{\beta_{u',v}}(A) = \mathrm{ad}v\mathrm{ad}u'(A) = \mathrm{ad}u'\mathrm{ad}v(A).$$

We have seen previously that there exists a constant $c_{\alpha_{\beta_{u',v}}} > 0$ such that for all $f^* \in \mathcal{S}(E^*)$

$$\int_{E^*} f^*(B)dB = c_{\alpha_{\beta_{u',v}}} \int_E f^*(\alpha_{\beta_{u',v}}(x))dx.$$

Using Lemma 2.3.1 we have:

$$\begin{aligned}
\int_{K_p(1,-1)} f^*(\alpha_{\beta_{u',v}}(A))dA &= \int_{K_p(1,-1)} f^*(\mathrm{ad}u'\mathrm{ad}v(A))dA \\
&= |\Delta_{p+1}(v)|^{-\frac{(p+1)d}{\kappa}} \int_{K_p(1,1)} f^*(\mathrm{ad}u'(w))dw \\
&= |\Delta_{p+1}(v)|^{-\frac{(p+1)d}{\kappa}} |\nabla_{k-p}(u')|^{-\frac{(k-p)d}{\kappa}} \int_{K_p(-1,1)} f^*(B)dB
\end{aligned}$$

Hence

$$c_{\alpha_{\beta_{u',v}}} = |\Delta_{p+1}(v)|^{-\frac{(p+1)d}{\kappa}} |\nabla_{k-p}(u')|^{-\frac{(k-p)d}{\kappa}}$$

Then if we set $\gamma(u',v) = \gamma_\psi(Q_{u',v})$, Theorem 2.4.2, can be specialized to:

Theorem 2.4.3. (See Corollary 4.27 p. 107 of [2], for the case $p = 0$ and $F = \mathbb{R}$)
Let $u' \in K_p(-2,0)'$ and $v \in K_p(0,2)'$. Then for $f \in \mathcal{S}(K_p(1,-1))$ we have:

$$\begin{aligned}
&\int_{K_p(1,-1)} \mathcal{F}f(\mathrm{ad}u'\mathrm{ad}v(A))\psi(Q_{u',v}(A))dA \\
&= \gamma(u',v)|\Delta_{p+1}(v)|^{-\frac{(p+1)d}{2\kappa}} |\nabla_{k-p}(u')|^{-\frac{(k-p)d}{2\kappa}} \int_{K_p(1,-1)} f(A)\psi(-Q_{u',v}(A))dA
\end{aligned}$$

Theorem 2.4.4. (I. Muller ([12]), for the case $\ell = 1$, and $p = k-1$, see also Theorem 4.28 p. 107 of [2] for the case $p = 0$ and $F = \mathbb{R}$).
If $f \in \mathcal{S}(V^+)$ and if $(u',v) \in K_p(-2,0)' \times K_p(0,-2)$ then

$$T^{p,-}_{\mathcal{F}f}(u',v') = \mathcal{F}_v(\gamma(u',v)(\mathcal{F}_u T^{p,+}_f)(u',v))(v').$$

Proof.
We know from Theorem 2.3.6, 1)a) and 1)b) that $T^{p,+}_f(u,v)$ is everywhere defined for $u \in K_p(2,0)$ and $v \in K_p(0,2)'$ and that $T^{p,-}_{\mathcal{F}f}(u',v')$ is everywhere defined for $u' \in K_p(-2,0)'$ and $v' \in K_p(0,-2)$. The same Theorem tells us also that $T^{p,+}_f(u,v)$ is almost everywhere defined on $K_p(2,0) \times K_p(0,2)$ and that the function $u \longmapsto T^{p,+}_f(u,v)$ is integrable over $K_p(2,0)$ for almost all $v \in K_p(0,2)$.
We note also that here, although the base field F is p-adic and that we work with a general maximal parabolic P_p, the main steps of the proof are the same as in ([2] Theorem 4.28, p.107) where $F = \mathbb{R}$ and where $p = 0$.
In the rest of the proof we will omit the spaces where the integrations are performed, and we make the following convention: the integration in the variables A, A' will be on $K_p(1,-1)$, in u on $K_p(2,0)$, in v on $K_p(0,2)$.
Let $f \in \mathcal{S}(V^+)$. As we have already seen we can suppose that

$$f(u+w+v) = f_1(u)f_2(w)f_3(v) \quad \text{where} \quad f_1 \in \mathcal{S}(K_p(2,0)), f_2 \in \mathcal{S}(K_p(1,1)), f_3 \in \mathcal{S}(K_p(0,2)).$$

We suppose now that v is fixed in $K_p(0,2)'$. We have then

$$\begin{aligned}\int_u |T_f^{p,+}(u,v)|du &\leq |\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} \int_u \int_A |f(e^{\mathrm{ad}A}(u+v))|dAdu \\ &= |\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} |f_3(v)| \int_u \int_A |f_1(u+\frac{1}{2}(\mathrm{ad}A)^2 v)|\,|f_2([A,v])|dudA.\end{aligned}$$

We make the change of variable $u+\frac{1}{2}(\mathrm{ad}A)^2 v \to u$ in the u-integral and we note that the A-integral is over a compact set (due to the fact that $A \longmapsto [A,v]$ is an isomorphism from $K_p(1,-1)$ onto $K_p(1,1)$). Hence, for any $v \in K_p(0,2)'$

$$\int_u |T_f^{p,+}(u,v)|du \leq \int_u T_{|f|}^{p,+}(u,v)du < +\infty. \tag{2.13}$$

Therefore

$$u \longmapsto T_f^{p,+}(u,v) \in L^1(K_p(2,0)) \quad \text{for } v \in K_p(0,2)'.$$

Let us now compute the u-Fourier transform of $T_f^{p,+}$ given by

$$\mathcal{F}_u T_f^{p,+}(u',v) = |\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} \int_{u,A} f(u+\frac{1}{2}(\mathrm{ad}A)^2 v + [A,v]+v)\psi(b(u,u'))dAdu.$$

Due to (2.13) the order of the integrations does not matter. Making the change of variable $u+\frac{1}{2}(\mathrm{ad}A)^2 v \to u$ we obtain

$$\mathcal{F}_u T_f^{p,+}(u',v) = |\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} \int_{u,A} f(u+[A,v]+v)\psi(b(u-\frac{1}{2}(\mathrm{ad}A)^2 v,u'))dAdu.$$

We introduce now the function

$$J(u',A,v) = \int_u f(u+[A,v]+v)\psi(b(u,u'))du = (\mathcal{F}_u f)(u',[A,v],v).$$

Then from the definition of the form $Q_{u',v}$

$$\mathcal{F}_u T_f^{p,+}(u',v) = |\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} \int_A J(u',A,v)\psi(-Q_{u',v}(A))dA. \tag{2.14}$$

We suppose now that $u' \in K_p(-2,0)'$.

As v is generic in $K_p(0,2)$ the function $(u,A) \longmapsto f(u+[A,v]+v)$ belongs to $\mathcal{S}(K_p(2,0)\times K_p(1,-1))$ and the function $A \longmapsto J(u',A,v)$ belongs to $\mathcal{S}(K_p(1,-1))$. As u' is generic too, the quadratic form $Q_{u',v}$ is non degenerate and we can apply the Weil formula as in Theorem 2.4.3 to the integral in the right hand side of (4). We obtain

$$\int_A J(u',A,v)\psi(-Q_{u',v}(A))dA$$

$$= \gamma(u',v)^{-1}|\Delta_{p+1}(v)|^{\frac{(p+1)d}{2\kappa}} |\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} \int_{A'} (\mathcal{F}_A J)(u',\mathrm{ad}u'\mathrm{ad}v(A'),v)\psi(Q_{u',v}(A'))dA', \tag{2.15}$$

where

$$(\mathcal{F}_A J)(u',\mathrm{ad}u'\mathrm{ad}v\,A',v) = \int_A (\mathcal{F}_u f)(u',[A,v],v)\psi(b((A,\mathrm{ad}u'\mathrm{ad}v\,A'))dA.$$

If we make the change of variable $w=[A,v]$, and if we use Lemma 2.3.1, together with the fact that $b((A,\mathrm{ad}u'\mathrm{ad}v\,A') = b([A,v],[u',A'])$ the preceding formula becomes

$$(\mathcal{F}_A J)(u',\mathrm{ad}u'\mathrm{ad}v\,A',v) = |\Delta_{p+1}(v)|^{-\frac{(p+1)d}{\kappa}} (\mathcal{F}_w \mathcal{F}_u f)(u',[u',A'],v). \tag{2.16}$$

From (2.14), (2.15) and (2.16), we obtain

$$\gamma(u',v)(\mathcal{F}_u T_f^{p,+})(u',v) = |\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} \int_{A'} (\mathcal{F}_w\mathcal{F}_u f)(u',[u',A'],v)\psi(Q_{u',v}(A'))dA'. \quad (2.17)$$

From now on we only suppose that u' is generic in $K_p(-2,0)$ (v may be singular).

The Fourier transform $\mathcal{F}_w\mathcal{F}_u f \in \mathcal{S}(K_p(-2,0)\times K_p(-1,-1)\times K_p(0,2))$ and as u' is generic we obtain, as before, that

$$\int_{v,A'} |(\mathcal{F}_w\mathcal{F}_u f)(u',[u',A'],v)|dA'dv < +\infty.$$

It follows than from (2.17) that the function $v \longmapsto \gamma(u',v)(\mathcal{F}_u T_f^{p,+})(u',v)$ belongs to $L^1(K_p(0,2)$ for u' generic in $K_p(-2,0)$ and this allows, by Fubini's Theorem, to change the order of the integrations in A' and v in the following computation of its Fourier transform.
We have

$$\begin{aligned}
&\mathcal{F}_v(\gamma(u',v)(\mathcal{F}_u T_f^{p,+})(u',v))(v') \\
&= |\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} \int_v \int_{A'} (\mathcal{F}_w\mathcal{F}_u f)(u',[u',A'],v)\psi(b(\frac{1}{2}(\mathrm{ad}A')^2u',v)+b(v',v))dA'dv \\
&= |\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} \int_{A'} \int_v (\mathcal{F}_w\mathcal{F}_u f)(u',[u',A'],v)\psi(b(\frac{1}{2}(\mathrm{ad}A')^2u',v)+b(v',v))dvdA' \\
&= |\nabla_{k-p}(u')|^{\frac{(k-p)d}{2\kappa}} \int_{A'} (\mathcal{F}_v\mathcal{F}_w\mathcal{F}_u f)(u',[u',A'],v'+\frac{1}{2}(\mathrm{ad}A')^2u')dA'.
\end{aligned}$$

But $u'+[u',A']+v'+\frac{1}{2}(\mathrm{ad}A')^2u' = e^{\mathrm{ad}(-A')(u'+v')}$ and $\mathcal{F}_v\mathcal{F}_w\mathcal{F}_u f = \mathcal{F}f$ and we recognize the definition of $T_{\mathcal{F}f}^{p,-}$ (Definition 2.3.4, 2)). Hence

$$T_{\mathcal{F}f}^{p,-}(u',v') = \mathcal{F}_v(\gamma(u',v)(\mathcal{F}_u T_f^{p,+})(u',v))(v')$$

for u' generic in $K_p(-2,0)$.

□

Remark 2.4.5.
Godement and Jacquet have used a mean function φ_Φ ([7] p.37) which looks like our mean function $T_f^{p,+}$ (or $T_g^{p,-}$) but which is not equal to our's in the case of $GL_n(F)\times GL_n(F)$ acting on $M_n(F)$ which corresponds to their situation. Also they proved that $\mathcal{F}(\varphi_\Phi) = \varphi_{\mathcal{F}(\Phi)}$ (Loc. cit Lemma 3.4.0 p. 38), which again looks similar to the preceeding Theorem 2.4.4.

2.5. **Computation of the Weil constant $\gamma(u',v)$.**

(Remember that in this section, we suppose that $\ell = 1$).

The aim of this paragraph is to compute the Weil constant $\gamma(u',v) = \gamma_\psi(Q_{u',v})$ introduced in Theorem 2.4.2, in the case where $u' \in K_{k-1}(-2,0)'$ and $v \in K_{k-1}(0,2)' = \tilde{\mathfrak{g}}^{\lambda_k} \setminus \{0\}$. We recall first some general results of A. Weil [17] and of S. Rallis et G. Schiffmann [13] concerning $\gamma_\psi(Q)$.

First of all, if Q and Q' are two equivalent non degenerate quadratic forms ,we have

$$\gamma_\psi(Q) = \gamma_\psi(Q').$$

Remember that there are four classes modulo the squares in F, namely $\mathscr{C} = F^*/F^{*2} = \{1, \varepsilon, \pi, \varepsilon\pi\}$ where ε is a unit which is not a square, and where π is a uniformizer for F.

For $a, b \in F^*$, let us denote by $(a,b)_{\mathscr{C}}$ the Hilbert symbol defined by

$$(a,b)_{\mathscr{C}} = \begin{cases} 1 & \text{if } b \text{ belongs to the norm group of } F[\sqrt{a}] \\ -1 & \text{if not} \end{cases}.$$

Denote by χ_a the quadratic character of F^* (associated to the quadratic extension $F[\sqrt{a}]$) defined by

$$\chi_a(b) = (a,b)_{\mathscr{C}}.$$

If $\mathfrak{q}_1$ is the quadratic form on F defined by $\mathfrak{q}_1(x) = x^2$ we set:

$$\alpha(a) = \gamma_\psi(a\mathfrak{q}_1), \quad \varphi(a) = \frac{\alpha(a)}{\alpha(1)} \quad a \in F^*. \tag{2.18}$$

As $a^2\mathfrak{q}_1$ is equivalent to $\mathfrak{q}_1$ one can consider α and φ as functions on $\mathscr{C}$.
A fundamental result of A. Weil ([13] (1-6)) tells us that

$$\varphi(ab) = \varphi(a)\varphi(b)(a,b)_{\mathscr{C}}, \quad a, b \in F^*.$$

As $\gamma_\psi(Q)\gamma_\psi(-Q) = 1$ for any non degenerate quadratic form Q, one has $\alpha(1)^{-1} = \alpha(-1)$ and hence

$$\alpha(ab) = \alpha(a)\alpha(b)(a,b)_{\mathscr{C}}\alpha(-1).$$

One has also the following result:

Lemma 2.5.1. *([13] Proposition I-7) Let Q be a non degenerate quadratic form on a vector space E of dimension n, and discriminant $disc(Q)$.*

(1) *If $n = 2r$ is even then*

$$\gamma_\psi(xQ) = \gamma_\psi(Q)(x, (-1)^r disc(Q))_{\mathscr{C}}, \quad x \in F^*$$

(2) *If $n = 2r+1$ is odd then*

$$\gamma_\psi(xQ) = \gamma_\psi(Q)(x, (-1)^r disc(Q))_{\mathscr{C}}\alpha(x)\alpha(-1).$$

Proof. The first assertion is exactly the assertion of Proposition I-7 of [13]. As our formulation of assertion 2) is slightly different from that in [13], we give some details.

From ([13] Proposition 1-3), we know that if $Q(x_1, \ldots, x_n) = \sum_{s=1}^n a_s x_s^2$ then $\gamma_\psi(Q) = \prod_{s=1}^n \alpha(a_s)$. Using the fact that $\alpha(a)\alpha(b) = \alpha(ab)(a,b)_{\mathscr{C}}\alpha(1)$, one obtains by induction that

$$\gamma_\psi(Q) = \alpha(1)^{n-1}\alpha(a_1 \ldots a_n)\prod_{i<j}(a_i, a_j)_{\mathscr{C}}.$$

Set $D = disc(Q) = a_1 \ldots a_n \bmod F^{*2}$. As $(xa_i, xa_j)_{\mathscr{C}} = (x,x)_{\mathscr{C}}(x,a_i)_{\mathscr{C}}(x,a_j)_{\mathscr{C}}(a_i,a_j)_{\mathscr{C}}$, $(x,x)_{\mathscr{C}} = (-1,x)_{\mathscr{C}}$, and $\prod_{i<j}(x,a_i)_{\mathscr{C}}(x,a_j)_{\mathscr{C}} = (x^{n-1}, D)_{\mathscr{C}}$, one obtains

$$\gamma_\psi(xQ) = \gamma_\psi(Q)(x, (-1)^{\frac{n(n-1)}{2}})_{\mathscr{C}}(x^{n-1}, D)_{\mathscr{C}}\frac{\alpha(x^n D)}{\alpha(D)}.$$

If n is even then $(x^{n-1},D)_{\mathscr{C}}=(x,D)_{\mathscr{C}}$ and $\alpha(x^nD)=\alpha(D)$. This gives the first assertion . And if n is odd, we have $(x^{n-1},D)_{\mathscr{C}}=1$ and $\alpha(x^nD)=\alpha(xD)=\alpha(x)\alpha(D)(x,D)_{\mathscr{C}}\alpha(-1)$. This gives the second assertion. □

In what follows we assume that $k\geq 1$.

Notation 2.5.2. We denote by $\tilde{\mathfrak{l}}_k$ the Lie algebra generated by $\tilde{\mathfrak{g}}^{\pm\lambda_k}$, that is

$$\tilde{\mathfrak{l}}_k=\tilde{\mathfrak{g}}^{-\lambda_k}\oplus FH_{\lambda_k}\oplus\tilde{\mathfrak{g}}^{\lambda_k}.$$

Let $\underline{\tilde{\mathfrak{g}}}$ be the centralizer of $\tilde{\mathfrak{l}}_k$ in $\tilde{\mathfrak{g}}$. We will denote by underlined letters the elements and algebras associated to $\underline{\tilde{\mathfrak{g}}}$. The algebra $\underline{\tilde{\mathfrak{g}}}$ is 3-graded by the element $\underline{H}_0=\sum_{j=0}^{k-1}H_{\lambda_j}$, and is regular of rank k. We denote by $\underline{\tilde{\mathfrak{g}}}=\underline{V}^-\oplus\underline{\mathfrak{g}}\oplus\underline{V}^+$ the corresponding decomposition. It can be noted that in the notations of §2.1 we have $\underline{V}^{\pm}=K_{k-1}(\pm 2,0)$ and $\underline{\mathfrak{g}}=K_{k-1}(0,0)$ and $\tilde{\mathfrak{g}}^{\pm\lambda_k}=K_{k-1}(0,\pm 2)$. We have also $\underline{\mathfrak{g}}=Z_{\underline{\tilde{\mathfrak{g}}}}(\mathfrak{a}^0)\oplus(\oplus_{i\neq j<k}E_{i,j}(1,-1))$.

If $\underline{\mathfrak{a}}^0=\oplus_{j<k}FH_{\lambda_j}$, then $\underline{G}$ stands for the group $\mathcal{Z}_{\mathrm{Aut}_0(\underline{\tilde{\mathfrak{g}}})}(\underline{H}_0)$ whose Lie algebra is $\underline{\mathfrak{g}}$ and $\underline{P}=\underline{LN}$ is the parabolic subgroup of $\underline{G}$ corresponding to the Lie algebra

$$\underline{\mathfrak{p}}=\mathcal{Z}_{\underline{\mathfrak{g}}}(\underline{\mathfrak{a}}^0)\oplus\oplus_{r<s<k}E_{r,s}(1,-1).$$

Let $\underline{\Delta}_j$ be the polynomial relative invariants of $(\underline{P},\underline{V}^+)$ normalized such $\underline{\Delta}_j(X_0+\ldots+X_{k-1})=1$, etc... (in the notations of Definition 2.1.1, we have for example $\underline{\Delta}_0=\tilde{\Delta}_1$).

Lemma 2.5.3. *(i) Let u,v and A be respectively in $\underline{V}^+,\tilde{\mathfrak{g}}^{\lambda_k}$ and $K_{k-1}(1,-1)$, then*

$$\Delta_j(e^{\mathrm{ad}A}(u+v))=\begin{cases}\underline{\Delta}_j(u)\Delta_k(v) & \text{if} \quad j=0,\ldots,k-1,\\ \Delta_k(v) & \text{if} \quad j=k.\end{cases}$$

(ii) If $v'=\nu Y_k$, we set $\tilde{\nabla}_k(v')=\nu$. Then , for u',v' and A respectively in $\underline{V}^-,\tilde{\mathfrak{g}}^{-\lambda_k}$ and $K_{k-1}(1,-1)$, we have

$$\nabla_j(e^{\mathrm{ad}A}(u'+v'))=\begin{cases}\underline{\nabla}_0(u')\tilde{\nabla}_k(v') & \text{if} \quad j=0,\\ \underline{\nabla}_{j-1}(u') & \text{if} \quad j=1,\ldots,k.\end{cases}$$

Proof. See ([2] Lemma 5.18).

□

Corollary 2.5.4. *Let $k\geq 1$ and $a=(a_0,\ldots,a_k)\in\mathscr{S}_e^{k+1}$.*

(1) *Let $u\in\underline{V}^+$ and $v\in\tilde{\mathfrak{g}}^{\lambda_k}$ then*

$$u+v\in\mathcal{O}^+(a_0,\ldots a_k)\Longleftrightarrow u\in\underline{\mathcal{O}}^+(a_0,\ldots a_{k-1}),\ \text{and}\ \Delta_k(v)a_k\in S_e.$$

(2) *Let $u'\in\underline{V}^-$ and $v'\in\tilde{\mathfrak{g}}^{-\lambda_k}$ then*

$$u'+v'\in\mathcal{O}^-(a_0,\ldots a_k)\Longleftrightarrow u'\in\underline{\mathcal{O}}^-(a_0,\ldots a_{k-1})\ \text{and}\ \tilde{\nabla}_k(v')a_k\in S_e.$$

Proof. Let $u \in \underline{V}^+$ and $v \in \tilde{\mathfrak{g}}^{\lambda_k}$. By Lemma 2.5.3, we have $\Delta_j(u+v) = \underline{\Delta}_j(u)\Delta_k(v)$ for $0 \leq j \leq k-1$ and $\Delta_k(u+v) = \Delta_k(v)$. Then the definition of the open sets $\mathcal{O}^+(a_0, \ldots, a_k)$ implies immediately the first assertion.

Let $u' \in \underline{V}^-$ and $v' \in \tilde{\mathfrak{g}}^{-\lambda_k}$. By Lemma 2.5.3 we have $\nabla_j(u'+v') = \underline{\nabla}_{j-1}(u')$ if $j = 1, \ldots, k$ and $\nabla_0(u'+v') = \underline{\nabla}_0(u')\tilde{\nabla}_k(v')$, where $\tilde{\nabla}_k(y_k Y_k) = y_k$. Therefore $u'+v' \in \mathcal{O}^-(a_0, \ldots a_k)$ if and only if $\underline{\nabla}_{j-1}(u')a_0 \ldots a_{k-j} \in S_e$ for $j = 1, \ldots, k$ et $\underline{\nabla}_0(u')\tilde{\nabla}_k(v')a_0 \ldots a_k \in S_e$. These conditions are equivalent to $\underline{\nabla}_j(u')a_0 \ldots a_{k-1-j} \in S_e$ for $j = 0, \ldots, k-1$ and $\tilde{\nabla}_k(v')a_k \in S_e$. This gives the second assertion.

□

Proposition 2.5.5. *Let $k \geq 1$. Fix $\underline{a} = (a_0, \ldots, a_{k-1}) \in \mathscr{S}_e^k$ and $c_k \in \mathscr{S}_e$. Let $u' \in \underline{\mathcal{O}}^-(\underline{a})$ and let $v \in \tilde{\mathfrak{g}}^{\lambda_k} \setminus \{0\}$such that $\Delta_k(v)c_k \in S_e$. Then $\gamma(u', v)$ depends only on $\underline{a}$ and on c_k.*
Therefore let us set:

$$\gamma_k(\underline{a}, c_k) = \gamma(u', v), \quad \textit{for } u' \in \underline{\mathcal{O}}^-(\underline{a}),\ v \in K_p(0,2),\ \Delta_k(v)c_k \in S_e.$$

Remember that $\alpha(a) = \gamma_\psi(a\mathfrak{q}_1)$ for $a \in F^*$, where $\mathfrak{q}_1(x) = x^2$, $x \in F$. Remember also that if $e = 2$, we have $S_e = N_{E/F}(E^*)$ where E is the quadratic extension of F such that $N_{E/F}(E^*) = Im(q_{an,2})^*$ (see Definition 1.1.4). Denote by ϖ_E the quadratic character associated to E, that is $\varpi_E(a) = 1$ if $a \in N_{E/F}(E^*)$ and $\varpi_E(a) = -1$ if $a \notin N_{E/F}(E^*)$ (hence if $E = F[\sqrt{\xi}]$, one has $\varpi_E = \chi_\xi$). Then $\gamma_k(\underline{a}, c_k) =$

$-\ \gamma_\psi(q_e)^k$ if $e = 0$ or $e = 4$ (*case where* $\mathscr{S}_e = \{1\}$)

$-\ \gamma_\psi(q_e)^k \varpi_E(c_k)^k \prod_{j=0}^{k-1} \varpi_E(a_j)$ if $e = 2$ (*case where* $\mathscr{S}_e = F^*/N_{E/F}(E^*)$)

$-\ \prod_{j=0}^{k-1} \gamma_\psi(q_e)((-1)^{\frac{d-1}{2}} disc(q_e), a_j c_k)_{\mathscr{C}} \alpha(a_j c_k)\alpha(-1)$ if $e = 1$ or $e = 3$, (*case where* $\mathscr{S}_e = \mathscr{C} = F^*/F^{*2}$).

($disc(q_e)$ is the discriminant of q_e).

Proof. Let $u' \in \underline{\mathcal{O}}^-(\underline{a})$ and $v \in \tilde{\mathfrak{g}}^{\lambda_k} = FX_k$ such that $\Delta_k(v)c_k \in S_e$. This implies that $v = c_k \nu X_k$ with $\nu \in S_e$. Set $Y_{\underline{a}} = a_0 Y_0 + \ldots a_{k-1}Y_{k-1}$. We will prove simultaneously the two assertions.

Let us first show that there exists $p \in \tilde{P}$ such that $pu' = Y_{\underline{a}}$ and $pX_k = X_k$.
From Lemma 4.3.3) in [8], applied to the algebra $\tilde{\mathfrak{g}}$, there exists $\underline{n} \in \underline{N}$ and scalars $y_0, \ldots y_{k-1}$ such that $\underline{n}u' = y_0 Y_0 + \ldots + y_{k-1}Y_{k-1}$. From the definition of $\underline{N}$, one has $\underline{N} \subset N$ and $\underline{n}.v = v$.
As $y_0 Y_0 + \ldots + y_{k-1}Y_{k-1} \in \underline{\mathcal{O}}^-(\underline{a})$, there exists $\mu_j \in S_e$ such that $y_j = \mu_j a_j$ for $j = 0, \ldots, k-1$.
From Remark 1.2.2, the elements $\mu_0^{-1}X_0 + \ldots + \mu_{k-1}^{-1}X_{k-1} + X_k$ and $X_0 + \ldots + X_k$ are $\tilde{L}$-conjugated. Hence, there exists $l \in \tilde{L}$ such that $x_j(l) = \mu_j^{-1}$ for $j = 0, \ldots, k-1$ and $x_k(l) = 1$. The element $p = l\underline{n}$ satisfies then the required property.

Then $pu' = Y_{\underline{a}}$ and $pv = v = \nu c_k X_k$ with $p \in \tilde{P}$.

Hence the quadratic form $Q_{u',v}$ is equivalent to the form $c_k\nu Q_{Y_{\underline{a}},X_k}$, and therefore

$$\gamma(u',v)=\gamma_\psi(c_k\nu Q_{Y_{\underline{a}},X_k}).$$

The space $K_{k-1}(1,-1)$ (on which the forms live) is the direct sum of the spaces $E_{j,k}(1,-1)$ for $j=0,\dots,k-1$. These spaces are orthogonal for $Q_{a_0Y_0+\dots+a_{k-1}Y_{k-1},X_k}$. Moreover, if $A\in E_{j,k}(1,-1)$, we have $b((\mathrm{ad}A)^2Y_i,X_k)=0$ for $j\neq i$. Therefore

$$Q_{u',v}\sim(\oplus_{j=0}^{k-1}c_k\nu a_jQ_{j,k}),\quad \text{where } Q_{j,k}(A)=\frac{1}{2}b((\mathrm{ad}A)^2Y_j,X_k),\ \text{for } A\in E_{j,k}(1,-1).$$

Using ([8] Remark 3.5.3) we see that for $Y\in E_{j,k}(-1,-1)$, one has $Q_{j,k}([X_j,Y])=q_{X_j,X_k}(Y)$ where q_{X_i,X_j} is defined as in (1.4). As $\mathrm{ad}X_j$ is an isomorphism from $E_{j,k}(-1,-1)$ onto $E_{j,k}(1,-1)$, the quadratic forms $Q_{j,k}$ and q_{X_j,X_k} are equivalent. And as all the forms q_{X_i,X_j} are equivalent, we obtain that $Q_{j,k}\sim q_{X_0,X_1}=q_e$.
From ([13] Proposition I-3) we obtain that

$$\gamma(u',v)=\prod_{j=0}^{k-1}\gamma_\psi(c_ka_j\nu q_e).$$

If e is odd then $\nu\in S_e=F^{*2}$ and hence $\gamma_\psi(c_ka_j\nu q_e)=\gamma_\psi(c_ka_jq_e)$. Moreover is e is odd then d is odd too (see table 1 in [2]). As the rank of q_e is d, the result is a consequence of Lemma 2.5.1 (2).

If $e=0$ or 4, the form q_e is the sum of an anisotropic quadratic form of rank e and a hyperbolic form of rank $d-e$. As all the anisotropic quadratic forms of rank 4 are equivalent, and as the same is true for the hyperbolic forms of rank $d-e$, we obtain that $c_ka_j\nu q_e$ is equivalent to q_e, and therefore $\gamma_\psi(c_ka_j\nu q_e)=\gamma_\psi(q_e)$. This gives the result in that case.

Suppose now that $e=2$. Then q_2 is equivalent to $q_{an,2}+q_{hyp,2}$ where $q_{hyp,2}$ is a hyperbolic form of rank $d-2$ and $q_{an,2}$ is an anisotropic form of rank 2 such that $\mathrm{Im}(q_{an,2})^*=N_{E/F}(E^*)$. It follows that $q_{an,2}$ is equivalent to $x^2-\xi y^2$. Therefore $(-1)^{d/2}disc(q_2)=-disc(q_{an,2})=\xi$. As χ_ξ coincides with the quadratic character ϖ_E associated to E and as $\nu\in S_e=N_{E/F}(E^*)$, Lemma 2.5.1 (1) implies that

$$\gamma_\psi(c_ka_j\nu q_2)=\gamma_\psi(q_2)(c_ka_j\nu,\xi)_{\mathscr{C}}=\gamma_\psi(q_2)\varpi_E(c_ka_j).$$

Hence

$$\gamma(u',v)=\prod_{j=0}^{k-1}\gamma_\psi(q_2)\varpi_E(c_ka_j),$$

and this ends the proof.

□

3. Functional equation of the zeta functions associated to $(\tilde{P},V^+)$

3.1. The (A2') Condition in the Theorem k_p of F. Sato.

In this paragraph, we suppose that F is algebraically closed.

A maximal split abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is then a Cartan subalgebra of $\tilde{\mathfrak{g}}$ and of $\mathfrak{g}$. We denote by $\tilde{\Sigma}$ and Σ the corresponding root systems. Let $\Sigma^+\subset\tilde{\Sigma}^+$ the positive subsystems defined in Theorem

1.2.1. in [8]. Then $\mu \in \tilde{\Sigma}^+ \setminus \Sigma^+$ if and only if $\tilde{\mathfrak{g}}^\mu \subset V^+$ and $\mu \in \Sigma^+$ if and only if $\tilde{\mathfrak{g}}^\mu = \mathfrak{g}^\mu \subset E_{i,j}(1,-1) \subset \mathfrak{g}$ for $i > j$ (cf. [8] Proposition 1.9.1).

Proposition 3.1.1. *Let $k \geq 1$. Let $\mu \in \tilde{\Sigma}^+ \setminus \Sigma^+$. Let H_μ be the coroot of μ and let $\{X_{-\mu}, H_\mu, X_\mu\}$ be an $\mathfrak{sl}_2$-triple. Let us denote by $\tilde{\mathfrak{l}}[\mu]$ the subalgebra (isomorphic to $\mathfrak{sl}_2(F)$) generated by $FX_{\pm\mu}$ and set*

$$\tilde{\mathfrak{g}}[\mu] = Z_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{l}}[\mu]).$$

If $\tilde{\mathfrak{g}}[\mu] \cap V^+ \neq \{0\}$ then

(1) *the Lie algebra $\tilde{\mathfrak{g}}[\mu]$ is reductive and 3-graded by the element $H_0 - H_\mu$. The corresponding grading will be denoted by*

$$\tilde{\mathfrak{g}}[\mu] = V[\mu]^- \oplus \mathfrak{g}[\mu] \oplus V[\mu]^+.$$

(2) *The algebra $\mathfrak{a}[\mu] = \{H \in \mathfrak{a}, \mu(H) = 0\}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}[\mu]$ and the corresponding root system is given by*

$$\tilde{\Sigma}[\mu] = \{\alpha \in \tilde{\Sigma}; (\alpha, \mu) = 0\}.$$

We will set: $\tilde{\Sigma}[\mu]^+ = \tilde{\Sigma}[\mu] \cap \tilde{\Sigma}^+$.

(3) *Moreover, the Lie algebra $\tilde{\mathfrak{g}}[\mu]$ is regular of rank k if μ is a long root, and of rank of $k-1$ if μ is a short root.*

Proof. Although the root μ is here arbitrary, the proof of the 2 first points is similar to ([8] Proposition 1.5.3).

If μ is a long root of $\tilde{\Sigma}$ then by ([8] Proposition 1.7.7), there exists a element w of the Weyl group such that $w\mu = \lambda_0$. The third assertion follows from ([8] Corollary 1.8.4).

Suppose that μ is a short root in $\tilde{\Sigma}$. From Table 1 in [8], the root system $\tilde{\Sigma}$ is of type B_n or C_n with $n \geq 2$.

If $\tilde{\Sigma}$ is of type B_n then $k = 1$ and $\mu = \dfrac{\lambda_0 + \lambda_1}{2}$. As λ_0 and λ_1 are long and orthogonal, we have $H_\mu = H_0$ and then $\tilde{\mathfrak{g}}[\mu] \cap V^+ = \{0\}$.

As we suppose that $\tilde{\mathfrak{g}}[\mu] \cap V^+ \neq \{0\}$ the root system $\tilde{\Sigma}$ is of type C_n with $n = k+1 \geq 3$ and we have

$$\tilde{\Sigma}^+ = \{\frac{\lambda_i + \lambda_j}{2}; 0 \leq i \leq j \leq k\}.$$

For our purpose, we can suppose that $\mu = \dfrac{\lambda_k + \lambda_{k-1}}{2}$. Hence $H_\mu = H_{\lambda_k} + H_{\lambda_{k-1}}$ and $H_0 - H_\mu = H_{\lambda_0} + \ldots + H_{\lambda_{k-2}}$. Then $\tilde{\Sigma}[\mu]$ is equal to $\{\pm\lambda_0\}$ if $k = 2$ and $\tilde{\Sigma}[\mu]$ is of type C_{k-1} if $k \geq 3$. This gives the last assertion.

□

Define

$$N_0 = \exp \operatorname{ad} \mathfrak{n}_0, \qquad \text{where } \mathfrak{n}_0 = \oplus_{\mu \in \Sigma^+} \tilde{\mathfrak{g}}^{-\mu}.$$

Proposition 3.1.2. *(compare with* [12] *Lemme 2.2.1 and Proposition 2.3.1).*
Let $x \in V^+$. Then there exists a family of two by two strongly orthogonal roots $\{\mu_1, \dots, \mu_r\}$ such that

$$N_0.x \cap (\oplus_{j=1}^r \tilde{\mathfrak{g}}^{\mu_j}) \neq \emptyset.$$

Proof. The proof goes by induction on the rank of $\tilde{\mathfrak{g}}$. The result is clear for $k = 0$ since $V^+ = \tilde{\mathfrak{g}}^{\lambda_0}$. Suppose now that the result is true for any irreducible graded regular algebra of rank strictly less than $k + 1$.

If $\mu \in \tilde{\Sigma}^+ \setminus \Sigma^+$, then $\mu(H_0) = 2$, and hence for all μ, μ' in $\tilde{\Sigma}^+ \setminus \Sigma^+$, $\mu + \mu'$ is never a root. Therefore $n(\mu, \mu') \geq 0$. From the classification of the 3-graded Lie algebras we consider, we know that G_2 does never occur and hence $n(\mu, \mu') \in \{0, 1, 2\}$ for $\mu, \mu' \in \tilde{\Sigma}^+ \setminus \Sigma^+$.

We will also use the fact that for μ, μ' in $\tilde{\Sigma}^+ \setminus \Sigma^+$, we have $\mu \perp \mu' \iff \mu \perp\!\!\!\perp \mu'$ (the proof is similar to the proof of Corollary 1.8.2. in [8]).

Let $x \in V^+$. Then $x = \sum_{\mu \in \tilde{\Sigma}^+ \setminus \Sigma^+} X_\mu$ with $X_\mu \in \tilde{\mathfrak{g}}^\mu \subset V^+$. We denote

$$s(x) = \{\mu \in \tilde{\Sigma}^+ \setminus \Sigma^+, X_\mu \neq 0\}.$$

1st case: We suppose that there exists a long root μ_0 among the roots of maximal height in $s(x)$.

For $j = 0, 1, 2$, we set $s(x)_j = \{\mu \in s(x); n(\mu, \mu_0) = j\}$. As μ_0 is a long root, $n(\mu, \mu_0) = 2$ if and only if $\mu = \mu_0$. Hence

$$s(x) = \{\mu_0\} \cup s(x)_0 \cup s(x)_1. \tag{3.1}$$

Let $x = X_{\mu_0} + \sum_{\mu \in s(x)_0} X_\mu + \sum_{\mu \in s(x)_1} X_\mu$.

If $s(x)_1 = \emptyset$ then $s(x - X_{\mu_0}) \subset \mu_0^\perp$. From Proposition 3.1.1, one has $x - X_{\mu_0} \in V[\mu_0]^+$ and we obtain the result by induction from $\tilde{\mathfrak{g}}[\mu_0]$, which is of rank k (as $\mathfrak{n}[\mu_0] \subset \mathfrak{n}$ from our definition of $\tilde{\Sigma}[\mu_0]^+$).
We suppose that $s(x)_1 \neq \emptyset$.
For $\mu \in s(x)_1$, we have $\mu - \mu_0 \in -\Sigma^+$ (it is a root because $n(\mu, \mu_0) = 1$ and it is negative because μ_0 is of maximal height). Let us fix $Z_{\mu-\mu_0} \in \tilde{\mathfrak{g}}^{\mu-\mu_0} \subset \mathfrak{n}_0$ such that $[Z_{\mu-\mu_0}, X_{\mu_0}] = -X_\mu$.

Define $A = \sum_{\mu \in s(x)_1} Z_{\mu-\mu_0} \in \mathfrak{n}_0$.

Let $\mu \in s(x)_1$ and $\mu' \in s(x)_0$. Suppose that $\mu - \mu_0 + \mu' \in \tilde{\Sigma}$. As $n(\mu - \mu_0 + \mu', \mu_0) = -1$, then the linear form $\mu - \mu_0 + \mu' + \mu_0 = \mu + \mu'$ would be a root of $\tilde{\Sigma}^+$ and this is not possible. Therefore $\mu - \mu_0 + \mu' \notin \tilde{\Sigma}$ and hence $[Z_{\mu-\mu_0}, X_{\mu'}] = 0$. Then $\sum_{\mu' \in s(x)_0} [A, X_{\mu'}] = 0$, and this implies that

$$e^{\mathrm{ad}A}\Big(\sum_{\mu' \in s(x)_0} X_{\mu'}\Big) = \sum_{\mu' \in s(x)_0} X_{\mu'}.$$

Take μ and μ' in $s(x)_1$. Then $n(\mu - \mu_0 + \mu', \mu_0) = 0$. And therefore the element $y = [A, \sum_{\mu' \in s(x)_1} X_{\mu'}] = \sum_{\mu \in s(x)_1} \sum_{\mu' \in s(x)_1} [Z_{\mu-\mu_0}, X_{\mu'}]$ is such that $s(y) \subset \mu_0^\perp$. Considering the preceding case one obtains $\mathrm{ad}(A)^2 \sum_{\mu' \in s(x)_1} X_{\mu'} = [A, y] = 0$. Then

$$e^{\mathrm{ad}A}\Big(\sum_{\mu' \in s(x)_1} X_{\mu'}\Big) = \sum_{\mu' \in s(x)_1} X_{\mu'} + y, \text{ where } s(y) \subset \mu_0^\perp.$$

As $[A, X_{\mu_0}] = \sum_{\mu\in s(x)_1}[Z_{\mu-\mu_0}, X_{\mu_0}] = -\sum_{\mu\in s(x)_1} X_\mu$, we have

$$e^{\mathrm{ad}\ A}X_{\mu_0} = X_{\mu_0} - \sum_{\mu\in s(x)_1} X_\mu - \frac{1}{2}[A, \sum_{\mu\in s(x)_1} X_\mu] = X_{\mu_0} - \sum_{\mu\in s(x)_1} X_\mu - \frac{y}{2}.$$

Finally we get

$$\begin{aligned} e^{\mathrm{ad}\ A}x &= e^{\mathrm{ad}\ A}X_{\mu_0} + e^{\mathrm{ad}\ A}(\sum_{\mu'\in s(x)_0} X_{\mu'}) + e^{\mathrm{ad}\ A}(\sum_{\mu'\in s(x)_1} X_{\mu'}) \\ &= (X_{\mu_0} - \sum_{\mu\in s(x)_1} X_\mu - \frac{y}{2}) + (\sum_{\mu'\in s(x)_0} X_{\mu'}) + (\sum_{\mu'\in s(x)_1} X_{\mu'} + y) \\ &= X_{\mu_0} + \sum_{\mu'\in s(x)_0} X_{\mu'} + \frac{y}{2}, \quad \text{where } s(y) \subset \mu_0^\perp. \end{aligned}$$

If we set $y_0 = \sum_{\mu'\in s(x)_0} X_{\mu'} + \frac{y}{2}$, then $s(y_0) \subset \mu_0^\perp$ and hence $y_0 \in V[\mu_0]^+$. Again the result is obtained by induction from $\tilde{\mathfrak{g}}[\mu_0]$.

3rd case: We suppose that all roots of maximal height in $s(x)$ are short. From Table 1 in [8], we know that the only cases were there are roots of different length correspond to cases where $\tilde{\Sigma}$ is of type B_n or of type C_n for $n \geq 2$.

• If $\tilde{\Sigma}$ is of type C_n, we have $\tilde{\Sigma}^+ \setminus \Sigma^+ = \{\lambda_j; 0 \leq j \leq k\} \cup \{\frac{\lambda_j+\lambda_i}{2}; 0 \leq i < j \leq k\}$ where $n = k+1$. We also have $\lambda_k \notin s(x)$ (because the long root λ_k is the greatest root in $\tilde{\Sigma}^+$).

If $s(x) \subset \lambda_k^\perp$ then $x \in V[\lambda_k]^+$. Again the result is obtained by induction from $\tilde{\mathfrak{g}}[\lambda_k]$.

We suppose that $s(x) \not\subset \lambda_k^\perp$. This implies that there exists $i < k$ such that $\frac{\lambda_k+\lambda_i}{2} \in s(x)$.
If $k = 1$, then $x \in \tilde{\mathfrak{g}}^{\mu_0}$ and the result follows.
From know on, we suppose that $k \geq 2$. Define

$$i_0 = \max\{i; \frac{\lambda_k+\lambda_i}{2} \in s(x)\} \text{ and } \mu_0 = \frac{\lambda_k+\lambda_{i_0}}{2}.$$

Hence

$$s(x) \subset \{\mu_0\} \cup \{\frac{\lambda_i+\lambda_k}{2}, i < i_0\} \cup \{\frac{\lambda_i+\lambda_j}{2}; 0 \leq i \leq j < k\}.$$

And therefore one can write

$$x = X_{\mu_0} + x_1 + x_2, \text{ with } s(x_1) \subset \{\frac{\lambda_i+\lambda_k}{2}, i < i_0\} \text{ and } s(x_2) \subset \{\frac{\lambda_i+\lambda_j}{2}; 0 \leq i \leq j < k\}.$$

If $\mu = \frac{\lambda_i+\lambda_k}{2}$, with $i < i_0$, belongs to $s(x)$, one has $\mu - \mu_0 \in \tilde{\Sigma}$ (as $n(\mu, \mu_0) = 1$).
Then we fix the elements $Z_{\mu-\mu_0} \in \tilde{\mathfrak{g}}^{\mu-\mu_0} \subset E_{i,i_0}(1,-1) \subset \mathfrak{n}_0$ such that $[Z_{\mu-\mu_0}, X_{\mu_0}] = -X_\mu$.

Let us define $A = \sum_{\mu=\frac{(\lambda_i+\lambda_k)}{2}, 0\leq i<i_0} Z_{\mu-\mu_0} \in \mathfrak{n}_0$ and $n_0 = e^{\mathrm{ad}A}$. One has then $[A, X_{\mu_0}] = -x_1$ and $[A, x_1] \in \oplus_{i<i_0, j<i_0}[E_{i,i_0}(1,-1), E_{j,k}(1,1)] = \{0\}$. Therefore $(\mathrm{ad}A)^2 x_1 = 0$.
Let $i < i_0$. For $r, s \leq k$, one has $[E_{i,i_0}(1,-1), E_{r,i_0}(1,1)] \subset E_{i,r}(1,1)$ and $[E_{i,i_0}(1,-1), E_{r,s}(1,1)] = \{0\}$ for $r \neq i_0$ and $s \neq i_0$. As $A \in \oplus_{i<i_0} E_{i,i_0}(1,-1)$, we deduce that $e^{\mathrm{ad}A}$ normalizes $\oplus_{r\leq s<k} E_{r,s}(1,1)$
And therefore

$$n_0.x = X_{\mu_0} + y \text{ with } s(y) \subset \{\frac{\lambda_i+\lambda_j}{2}; 0 \leq i \leq j < k\}.$$

Set $y = y_0 + y_1$ with $s(y_0) \subset \mu_0^\perp = \{\frac{\lambda_i+\lambda_j}{2}; \{i,j\} \cap \{i_0, k\} = \emptyset\}$ and $s(y_1) \subset \{\frac{\lambda_i+\lambda_j}{2}; 1 \leq i \leq j < k; i_0 \in \{i,j\}\}$.

Under the action of H_{μ_0} the elements in $E_{i,k}(1,-1)$ are of weight 0 if $i = i_0$ and of weight -1 if $i \neq i_0$. This implies that $\mathrm{ad}(X_{\mu_0})$ is surjective from $\oplus_{i=0}^{k-1} E_{i,k}(1,-1)$ onto $\oplus_{i=0}^{k-1} E_{i,i_0}(1,1)$. Hence there exists $B \in \oplus_{i=0}^{k-1} E_{i,k}(1,-1)$ such that $[X_{\mu_0}, B] = y_1$.
As $[E_{i,k}(1,-1), E_{r,s}(1,1)] = \{0\}$ for $i < k$ and $r \leq s < k$, one has $[B, y_1] = [B, y_0] = 0$.
Therefore the element $n_1 = e^{\mathrm{ad}B} \in N_0$ is such that $n_1 n_0 x = n_1(X_{\mu_0} + y_0 + y_1) = X_{\mu_0} + y_0$ with $s(y_0) \subset \mu_0^{\perp}$. We obtain the result by applying the induction to the element $n_1 n_0 x - X_{\mu_0} \in \tilde{\mathfrak{g}}[\mu_0]$.

• If $\tilde{\Sigma}$ is of type B_n for $n \geq 2$, then $k = 1$ and $V^+ = \tilde{\mathfrak{g}}^{\lambda_0} \oplus E_{0,1}(1,1) \oplus \tilde{\mathfrak{g}}^{\lambda_1}$.
From the tables and notations in ([3] Chapitre VI) , if we take $\tilde{\Sigma}^+ = \{\varepsilon_i \pm \varepsilon_j; 1 \leq i < j \leq n\} \cup \{\varepsilon_i; 1 \leq i \leq n\}$, then $\lambda_0 = \varepsilon_1 - \varepsilon_2 = \alpha_1$, and $\lambda_1 = \varepsilon_1 + \varepsilon_2$ is the greatest root. Therefore $\tilde{\mathfrak{g}}^{\mu} \subset V^+$ if and only if $\mu = \varepsilon_1$ or $\mu = \varepsilon_1 \pm \varepsilon_j$ for $j \geq 2$.

Moreover, $\mu_0 = \dfrac{\lambda_0 + \lambda_1}{2} = \varepsilon_1$ is the unique short root in $\tilde{\Sigma}^+$ such that $\tilde{\mathfrak{g}}^{\mu_0} \subset V^+$. As λ_0 and λ_1 are orthogonal, one has $H_{\mu_0} = H_{\lambda_0} + H_{\lambda_1} = H_0$.

From the assumption on $s(x)$, the root μ_0 is of maximal height in $s(x)$. Let $\mu \in s(x)$, $\mu \neq \mu_0$. As $n(\mu, \mu_0) = 2$, one has $\mu - \mu_0 \in \tilde{\Sigma}^-$. Let us fix $Z_{\mu-\mu_0} \in \tilde{\mathfrak{g}}^{\mu-\mu_0} \subset \mathfrak{n}_0$ such that $[Z_{\mu-\mu_0}, X_{\mu_0}] = -X_{\mu}$ and set $A = \sum_{\mu \in s(x), \mu \neq \mu_0} Z_{\mu-\mu_0}$.

As $\tilde{\Sigma}$ is of type B_n, we have that if $\mu \neq \mu_0$ is such that $\mu(H_0) = 2$, the root $\mu - \mu_0$ is short and orthogonal to μ_0 (in the preceding notations, one has $\mu - \mu_0 = \pm\varepsilon_i$ for $i \geq 2$). It follows that if μ and μ' are two elements of $s(x)$, distinct from μ_0, then $\mu - \mu_0$ is a short negative root and hence $\mu + \mu' - \mu_0 \notin \tilde{\Sigma}$ (because $\mu - \mu_0 = -\varepsilon_i$ with $i \geq 2$ and $\mu' = \varepsilon_1 - \varepsilon_j$ with $j \geq 2$).

Therefore $[A, \sum_{\mu \in s(x), \mu \neq \mu_0} X_{\mu}] = 0$. Then if $n = e^{\mathrm{ad}\, A}$, we have $n.x = X_{\mu_0}$.

This ends the proof.

□

As F is algebraically closed, the group P has a unique open orbit in V^+ given by

$$\mathcal{O}^+ = \{X \in V^+; \prod_{j=0}^{k} \Delta_j(X) \neq 0\}.$$

We define $\mathcal{S} = V^+ - \mathcal{O}^+$. If χ_j $(j = 0, \ldots, k)$ is the character of P corresponding to Δ_j, we denote by $\mathbb{X}_{V^+}(P)$ the group of characters of P generated by the χ_j's.

Theorem 3.1.3. (Condition (A2') of F. Sato)(see [15] §2.3)

(1) *P has a finite number of orbits in $\mathcal{S}$ (and hence in V^+).*
(2) *Let $x \in \mathcal{S}$ and let P_x^0 the identity component of the centralizer of x in P. Then there exists a non trivial character χ in $\mathbb{X}_{V^+}(P)$ and $p \in P_x^0$ such that $\chi(p) \neq 1$.*

Proof. Of course $N_0 \subset P$ and $P = LN_0$. From Proposition 3.1.2, for any element x in $\mathcal{S}$, there exists a family $\mathcal{R} \subset \tilde{\Sigma}_2^+$ of strongly orthogonal roots such that $N_0.X \cap (\oplus_{\mu \in \mathcal{R}} \tilde{\mathfrak{g}}^{\mu}) \neq \emptyset$.
As $\tilde{\Sigma}^+ \setminus \Sigma^+$ is finite, there are also only a finite number of families of strongly orthogonal roots in $\tilde{\Sigma}^+ \setminus \Sigma^+$. Therefore, in order to prove assertion 1), it will be enough to show that if $\mathcal{R} = \{\mu_1, \ldots \mu_r\} \subset$

$\tilde{\Sigma}^+ \setminus \Sigma^+$ is such a family of strongly orthogonal roots then $\oplus_{j=1}^r \tilde{\mathfrak{g}}^{\mu_j}$ is included in a finite number of L-orbits .
Fix such a family $\mathcal{R} = \{\mu_1, \dots \mu_r\}$. If $\mu \in \mathcal{R}$, we denote by H_μ the coroot μ and by $\tilde{\mathfrak{l}}[\mu]$ the algebra generated by $\tilde{\mathfrak{g}}^{\pm\mu}$, that is $\tilde{\mathfrak{l}}[\mu] = \tilde{\mathfrak{g}}^{-\mu} \oplus FH_\mu \oplus \tilde{\mathfrak{g}}^{\mu}$. Let $L[\mu]$ the group associated to this graded algebra. As F is algebraically closed, the group $L[\mu]$ is the centralizer in $\mathrm{Aut}_e(\tilde{\mathfrak{l}}[\mu])$ of H_μ, and hence $L[\mu] \subset \mathrm{Aut}_e(\tilde{\mathfrak{g}})$.

Let us show that $L[\mu] \subset L$. Let $l \in L[\mu]$. Then $l.H_\mu = H_\mu$. If $Z \in \mathfrak{a}$ is orthogonal to μ then $[\tilde{\mathfrak{l}}[\mu], Z] = \{0\}$ and therefore $l.Z = Z$. If μ is one of the roots λ_j, we get immediately that $l \in L$. If not, there exist $i < j$ such that $\tilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(1,1)$. Then $\mu(H_\mu - 2H_{\lambda_i}) = \mu(H_\mu - 2H_{\lambda_j}) = 0$, and then $l.H_{\lambda_i} = H_{\lambda_i}$ and $l.H_{\lambda_j} = H_{\lambda_j}$, and $\mu(H_{\lambda_s}) = 0$ for $s \neq i, j$. Therefore l centralizes $\mathfrak{a}^0$, and hence $L[\mu] \subset L$.

From ([8] Theorem 1.12.4), each $L[\mu_j]$ has a finite number of orbits in $\tilde{\mathfrak{g}}^{\mu_j}$. As the roots μ_j are strongly orthogonal, the subgroups $L[\mu_j]$ commute 2 by 2. Therefore the group $L[\mu_1] \dots L[\mu_r] \subset L$ has a finite number of orbits in $\oplus_{\mu \in \mathcal{R}} \tilde{\mathfrak{g}}^{\mu}$. This proves assertion 1).

Assertion 2) will be proved by induction on k. The case $k = 0$ is obvious. Suppose that $k \geq 1$.
We suppose that assertion 2) is true for all regular graded algebras (in the sense of Definition 1.1.1) of rank k and we will prove it if $\tilde{\mathfrak{g}}$ is of rank $k+1$.

Remember the notations 2.5.2, that is

$$\underline{\tilde{\mathfrak{g}}} = \tilde{\mathfrak{g}}[\lambda_k], \quad \text{et} \quad \underline{V}^{\pm} = V[\lambda_k]^{\pm}.$$

(where $\tilde{\mathfrak{g}}[\lambda_k] = Z_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{l}}_k)$ and $\tilde{\mathfrak{l}}_k = \tilde{\mathfrak{g}}^{-\lambda_k} \oplus [\tilde{\mathfrak{g}}^{-\lambda_k}, \tilde{\mathfrak{g}}^{\lambda_k}] \oplus \tilde{\mathfrak{g}}^{\lambda_k}$).
The Lie algebra $\underline{\tilde{\mathfrak{g}}}$, which is of rank k, is graded by the element $\underline{H}_0 = \sum_{j=0}^{k-1} H_{\lambda_j}$. We will denote by underlined letters all the elements or spaces attached to $\underline{\tilde{\mathfrak{g}}}$.
As we have supposed that F is algebraically closed, we have $\mathrm{Aut}_0(\tilde{\mathfrak{g}}) = \mathrm{Aut}_e(\tilde{\mathfrak{g}})$ and hence L is the centralizer of $\mathfrak{a}^0 = \oplus_{j=0}^k FH_{\lambda_j}$ in $\mathrm{Aut}_e(\tilde{\mathfrak{g}})$. Let $\underline{l} \in \mathrm{Aut}_e(\underline{\tilde{\mathfrak{g}}})$. As $[\tilde{\mathfrak{l}}_k, \underline{\tilde{\mathfrak{g}}}] = \{0\}$, we have $\underline{l}.H_{\lambda_k} = H_{\lambda_k}$ and $\underline{l}$ acts trivially on $\tilde{\mathfrak{g}}^{\lambda_k}$.
Therefore any element of $\underline{L}$ defines an element in L which acts trivially on $\tilde{\mathfrak{g}}^{\lambda_k}$.
Remember that χ_j is the character of P attached to Δ_j $(j = 0, \dots, k)$ and that $\mathbb{X}_{V^+}(P)$ is the group of characters of P generated by the $\chi_j{}'s$.
Let $l \in \underline{L} \subset L$ and $j = 0, \dots, k-1$. By Lemma 2.5.3, we have $\underline{\chi}_j(l) = \chi_j(l)$ and $\chi_k(l) = 1$.
As $\underline{\mathfrak{n}} = \oplus_{i<j<k} E_{i,j}(1,-1) \subset \mathfrak{n}$, we have $\underline{P} \subset P$ and any element of $\underline{P}$ acts trivially on $\tilde{\mathfrak{g}}^{\lambda_k}$. As any character in $\mathbb{X}_{V^+}(P)$ is trivial on N, the mapping $\underline{\chi}_j \longrightarrow \chi_j$ $(j = 0, \dots, k-1)$ can be seen as an inclusion $\mathbb{X}_{\underline{V}^+}(\underline{P}) \hookrightarrow \mathbb{X}_{V^+}(P)$.

Let $x \in \mathcal{S}$. By Proposition 3.1.2, there exists $n_0 \in N_0$ and a family $\mathcal{R} = \{\mu_1, \dots \mu_r\}$ of stronly orthogonal roots in $\tilde{\Sigma}^+ \setminus \Sigma^+$ such that the element $X = n_0.x$ belongs to $\tilde{\mathfrak{g}}^{\mu_1} \oplus \dots \oplus \tilde{\mathfrak{g}}^{\mu_r}$. We suppose that r is minimal, that is $X = X_{\mu_1} + \dots + X_{\mu_r}$ with $X_{\mu_j} \in \tilde{\mathfrak{g}}^{\mu_j} - \{0\}$. As $n_0^{-1} P_X^0 n_0 = P_x^0$, it is enough to prove assertion 2) for X.

For $j = 1, \dots, r$, let H_{μ_j} be the coroot of μ_j and fix $X_{-\mu_j} \in \tilde{\mathfrak{g}}^{-\mu_j}$ such that $\{X_{-\mu_j}, H_{\mu_j}, X_{\mu_j}\}$ is an $\mathfrak{s}l_2$-triple. Define

$$h = H_{\mu_1} + \dots H_{\mu_r}.$$

As the roots μ_j are strongly orthogonal, if we set $Y = X_{-\mu_1} + \cdots + X_{-\mu_r} \in V^-$ and $X = X_{\mu_1} + \cdots + X_{\mu_r} \in V^+$, then $\{Y, h, X\}$ is an $\mathfrak{sl}_2$-triple. If $Z \in V^+$, then $[X, Z] = 0$ and $\mathrm{ad}(Y)^3 Z = 0$. Therefore the element Z generates a submodule of dimension at most 3 under the action of this triple. It follows that the weights of $\mathrm{ad}(h)$ in V^+ belong to $\{0, 1, 2\}$.

Remember from Definition 1.1.5, that if $\{Z^-, u, Z\}$ is an $\mathfrak{sl}_2$-triple where $Z^- \in V^-$, $u \in \mathfrak{a}$, and $Z \in V^+$, we denote by $h_Z(t)$ the element in L which acts by t^m on the space of weight m under ad u.

1st case : If $\Delta_k(X) \neq 0$ then $\lambda_k \in \mathcal{R}$. Hence $X = Z_0 + Z_2$ with $Z_0 \in \underline{V}^+$ and $Z_2 \in \tilde{\mathfrak{g}}^{\lambda_k}$. As $X \in \mathcal{S}$ and $\Delta_k(X) \neq 0$, there exists $j < k$ such that $\Delta_j(X) = 0$. By Lemma 2.5.3, we have $\Delta_k(X) = \Delta_k(Z_2) \neq 0$ and $\Delta_j(Z_0 + Z_2) = \underline{\Delta}_j(Z_0)\Delta_k(Z_2) = 0$, and hence $Z_0 \in \underline{\mathcal{S}}$. By induction, there exists $\chi \in \mathbb{X}_{\underline{V}^+}(\underline{P}) \subset \mathbb{X}_{V^+}(P)$ and $p \in \underline{P}^0_{Z_0} \subset P$ such that $\chi(p) \neq 1$. As $p \in \underline{P}$, one has $p.Z_2 = Z_2$, and hence $p \in P^0_X$. This gives the result in that case.

2nd case. Suppose $\Delta_k(X) = 0$ and $X \in \underline{V}^+$ (this is equivalent to the condition $\lambda_k(h) = 0$). Then for all $t \in F^*$, the element $l_t = h_{X_{\lambda_k}}(t) \in L$ satisfies $l_t.X = X$ and $\chi_k(l_t) = t^2$. Hence $l_t \in P^0_X$ and for $t \neq \pm 1$, we have $\chi_k(l_t) \neq 1$.

3rd case. Suppose $\Delta_k(X) = 0$ and $X \notin \underline{V}^+$ (this is equivalent to $\lambda_k \notin \mathcal{R}$ and $\lambda_k(h) = 1$ or 2).

If $\lambda_k(h) = 1$ then there exists a unique root μ in $\mathcal{R}$ such that $\lambda_k(H_\mu) = n(\lambda_k, \mu) = 1$. One can suppose that $\mu = \mu_1$ and then, for $j \geq 2$, the root μ_j is orthogonal to λ_k. As λ_k is a long root, we have $n(\mu_1, \lambda_k) = 1$. It follows that $[H_{\lambda_k}, X_{\mu_1}] = X_{\mu_1}$ and $[H_{\lambda_k}, X_{\mu_j}] = [H_{\mu_1}, X_{\mu_j}] = 0$ for $j \geq 2$. For $t \in F^*$ we define $l_t = h_{X_{\lambda_k}}(t)^2 h_{X_{-\mu_1}}(t)$. Then $l_t.X = X$ (because $h_{X_{-\mu_1}}(t).(X_{\mu_1} + \ldots + X_{\mu_r}) = t^{-2}X_{\mu_1} + X_{\mu_2} + \ldots + X_{\mu_r}$, $h_{X_{\lambda_k}}(t)(X_{\mu_1}) = tX_{\mu_1}$ and $h_{X_{\lambda_k}}(t)(X_{\mu_j}) = X_{\mu_j}$ for $j \geq 2$). We have also $l_t.X_{\lambda_k} = t^3 X_{\lambda_k}$ (because $[H_{-\mu_1}, X_{\lambda_k}] = -X_{\lambda_k}$ implies $h_{X_{-\mu_1}}(t)(X_{\lambda_k}) = t^{-1}X_{\lambda_k}$ and $h_{X_{\lambda_k}}(t)^2(X_{\lambda_k}) = t^4 X_{\lambda_k}$). Hence $\chi_k(l_t) = t^3$. As $l_t \in P^0_X$ (product of exponentials), assertion 2) is proved.

If $\lambda_k(h) = 2$, there are two possible cases. Up to a permutation of the μ_j's, we can suppose that:
- either $n(\lambda_k, \mu_1) = n(\lambda_k, \mu_2) = 1$ and $n(\lambda_k, \mu_j) = 0$ for $j \geq 3$,
- or $n(\lambda_k, \mu_1) = 2$ and $n(\lambda_k, \mu_j) = 0$ for $j \geq 2$.

If $n(\lambda_k, \mu_1) = n(\lambda_k, \mu_2) = 1$ and $n(\lambda_k, \mu_j) = 0$ for $j \geq 3$, then, as λ_k is long, one has $n(\mu_1, \lambda_k) = \mu_1(H_{\lambda_k}) = n(\mu_2, \lambda_k) = \mu_2(H_{\lambda_k}) = 1$. Therefore the element $u = 2H_{\lambda_k} - H_{\mu_1} - H_{\mu_2}$ commutes with X and for $Z \in \tilde{\mathfrak{g}}^{\lambda_k}$, one has $[u, Z] = 2Z$. Then, as before, if $t \in F^*$, the element $l_t = h_{X_{\lambda_k}}(t)^2 h_{X_{-\mu_1}}(t) h_{X_{-\mu_2}}(t) \in L$ fixes X and $\chi_k(l_t) = t^2$. Again assertion 2) is proved.

If $n(\lambda_k, \mu_1) = 2$ and $n(\lambda_k, \mu_j) = 0$ for $j \geq 2$, then we have $n(\mu_1, \lambda_k) = 1$ (from the tables in [3] we know that if $n(\lambda_k, \mu_1) = 2$ then $n(\mu_1, \lambda_k) = 1$ or 2; $n(\mu_1, \lambda_k) = 2$ would imply $\mu_1 = \lambda_k$, but $\lambda_k \notin \mathcal{R}$). As $\mu_1(H_0) = 2$, there exists a unique $j < k$ such that $\mu_1(H_{\lambda_j}) = 1$. As λ_k and λ_j have the same length, we have $n(\lambda_j, \mu_1) = 2$ ($n(\lambda_k, \mu_1) = 2$ and $n(\mu_1, \lambda_k) = 1$ imply that $||\lambda_k|| = \sqrt{2}||\mu_1|| = ||\lambda_j||$, and then $n(\mu_1, \lambda_j) = 1$ implies $n(\lambda_j, \mu_1) = 2$). As $\lambda_j(h) \in \{0, 1, 2\}$, we have $n(\lambda_j, \mu_s) = 0$ for $s \geq 2$. Hence $[H_{\lambda_k} - H_{\lambda_j}, X] = 0$. Then, for $t \in F^*$, the element $l_t = h_{X_{\lambda_k}}(t) h_{X_{-\lambda_j}}(t) = h_{X_{\lambda_k} + X_{-\lambda_j}}(t)$ is such that $l_t.X = X$ and $\chi_k(l_t) = t^2$.
This ends the proof of the Theorem.

□

3.2. **Zeta functions associated to $(\tilde{P}, V^+)$: existence of a functional equation.**

We denote by $\widehat{F^*}$ and $\widehat{\mathscr{O}_F^*}$ the groups of characters of F^* and $\mathscr{O}_F^*$ respectively. A character ω in $\widehat{\mathscr{O}_F^*}$ will be identified with a character in $\widehat{F^*}$ by setting $\omega(\pi) = 1$. Then, for any $\omega \in \widehat{F^*}$, there exists a unique $\delta \in \widehat{\mathscr{O}_F^*}$ and $s \in \mathbb{C}$ such that $\omega(x) = \delta(x)|x|^s$ for $x \in F^*$. The complex number s is only defined modulo $\left(\frac{2\pi i}{\log q}\right)\mathbb{Z}$, but its real part $\mathrm{Re}(s)$ is uniquely defined by ω. We therefore set $\mathrm{Re}(\omega) = \mathrm{Re}(s)$.

Definition 3.2.1. *Let $\omega = (\omega_0, \ldots, \omega_k) \in \widehat{F^*}^{k+1}$ and $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$. Remember that we have defined $m = \frac{\dim V^+}{\deg \Delta_0}$. As we have said we consider here the case where $\ell = 1$, and then we have $m = 1 + \frac{kd}{2}$. We set*

$$\omega^\sharp = ((\omega_0 \ldots \omega_k)^{-1}, \omega_k, \ldots, \omega_1), \quad \text{and} \quad s^\sharp = (s_0^\sharp, s_1^\sharp, \ldots, s_k^\sharp) = (-(s_0 + \ldots + s_k), s_k, \ldots, s_1)$$

and $s^\sharp - m = (s_0^\sharp - m, s_1^\sharp, \ldots, s_k^\sharp) = (-(s_0 + \ldots + s_k) - m, s_k, \ldots, s_1)$.

For $\omega = (\omega_j)_j \in (\widehat{F^*})^{k+1}$ and $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$, we define

$$(\omega, s)(\Delta(X)) = \prod_{j=0}^{k} \omega_j(\Delta_j(X)) \, |\Delta_j(X)|^{s_j}, \quad X \in V^+$$

and

$$(\omega, s)(\nabla(Y)) = \prod_{j=0}^{k} \omega_j(\nabla_j(Y)) \, |\nabla_j(Y)|^{s_j}, \quad Y \in V^-.$$

Definition 3.2.2. *Let $a \in \mathscr{S}_e^{k+1}$ and $(\omega, s) \in \widehat{F^*}^{k+1} \times \mathbb{C}^{k+1}$. The zeta functions associated to $(\tilde{P}, V^+)$ and to $(\tilde{P}, V^-)$ are the functions:*

$$\mathcal{K}_a^+(f, \omega, s) = \int_{\mathcal{O}^+(a)} f(X)(\omega, s)(\Delta(X))dX, \quad f \in \mathcal{S}(V^+),$$

and

$$\mathcal{K}_a^-(g, \omega, s) = \int_{\mathcal{O}^-(a)} g(Y)(\omega, s)(\nabla(Y))dY, \quad g \in \mathcal{S}(V^-).$$

It is clear that the integrals defining these functions are absolutely convergent for Re(s_j)+Re$(\omega_j) > 0$, $j = 0, \ldots, k$.

Theorem 3.2.3.
Let $a \in \mathscr{S}_e^{k+1}$ and $(\omega, s) \in \widehat{F^*}^{k+1} \times \mathbb{C}^{k+1}$.
1) The zeta functions $\mathcal{K}_a^+(f, \omega, s)$ for $f \in \mathcal{S}(V^+)$ and $\mathcal{K}_a^-(g, \omega, s)$ for $g \in \mathcal{S}(V^-)$ are rational functions in q^{s_j} and q^{-s_j}, and hence they admit a meromorphic continuation on $\mathbb{C}^{k+1}$.

Moreover, if $\omega_j = \tau_j \otimes |\cdot|^{\nu_j}$ with $\tau_j \in \widehat{\mathscr{O}_F^*}$ and $\nu_j \in \mathbb{C}$, there exist polynomials $R^+(\omega, s)$ and $R^-(\omega, s)$ in the variables q^{-s_j}, product of polynomials of type $(1 - q^{-N - \sum_{j=0}^k N_j(s_j + \nu_j)})$ with $N, N_j \in \mathbb{N}$, such that, for all $f \in \mathcal{S}(V^+)$ and all $g \in \mathcal{S}(V^-)$, the functions $R^+(\omega, s)\mathcal{K}_a^+(f, \omega, s)$ and $R^-(\omega, s)\mathcal{K}_a^-(g, \omega, s)$ are polynomials in q^{s_j} et q^{-s_j}.

2) For $a \in \mathscr{S}_e^{k+1}$ and $g \in \mathcal{S}(V^-)$, the zeta functions satisfy the following functional equation:

$$\mathcal{K}_a^+(\mathcal{F}g, \omega^\sharp, s^\sharp - m) = \sum_{c \in \mathscr{S}_e^{k+1}} C_{(a,c)}^k(\omega, s)\mathcal{K}_c^-(g, \omega, s)$$

where $C_{(a,c)}^k(\omega, s)$ are rational functions in q^{s_j} and q^{-s_j}.
There is a similar result for $f \in \mathcal{S}(V^+)$ and $a \in \mathscr{S}_e^{k+1}$:

$$\mathcal{K}_a^-(\mathcal{F}(f), \omega^\sharp, s^\sharp - m) = \sum_{c \in \mathscr{S}_e^{k+1}} D_{(a,c)}^k(\omega, s)\mathcal{K}_c^+(f, \omega, s).$$

(again $D_{(a,c)}^k(\omega, s) \in \mathbb{C}(q^{\pm s_0}, \dots, q^{\pm s_k})$).

Proof.
1) This result was first proved, for integrals of the same kind as $\mathcal{K}_a^+(f, \omega, s)$ and $\mathcal{K}_a^-(g, \omega, s)$, and for one variable, by J-I Igusa (see Lemma 2 in [9]) and sketched for several variables by F. Sato (see Lemma 2.1 in [15]), both proofs using a result of J. Denef ([6]). But these authors did not point out the fact that the analogue of the integers N and N_j are positive. For the convenience of the reader, we sketch briefly the proof for $\mathcal{K}_a^+(f, \omega, s)$. This is essentially Igusa's proof.
First of all we identify V^+ with F^r ($r = \dim_F V^+$). We can assume that f is the characteristic function of an open compact subset C of F^r. We choose an integer m satisfying $\tau_j^m = 1$ for $j = 0, \dots, k$. For $x = \pi^\alpha u$ ($\alpha \in \mathbb{Z}, u \in \mathcal{O}_F^*$) we set $\tau_j(x) = \tau_j(u)$. We also define the map $\Delta : F^r \longrightarrow F^{k+1}$ by $\Delta(X) = (\Delta_0(X), \dots, \Delta_k(X))$ ($X \in V^+ = F^r$).
Let $\mathcal{A}$ be a complete set of representatives of F^*/F^{*m}, where F^{*m} is the set of elements a^m for $a \in F^*$. We denote by $c = (c_0, \dots, c_k)$ the elements of $\mathcal{A}^{k+1}$.Then we obtain

$$\mathcal{K}_a^+(f, \omega, s) = \sum_{c=(c_i) \in \mathcal{A}^{k+1}} \tau_0(c_0) \dots \tau_k(c_k) \int_{D_c} |\Delta_0(X)|^{s_0+\nu_0} \dots |\Delta_k(X)|^{s_k+\nu_k} dX$$

where $D_c = \mathcal{O}^+(a) \cap C \cap \Delta^{-1}(cF^{*m})$.
It is now enough to prove that assertion 1) is true for the integral $\int_{D_c} |\Delta_0(X)|^{s_0+\nu_0} \dots |\Delta_k(X)|^{s_k+\nu_k} dX$.
Let $\theta : F \longrightarrow \mathbb{Q}_p^\alpha$ be a $\mathbb{Q}_p$-linear isomorphism. We still denote by θ its extension as a $\mathbb{Q}_p$-linear isomorphism from $V^+ = F^r$ to $\mathbb{Q}_p^{r\alpha}$. Let dY be the measure on $\mathbb{Q}_p^{r\alpha}$ which is the image of the measure dX under θ and let $E_c = \theta(D_c)$. Let also N be the norm map from F to $\mathbb{Q}_p$ and denote by $|\ |_p$ the absolute value on $\mathbb{Q}_p$. Then

$$\int_{D_c} |\Delta_0(X)|^{s_0+\nu_0} \dots |\Delta_k(X)|^{s_k+\nu_k} dX = \int_{E_c} |N(\Delta_0(\theta^{-1}(Y))|_p^{s_0+\nu_0} \dots |N(\Delta_k(\theta^{-1}(Y))|_p^{s_k+\nu_k} dY.$$

Of course the functions $P_j(X) = \Delta_j(\theta^{-1}(Y))$ are polynomials.Therefore we must consider integrals of the form

$$\int_{E_c} |P_0(Y)|^{s_0+\nu_0} \dots |P_k(Y)|^{s_k+\nu_k} dY$$

where the P_j's are polynomials.
But a result of Denef ([6], Theorem 3.2)[1] tells us that, under the condition that E_c is a so-called *boolean combination of sets of type I, II and III* ([6], p. 2-3) and is contained in a compact set, this kind of integrals are always rational functions in the variables $p^{\pm s_j}$. The second condition on

[1]This result is still true for several polynomials with the same proof (private communication of J. Denef)

the integration domain is of course true for our set E. The proof that E is a boolean combination is exactly the same as the proof of Igusa in the one variable case (see [9], p.1018-1019).
It is easily seen that the integrals under consideration are absolutely convergent Laurent series in $q^{\pm s_j}$ (for $\mathrm{Re}(s_j)+\nu_j>0$. This implies that they are rational functions in $q^{\pm s_j}$.
Finally the fact that there exist polynomials polynomials $R^+(\omega,s)$ and $R^-(\omega,s)$ in the variables q^{-s_j} having the asserted properties is a consequence of the proof of Denef's result (see [6], p.5-6).
2) As the prehomogeneous vector space (G,V^+) is regular, the relative invariant Δ_0 is non degenerate in the sense of Definition 1.2 in [15]. As Δ_0 is also a relative invariant for (P,V^+), the prehomogeneous vector space $(\tilde{P},V^+)$ satisfies the regularity condition condition $(A.1)$ of F. Sato ([15]).
By Theorem 3.1.3, the prehomogeneous vector space $(\tilde{P},V^+)$ satisfies also condition $(A.2')$ in [15], and therefore the second assertion is a consequence of Theorem $k_{\mathfrak{p}}$ in *loc. cit* (§2.4).

□

3.3. **Explicit functional equation.**

We compute first the factors $D^k_{(a,c)}(\omega,s)$ appearing in Theorem 3.2.3 in the case where $k=0$ (rank 1). In that case, V^+ and V^- are isomorphic to F, and F^* is the unique open orbit of the group $G=P$ in V^+, and $\tilde{P}$ has $|\mathscr{S}_e|$ open orbits in V^+ and in V^-.

We will identify the spaces $V^\pm$ with F. Then the zeta functions $\mathcal{K}^\pm_a(f,\delta,s)$ appearing in Definition 3.2.2 are equal and we will omit the exponents $\pm$ in the notations below.
If dt is a (additive) Haar measure on F, $d^*t=\dfrac{dt}{|t|}$ is a Haar measure on F^*.
Tate's zeta function (see [16]) is then defined, for $f\in\mathcal{S}(F)$, $s\in\mathbb{C}$ and $\delta\in\widehat{F^*}$, by

$$Z(f,\delta,s)=\int_{F^*}f(t)\delta(t)|t|^s d^*t.$$

This integral is absolutely convergent for $s\in\mathbb{C}$ such that $\mathrm{Re}(s)+\mathrm{Re}(\delta)-1>0$ and $Z(f,\delta,s)$ extends to a meromorphic function on $\mathbb{C}$ which satisfies the following functional equation:

$$Z(f,\delta,s)=\rho(\delta,s)Z(\mathcal{F}(f),\delta^{-1},1-s),$$

where $\rho(\delta,s)$ is the so-called ρ factor of Tate (see [16]).
In the notation of Definition 3.2.2 we have for $f\in\mathcal{S}(F)$:

$$\mathcal{K}(f,\delta,s)=\int_F f(t)|t|^s\delta(t)dt=Z(f,\delta,s+1).$$

Therefore Tate's functional equation can be re-written as follows:

$$\mathcal{K}(f,\delta,s)=\rho(\delta,s+1)\mathcal{K}(\mathcal{F}(f),\delta^{-1},-s-1)\ \text{ and }\ \mathcal{K}(\mathcal{F}(f),\delta,s)=\delta(-1)\rho(\delta,s+1)\mathcal{K}(f,\delta^{-1},-s-1).$$

Let $\widehat{\mathscr{S}_e}$ be the group of characters of $\mathscr{S}_e$. Any element $\chi\in\widehat{\mathscr{S}_e}$ extends uniquely to a character of F^* which is trivial on S_e. We still denote by χ this extension.

Definition 3.3.1. *Let $\delta\in\widehat{F^*}$ and $s\in\mathbb{C}$. For $x\in\mathscr{S}_e$, we define*

$$\tilde{\rho}(\delta,s;x)=\frac{1}{|\mathscr{S}_e|}\sum_{\chi\in\widehat{\mathscr{S}_e}}\chi(x)\rho(\delta\chi,s).$$

As a function of x, $\tilde{\rho}(\delta, s; x)$ is the Fourier transform of the function $\chi \in \widehat{\mathscr{S}_e} \mapsto \rho(\delta\chi, s)$. Therefore $\rho(\delta\chi, s) = \sum_{x\in\mathscr{S}_e} \chi(x)\tilde{\rho}(\delta, s; x)$, for $\chi \in \widehat{\mathscr{S}_e}$.

Lemma 3.3.2. *(compare with* [12] *Corollaire 3.6.3).*
Let $\delta \in \widehat{F^*}$ and $a \in \mathscr{S}_e$. The functions $\mathcal{K}_a(f, \delta, s)$ satisfy the following functional equations:

$$\mathcal{K}_a(f, \delta, s) = \sum_{c\in\mathscr{S}_e} \tilde{\rho}(\delta, s+1; ac)\mathcal{K}_c(\mathcal{F}(f), \delta^{-1}, -s-1), \quad f \in \mathcal{S}(F),$$

and

$$\mathcal{K}_a(\mathcal{F}(g), \delta, s) = \sum_{c\in\mathscr{S}_e} \delta(-1)\tilde{\rho}(\delta, s+1; -ac)\mathcal{K}_c(g, \delta^{-1}, -s-1) \quad g \in \mathcal{S}(F).$$

Proof. :
From the definition $\mathcal{K}(f, \delta\chi, s) = \sum_{a\in\mathscr{S}_e} \chi(a)\mathcal{K}_a(f, \delta, s)$, By Fourier inversion on the group $\mathscr{S}_e$, we obtain:

$$\mathcal{K}_a(f, \delta, s) = \frac{1}{|\mathscr{S}_e|} \sum_{\chi\in\widehat{\mathscr{S}_e}} \chi(a)\mathcal{K}(f, \delta\chi, s), \quad a \in \mathscr{S}_e.$$

From Tate's functional equation, we obtain

$$\mathcal{K}_a(f, \delta, s) = \frac{1}{|\mathscr{S}_e|} \sum_{\chi\in\widehat{\mathscr{S}_e}} \chi(a)\rho(\delta\chi, s+1)\mathcal{K}(\mathcal{F}(f), \delta^{-1}\chi, -s-1))$$

$$= \frac{1}{|\mathscr{S}_e|} \sum_{c\in\mathscr{S}_e} \sum_{\chi\in\widehat{\mathscr{S}_e}} \chi(ac)\rho(\delta\chi, s+1)\mathcal{K}_c(\mathcal{F}(f), \delta^{-1}, -s-1))$$

$$= \sum_{c\in\mathscr{S}_e} \tilde{\rho}(\delta, s+1; ac)\mathcal{K}_c(\mathcal{F}(f), \delta^{-1}, -s-1)).$$

This is the first assertion.
For the second assertion we have similarly:

$$\mathcal{K}_a(\mathcal{F}(g), \delta, s) = \frac{1}{|\mathscr{S}_e|} \sum_{\chi\in\widehat{\mathscr{S}_e}} \chi(a)K(\mathcal{F}(g), \delta\chi, s)$$

$$= \frac{1}{|\mathscr{S}_e|} \sum_{\chi\in\widehat{\mathscr{S}_e}} \chi(a)\rho(\delta\chi, s+1)(\delta^{-1}\chi)(-1)\mathcal{K}(g, \delta^{-1}\chi, -s-1)$$

$$= \frac{1}{|\mathscr{S}_e|} \sum_{c\in\mathscr{S}_e} \sum_{\chi\in\widehat{\mathscr{S}_e}} \chi(-ac)\rho(\delta\chi, s+1)\delta^{-1}(-1)\mathcal{K}_c(g, \delta^{-1}, -s-1)$$

$$= \sum_{c\in\mathscr{S}_e} \delta(-1)\tilde{\rho}(\delta, s+1; -ac)\mathcal{K}_c(g, \delta^{-1}, -s-1).$$

□

If $x = (x_0, \ldots, x_k)$ is a $k+1$-tuple of elements (taken in any set), we will set $\underline{x} = (x_0, \ldots, x_{k-1})$.

Theorem 3.3.3.
Let $k \geq 0$. Let $\omega \in \widehat{F^*}^{k+1}$, $s \in \mathbb{C}^{k+1}$ and $f \in \mathcal{S}(V^+)$. For $a \in \mathscr{S}_e^{k+1}$, we have

$$\mathcal{K}_a^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m) = \sum_{c \in \mathscr{S}_e^{k+1}} D^k_{(a,c)}(\omega, s)\mathcal{K}_c^+(f, \omega, s),$$

where

$$D^0_{(a_0,c_0)}(\omega_0, s_0) = \omega_0(-1)\tilde{\rho}(\omega_0^{-1}, -s_0; -a_0c_0)$$

and where for $k \geq 1$:

$$\begin{aligned} &D^k_{(a,c)}(\omega, s) \\ &= \gamma_k(\underline{a}, c_k))(\omega_0 \dots \omega_k)(-1)\tilde{\rho}((\omega_0 \dots \omega_k)^{-1}, -(s_0 + \dots + s_k + \frac{kd}{2}); -a_kc_k)D^{k-1}_{(\underline{a},\underline{c})}(\underline{\omega}, \underline{s}) \\ &= \prod_{j=1}^{k} \gamma_j((a_0, \dots, a_{j-1}), c_j) \prod_{j=0}^{k}(\omega_0 \dots \omega_j)(-1)\tilde{\rho}((\omega_0 \dots \omega_j)^{-1}, -(s_0 + \dots + s_j + \frac{jd}{2}); -a_jc_j), \end{aligned}$$

where the constant $\gamma_j((a_0, \dots, a_{j-1}), c_j)$ is the one given in Proposition 2.5.5 (depending on e).

Proof.
If $k = 0$, the result is just Lemma 3.3.2.

The proof is by induction on k. Let us suppose that $k \geq 1$. From Theorem 3.2.3, the constants $D^k_{(a,c)}(\omega, s)$ do not depend on the function $f \in \mathcal{S}(V^+)$. Therefore to compute these constants we can take $f \in \mathcal{S}(\mathcal{O}^+)$ and suppose that $\mathrm{Re}((s^\sharp - m)_j) + \mathrm{Re}(\omega_j^\sharp) > 0$. Then the functions $\mathcal{K}_a^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m)$ and $\mathcal{K}_c^+(f, \omega, s)$ are defined by the converging integrals given in Definition 3.2.2.

Notations will be as in 2.5.2. From the integration formula in Theorem 2.3.5, we obtain

$$\begin{aligned} &\mathcal{K}_a^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m) \\ &= \int_{u' \in \underline{V}^-} \int_{v' \in \tilde{\mathfrak{g}}^{-\lambda_k}} T^{k-1,-}_{\mathcal{F}f}(u', v')\mathbf{1}_{\mathcal{O}^-(a)}(u' + v')\omega^\sharp(\nabla(u' + v'))\ |\nabla(u' + v')|^{s^\sharp - m}\ |\nabla_1(u')|^{\frac{d}{2}} du'dv' \end{aligned}$$

and then from Corollary 2.5.4, we get

$$\begin{aligned} &\mathcal{K}_a^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m) \\ &= \int_{u' \in \underline{\mathcal{O}}^-(a_0, \dots, a_{k-1})} \int_{v' \in \tilde{\mathfrak{g}}^{-\lambda_k}; \tilde{\nabla}_k(v')a_k \in S_e} T^{k-1,-}_{\mathcal{F}f}(u', v')\omega^\sharp(\nabla(u'+v'))\ |\nabla(u'+v')|^{s^\sharp - m}\ |\nabla_1(u')|^{\frac{d}{2}} du'dv'. \end{aligned}$$

From Lemma 2.5.3 we have

$$\nabla_j((u' + v')) = \begin{cases} \underline{\nabla}_0(u')\tilde{\nabla}_k(v') & \text{for} \quad j = 0 \\ \underline{\nabla}_{j-1}(u') & \text{for} \quad j = 1, \dots, k \end{cases}$$

Define $(\underline{\omega}, \underline{s}) = ((\omega_0, \dots, \omega_{k-1}), (s_0, \dots, s_{k-1}))$ and $[\omega] = \prod_{j=0}^k \omega_j$. We also note that $m = 1 + \frac{kd}{2} = \underline{m} + \frac{d}{2}$. Then:

$$|\nabla(u' + v')|^{s^\sharp - m}\ |\nabla_1(u')|^{\frac{d}{2}} = |\underline{\nabla}_0(u')\tilde{\nabla}_k(v')|^{s_0^\sharp - m} \prod_{j=1}^{k} |\underline{\nabla}_{j-1}(u')|^{s_j^\sharp}\ |\nabla_1(u')|^{\frac{d}{2}}$$

$$= |\underline{\nabla}_0(u')\tilde{\nabla}_k(v')|^{-(s_0 + \dots + s_k) - (1 + \frac{kd}{2})} |\underline{\nabla}_0(u')|^{s_k} |\underline{\nabla}_1(u')|^{s_{k-1}} \dots |\underline{\nabla}_{k-1}(u')|^{s_1}\ |\underline{\nabla}_0(u')|^{\frac{d}{2}}$$

$$= |\underline{\nabla}(u')|^{\underline{s}^\sharp - \underline{m}} |\tilde{\nabla}_k(v')|^{-(s_0+\ldots+s_k+\frac{kd}{2})-1},$$

and similarly we have

$$\omega^\sharp(\nabla(u'+v')) = \underline{\omega}^\sharp(\underline{\nabla}(u'))\big([\omega](\tilde{\nabla}_k(v'))\big)^{-1}.$$

Therefore

$$\begin{aligned}
&\mathcal{K}_a^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m) \\
&= \int_{u' \in \underline{\mathcal{O}}^-(a_0,\ldots,a_{k-1})} \int_{v' \in \tilde{\mathfrak{g}}^{-\lambda_k}; \tilde{\nabla}_k(v') a_k \in S_e} T_{\mathcal{F}f}^{k-1,-}(u',v') \underline{\omega}^\sharp(\underline{\nabla}(u'))\big([\omega](\tilde{\nabla}_k(v'))\big)^{-1} \\
&\qquad \times |\underline{\nabla}(u')|^{\underline{s}^\sharp - \underline{m}} |\tilde{\nabla}_k(v')|^{-(s_0+\ldots+s_k+\frac{kd}{2})-1} du' dv'
\end{aligned}$$

For $u' \in \underline{V}^-$, define the function

$$G_{u'}(v) = \gamma_k(u',v)(\mathcal{F}_u T_f^{k-1,+})(u',v), \quad v \in \tilde{\mathfrak{g}}^{\lambda_k}.$$

Theorem 2.4.4, says that

$$\mathcal{F}_v(G_{u'})(v') = T_{\mathcal{F}f}^{k-1,-}(u',v').$$

The maps $v \in \tilde{\mathfrak{g}}^{\lambda_k} \mapsto \Delta_k(v)$ and $v' \in \tilde{\mathfrak{g}}^{-\lambda_k} \mapsto \tilde{\nabla}_k(v')$ are isomorphisms from $\tilde{\mathfrak{g}}^{\lambda_k}$ and $\tilde{\mathfrak{g}}^{-\lambda_k}$ respectively onto F. Therefore the functional equation obtained in Lemma 3.3.2 for $k=0$ implies

$$\begin{aligned}
&\int_{v' \in \tilde{\mathfrak{g}}^{-\lambda_k}; \tilde{\nabla}_k(v') a_k \in S_e} T_{\mathcal{F}f}^{k-1,-}(u',v')\big([\omega](\tilde{\nabla}_k(v'))\big)^{-1} |\tilde{\nabla}_k(v')|^{-(s_0+\ldots+s_k+\frac{kd}{2})-1} dv' \\
&= \sum_{c_k \in \mathscr{S}_e} [\omega](-1)\tilde{\rho}([\omega]^{-1}, -(s_0+\ldots+s_k+\frac{kd}{2}); -a_k c_k) \\
&\qquad \times \int_{v \in \tilde{\mathfrak{g}}^{\lambda_k}; \Delta_k(v) c_k \in S_e} \gamma_k(u',v)(\mathcal{F}_u T_f^{k-1,+})(u',v)[\omega](\Delta_k(v))|v|^{s_0+\ldots+s_k+\frac{kd}{2}} dv.
\end{aligned}$$

We know from Proposition 2.5.5, that for $u' \in \underline{\mathcal{O}}^-(\underline{a})$ and $v \in \tilde{\mathfrak{g}}^{\lambda_k}$ with $\Delta_k(v)c_k \in S_e$, one has $\gamma(u',v) = \gamma_k(\underline{a}, c_k)$. Hence

$$\begin{aligned}
&\mathcal{K}_a^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m) \\
&= \sum_{c_k \in \mathscr{S}_e} [\omega](-1)\tilde{\rho}([\omega]^{-1}, -(s_0+\ldots+s_k+\frac{kd}{2}); -a_k c_k)\gamma_k(\underline{a}, c_k) \\
&\quad \times \int_{v \in \tilde{\mathfrak{g}}^{\lambda_k}; \Delta_k(v) c_k \in S_e} [\omega](\Delta_k(v))|v|^{s_0+\ldots+s_k+\frac{kd}{2}} \int_{u' \in \underline{\mathcal{O}}^-(\underline{a})} (\mathcal{F}_u T_f^{k-1,+})(u',v)\underline{\omega}^\sharp(\underline{\nabla}(u'))|\underline{\nabla}(u')|^{\underline{s}^\sharp - \underline{m}} du' dv.
\end{aligned}$$

The induction hypothesis applied to the function $u \mapsto T_f^{k-1,+}(u,v)$, gives the following equation:

$$\begin{aligned}
&\int_{u' \in \underline{\mathcal{O}}^-(\underline{a})} (\mathcal{F}_u T_f^{k-1,+})(u',v)\underline{\omega}^\sharp(\underline{\nabla}(u'))|\underline{\nabla}(u')|^{\underline{s}^\sharp - \underline{m}} du' \\
&= \sum_{\underline{c}=(c_0,\ldots c_{k-1}) \in \mathscr{S}_e^k} D_{(\underline{a},\underline{c})}^{k-1}(\underline{\omega}, \underline{s}) \int_{u \in \underline{\mathcal{O}}^+(\underline{c})} T_f^{k-1,+}(u,v)\underline{\omega}(\underline{\Delta})(u)|\underline{\Delta}(u)|^{\underline{s}} du.
\end{aligned}$$

The preceding equation becomes now:

$$\begin{aligned}&\mathcal{K}_a^-(\mathcal{F}f,\omega^\sharp,s^\sharp-m)\\&=\sum_{c\in\mathscr{S}_e^{k+1}}[\omega](-1)\tilde{\rho}([\omega]^{-1},-(s_0+\ldots+s_k+\frac{kd}{2});-a_kc_k)\gamma_k(\underline{a},c_k)D^{k-1}_{(\underline{a},\underline{c})}(\underline{\omega},\underline{s})\\&\quad\times\int_{v\in\tilde{\mathfrak{g}}^{\lambda_k};\Delta_k(v)c_k\in S_e}\int_{u\in\underline{\mathcal{O}}^+(\underline{c})}T_f^{k-1,+}(u,v)\underline{\omega}(\underline{\Delta})(u)|\underline{\Delta}(u)|^{\underline{s}}[\omega](\Delta_k(v))|v|^{s_0+\ldots+s_k+\frac{kd}{2}}dvdu.\end{aligned}$$

From Lemma 2.5.3 we have:

$$\Delta_j(u+v)=\begin{cases}\underline{\Delta}_j(u)\Delta_k(v) & \text{for} \quad j=0,\ldots,k-1\\ \Delta_k(v) & \text{for} \qquad j=k\end{cases},$$

which implies

$$\omega(\Delta)(u+v)=\prod_{j=0}^{k}\omega_j(\Delta_j(u+v))=\prod_{j=0}^{k-1}\omega_j(\underline{\Delta}_j(u))[\omega](\Delta_k(v))=\underline{\omega}(\underline{\Delta})(u)[\omega](\Delta_k(v))$$

and

$$|\Delta(u+v)|^s=\prod_{j=0}^{k}|\Delta_j(u+v)|^{s_j}=\prod_{j=0}^{k-1}|\underline{\Delta}_j(u)|^{s_j}|\Delta_k(v)|^{s_0+\ldots+s_k}.$$

Moreover from Corollary 2.5.4 we have

$$u\in\underline{\mathcal{O}}^+(\underline{c}),\ \text{and}\ \Delta_k(v)c_k\in S_e\Longleftrightarrow u+v\in\mathcal{O}^+(c).$$

Then, if we define

$$D^k_{(a,c)}(\omega,s)=\gamma_k(\underline{a},c_k)[\omega](-1)\tilde{\rho}([\omega]^{-1},-(s_0+\ldots+s_k+\frac{kd}{2});-a_kc_k)D^{k-1}_{(\underline{a},\underline{c})}(\underline{\omega},\underline{s})$$

(which is the announced relation between $D^k_{(a,c)}(\omega,s)$ and $D^{k-1}_{(\underline{a},\underline{c})}(\underline{\omega},\underline{s})$),
we obtain finally:

$$\begin{aligned}&\mathcal{K}_a^-(\mathcal{F}f,\omega^\sharp,s^\sharp-m)\\&=\sum_{c\in\mathscr{S}_e^{k+1}}D^k_{(a,c)}(\omega,s)\int_{u\in\underline{V}^+}\int_{v\in\tilde{\mathfrak{g}}^{\lambda_k}}\mathbf{1}_{\mathcal{O}^+(c)}(u+v)T_f^{k-1,+}(u,v)\omega(\Delta)(u+v)|\Delta(u+v)|^s|\Delta_k(v)|^{\frac{kd}{2}}dudv\\&=\sum_{c\in\mathscr{S}_e^{k+1}}D^k_{(a,c)}(\omega,s)\mathcal{K}_c^+(f,\omega,s).\end{aligned}$$

As by Lemma 3.3.2 , we have $D^0_{(a_0,c_0)}(\omega_0,s_0)=\omega_0(-1)\tilde{\rho}(\omega_0,-s_0;-c_0a_0)$, the explicit computation of $D^k_{(a,c)}(\omega,s)$ follows easily by induction.

□

In the case where e is even, the explicit value of $\gamma_k(\underline{a},c_k)$ allows to obtain a more simple expression for $D^k_{(a,c)}(\omega,s)$.

Corollary 3.3.4.
Let $\omega\in\widehat{F^*}^{k+1}$ *and* $s\in\mathbb{C}^{k+1}$.

1. *If $e = 0$ or 4, then $\mathscr{S}_e = \{1\}$ and we omit the dependance on the variables a, c. For $f \in \mathcal{S}(V^+)$, we have*

$$\mathcal{K}^-(\mathcal{F}f, \omega^\sharp, s^\sharp - m) = D^k(\omega, s)\mathcal{K}^+(f, \omega, s),$$

where

$$D^k(\omega, s) = \gamma_\psi(q)^{\frac{k(k+1)}{2}} \prod_{j=0}^{k} (\omega_0 \dots \omega_j)(-1)\rho\big((\omega_0 \dots \omega_j)^{-1}, -(s_0 + \dots + s_j + \frac{jd}{2})\big).$$

2. *If $e = 2$, remember that $S_e = N_{E/F}(E^*)$ where E is a quadratic extension of F with ϖ_E as associated quadratic character . Let $a = (a_0, \dots, a_k)$ and $c = (c_0, \dots, c_k)$ be elements of $\mathscr{S}_e^{k+1}$. Then*

$$D^k_{(a,c)}(\omega, s)$$
$$= \gamma_\psi(q_e)^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \varpi_E(c_j)^j \varpi_E(a_0 \dots a_{j-1}) \prod_{j=0}^{k} (\omega_0 \dots \omega_j)(-1)\tilde{\rho}((\omega_0 \dots \omega_j)^{-1}, -(s_0 + \dots + s_j + \frac{jd}{2}); -a_j c_j).$$

Proof.
If $e = 0$ or 4 then $\mathscr{S}_e = \{1\}$. In that case, for all $x \in \mathscr{S}_e$, $z \in \mathbb{C}$ and $\delta \in \widehat{F^*}$, one has $\tilde{\rho}(\delta, z; x) = \rho(\delta, z)$. The corollary is the an easy consequence of Proposition 2.5.5 and of Theorem 3.3.3.

□

4. Zeta functions associated to the H-distinguished minimal principal series

4.1. Characters of the group L which are trivial on $L \cap \mathcal{H}_p$.

Let $(\Omega_p^\pm)$, $p \in \{1, \dots, r_0\}$ be the open G-orbits in $V^\pm$ ([8] Theorem 3.6.3), The integer r_0 depends on k and on e, and $r_0 \leq 5$ (see([8], Theorem 3.6.3).
For $1 \leq p \leq r_0$, we fix an element I_p^+ in Ω_p^+ which is in the "diagonal" $\oplus_{j=0}^k \tilde{\mathfrak{g}}^{\lambda_j}$. We suppose moreover that $I_1^+ = I^+$. We set $I_p^- = \iota(I_p^+) \in \oplus_{j=1}^k \tilde{\mathfrak{g}}^{-\lambda_j}$ and hence $\{I_p^-, H_0, I_p^+\}$ is an $\mathfrak{s}l_2$-triple.
The non trivial element of the Weyl group associated to this $\mathfrak{s}l_2$-triple acts on $\tilde{\mathfrak{g}}$ by the element $w_p \in Aut_0(\tilde{\mathfrak{g}})$ given by $w_p = e^{\mathrm{ad}I_p^+} e^{\mathrm{ad}I_p^-} e^{\mathrm{ad}I_p^+}$. Let σ_p be the automorphism of $\mathrm{Aut}_0(\tilde{\mathfrak{g}})$ defined by $\sigma_p(g) = w_p g w_p^{-1}$ (for $g \in Aut_0(\tilde{\mathfrak{g}})$).

We know that σ_p is an involution of G ([8] Theorem 4.1.1). Moreover the group $\mathcal{H}_p = Z_G(I_p^+) = Z_G(I_p^-)$ is an open subgroup of the group of fixed points of σ_p. Therefore $\Omega_p^\pm$ is isomorphic to the symmetric space $G/\mathcal{H}_p$.

We recall also that we denote by P the parabolic subgroup defined by $P = LN$ where $L = Z_G(\mathfrak{a}^0)$, $N = \exp \mathrm{ad}\mathfrak{n}$, and $\mathfrak{n} = \oplus_{0 \leq i < j \leq k} E_{i,j}(1, -1)$ (cf. 1.2).
The key point is that the group P is a minimal σ_p-split parabolic subgroup for $p = 1, \dots, r_0$ (see [8] §4.2) (this means that $\sigma_p(P)$ is the parabolic subgroup opposite to P, and P is minimal for this property). The subgroup L is then the unique Levi subgroup of P which is stable under σ_p.

As $\ell = 1$, the group L acts by a character $x_j(\cdot)$ on each $\tilde{\mathfrak{g}}^{\lambda_j}$. Thus, for $p \in \{1, \dots, r_0\}$, as I_p^+ belongs to the "diagonal", the group $L \cap \mathcal{H}_p$ is the group of elements $l \in L$ such that $x_j(l) = 1$ for $j = 0, \dots, k$,

hence $L \cap \mathcal{H}_p$ does not depend on p. We set

$$L_H = \{l \in L; x_j(l) = 1, \text{ for } j = 0, \ldots, k\} = L \cap \mathcal{H}_p, \ p \in \{1, \ldots, r_0\}. \tag{4.1}$$

Let A_L the maximal split torus of the center of L. We denote by $\mathbb{X}(L)$ (resp. $\mathbb{X}(A_L)$) the group of rational characters of L (resp. de A_L) which are defined over F. The real vector space a_L is defined by $a_L = \text{Hom}_{\mathbb{Z}}(\mathbb{X}(L), \mathbb{R})$. The restriction map from L to A_L induces an injection from $\mathbb{X}(L)$ on $\mathbb{X}(A_L)$, and these 2 lattices have the same rank, which is equal to the dimension of $\mathfrak{a}_L$ (see [14] V.2.6). Then, the dual space a_L^* satisfies $a_L^* \simeq \mathbb{X}(L) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{X}(A_L) \otimes_{\mathbb{Z}} \mathbb{R}$ and each root $\alpha \in \Sigma$ induces a linear form of a_L^*, which we also denote by α.

The canonical map $H_L : L \to a_L$ is then defined by

$$e^{\langle H_L(l), \chi \rangle} = |\chi(l)|, \quad l \in L, \chi \in \mathbb{X}(L).$$

The group of non ramified characters of L is then defined by:

$$X(L) = \text{Hom}(L/\text{Ker}H_L, \mathbb{C}^*),$$

The map $\nu \longmapsto \chi_\nu$ from $(a_L^*)_{\mathbb{C}}$ to $X(L)$ defined by $\chi_\nu : l \mapsto e^{\langle \nu, H_L(l) \rangle}$ (or equivalently the map $\chi \otimes s \in \mathbb{X}(L) \otimes_{\mathbb{Z}} \mathbb{C} \longmapsto (l \mapsto |\chi(l)|^s)$) is surjective and its kernel is a lattice in $(a_L^*)_{\mathbb{C}}$. This defines a structure of algebraic variety on $X(L)$. More precisely $X(L)$ is a complex torus whose Lie algebra is $(a_L^*)_{\mathbb{C}}$.
Let $p \in \{1, \ldots, r_0\}$. The involution σ_p on L induces an involution on a_L and a_L^*, which we denote by the same letter. From the proof of Lemma 4.1.5 in [8], the Lie algebra of A_L decomposes as $\mathfrak{a}_L = \mathfrak{a}_L \cap \mathfrak{h}_p \oplus \mathfrak{a}^0$ where $\mathfrak{h}_p$ is the Lie algebra of $\mathcal{H}_p$. Since the groups $L \cap \mathcal{H}_p, \ p = 1, \ldots, r_0$ are equal (see (4.1)), this decomposition does not depend on σ_p. Hence, the induced involution on a_L does not depend on p. We set $a_L = a_L^\sigma \oplus a^0$ and $a_L^* = (a_L^\sigma)^* \oplus a^{0*}$ the decompositions of a_L and a_L^* in invariant and anti-invariant spaces under the action of σ_p for all p. By definition of $\mathfrak{a}^0$, a basis of $(a^{0,*})_{\mathbb{C}}$ is given by $\lambda_0, \ldots, \lambda_k$.

Let $X(L)_\sigma$ be the image of $(a^{0*})_{\mathbb{C}}$ by the map $\nu \mapsto \chi_\nu$. Then, the torus $X(L)_\sigma$ is the group of non ramified characters given by

$$l \mapsto \prod_{j=0}^{k} |x_j(l)|^{\mu_j} \text{ for } \mu = \mu_1\lambda_1 + \ldots + \mu_k\lambda_k \in (a^{0*})_{\mathbb{C}}.$$

Let δ_P be the modular function of P, which is given by $\delta_P(ln) = e^{2\langle \rho_P, H_L(l) \rangle}$. Here ρ_P is half the sum of the roots of A_L in N (counted with multiplicity).

Lemma 4.1.1. *We have*

$$2\rho_P = \frac{d}{2} \sum_{j=0}^{k} (k - 2j)\lambda_j.$$

This implies that for $l \in L$,

$$\delta_P(l)^{1/2} = \prod_{j=0}^{l} |x_j(l)|^{\rho_j}, \quad \text{where } \rho_j = \frac{d}{4}(k - 2j).$$

Proof. The Lie algebra of P decomposes as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ with $\mathfrak{l} = Z_{\mathfrak{g}}(\mathfrak{a}^0)$ and $\mathfrak{n} = \oplus_{i<j} E_{i,j}(1,-1)$. Let $S_{i,j}$ be the set of roots μ such that $\mu \neq (\lambda_i - \lambda_j)/2$ and $\tilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(1,-1)$. Let $m_\mu = \dim \tilde{\mathfrak{g}}^{\mu}$ be the multiplicity of μ. We know from ([8] Remark 3.3.3), that if $\mu \in S_{i,j}$ then $\mu' = \lambda_i - \lambda_j - \mu$ is a root in $S_{i,j}$ which is not equal to μ but has the same multiplicity, and of course $\mu + \mu' = \lambda_i - \lambda_j$. As $\dim(\tilde{\mathfrak{g}}^{(\lambda_i-\lambda_j)/2}) = e$, and $d = \dim\, E_{i,j}(1,-1) = e + \sum_{\mu \in S_{i,j}} m_\mu$, we obtain

$$\sum_{\mu \in S_{i,j}} m_\mu \mu = \frac{1}{2} \sum_{\mu \in S_{i,j}} m_\mu (\lambda_i - \lambda_j) = \frac{d-e}{2}(\lambda_i - \lambda_j).$$

Therefore

$$2\rho_P = e \sum_{i<j} \frac{(\lambda_i - \lambda_j)}{2} + \frac{d-e}{2} \sum_{i<j} (\lambda_i - \lambda_j) = \frac{d}{2} \sum_{i<j} (\lambda_i - \lambda_j).$$

A simple computation gives then the first assertion. The second assertion is then an immediate consequence of the first one.

□

From Remark 1.2.2 we know now that the map $\mathcal{T} : l \mapsto (x_0(l), \dots, x_k(l))$ induces an isomorphism of groups from $L/L \cap \mathcal{H}_p = L/L_H$ onto $\cup_{x \in \mathscr{S}_e} x S_e^{k+1}$.

Hence, any character of F^{*k+1} induces, by restriction to $\mathrm{Im}\mathcal{T}$, a character of L trivial on $L \cap \mathcal{H}_p$ for any $p \in \{1, \dots, N\}$. If $\delta = (\delta_0, \dots, \delta_k)$ is a character of $(F^*)^{k+1}$ the corresponding character of L, which is invariant on $L/L \cap \mathcal{H}_p = L/L_H$, is still denoted by δ and is given by

$$\delta(l) = \prod_{i=0}^{k} \delta_i(x_j(l)).$$

This is made more precise in the following Lemma.

Lemma 4.1.2.

1. *Any character of* $\mathrm{Im}\mathcal{T} = \cup_{x \mathscr{S}_e} x S_e^{k+1}$ *is the restriction to* $\mathrm{Im}\mathcal{T}$ *of a (non unique) character of* $(F^*)^{k+1}$.

2. *Two characters* $\delta = (\delta_0, \dots, \delta_k)$ *and* $\delta' = (\delta'_0, \dots, \delta'_k)$ *of* $(F^*)^{k+1}$ *coincide on* $\mathrm{Im}\mathcal{T} = \cup_{x \in \mathscr{S}_e} x S_e^{k+1}$, *if and only if for all* $j = 0 \dots k$, *there exists a character* χ^j *of* F^* *with values in* $\{\pm 1\}$, *and trivial on* S_e *such that*

$$\delta_j(x) = \chi^j(x)\delta'_j(x), \quad \text{et} \quad \prod_{j=0}^{k} \chi^j(x) = 1, \quad x \in F^*.$$

Proof. Note first that as $F^{*2} \subset S_e$, the group $(F^*)^{k+1}/\mathrm{Im}\mathcal{T}$ is a finite abelian group whose non trivial elements are of order 2.

Let $\tilde{\delta}$ be a character of $\mathrm{Im}\mathcal{T}$ and take $x \in (F^*)^{k+1} \setminus \mathrm{Im}\mathcal{T}$. Then, as $F^{*2} \subset \mathrm{Im}\mathcal{T}$, we have $x^2 \in \mathrm{Im}\mathcal{T}$. Let ϖ_x be a square root of $\tilde{\delta}(x^2)$, and let $\mathcal{M}_1$ be the subgroup of $(F^*)^{k+1}$ generated by $\mathrm{Im}\mathcal{T}$ and x. One can now define a character of $\mathcal{M}_1$ by setting $\tilde{\delta}_1(ax) = \tilde{\delta}(a)\varpi_x$ for $a \in \mathrm{Im}\mathcal{T}$. As $(F^*)^{k+1}/\mathrm{Im}\mathcal{T}$ is finite, we obtain the first assertion by induction.

As $\mathscr{S}_e = F^*/S_e$ and $F^{*2} \subset S_e$, the second assertion is clear.

□

4.2. **The minimal principal H-distinguished series.**
Let $p \in \{1, \dots r_0\}$. A smooth representation (π, V_π) of G is said to be $\mathcal{H}_p$-distinguished if the space $(V_\pi^*)^{\mathcal{H}_p}$ of $\mathcal{H}_p$-invariant linear forms on V_π is non-trivial. This is also equivalent to the condition $\mathrm{Hom}_{\mathcal{H}_p}(V_\pi, C^\infty(G/\mathcal{H}_p)) \neq \{0\}$ (where $C^\infty(G/\mathcal{H}_p)$ is the space of locally constant functions on $G/\mathcal{H}_p$).

If χ is a character of P we define as usual the space I_χ to be the space of functions $v : G \to \mathbb{C}$ which are right invariant under a compact open subgroup of G and which satisfy the condition:

$$v(nlg) = \delta_P(l)^{1/2}\chi(l)v(g), \quad n \in N, l \in L, g \in G.$$

The induced representation $I_P^G(\chi)$ is the right regular representation of G on I_χ.
One knows from ([1] Théorème 2.8), that if there exists an open P- orbit $P\gamma\mathcal{H}_p$ in $G/\mathcal{H}_p$ (or equivalently an open orbit $P\gamma I_p^+$ in Ω_p^+) with $\gamma \in G$, such that χ is trivial on $L \cap \gamma\mathcal{H}_p\gamma^{-1}$, then (under some additional conditions, see below), the representation $I_P^G(\chi)$ is $\mathcal{H}_p$-distinguished.
But as any open P-orbit in Ω_p meets the diagonal we can suppose that $\gamma.I_p^+ \in \oplus_{j=0}^k \tilde{\mathfrak{g}}^{\lambda_j}$ and hence, using (4.1), the character χ is trivial on $L \cap \gamma\mathcal{H}_p\gamma^{-1} = L_H = L \cap \mathcal{H}_p$. Therefore we make the following definition.

Definition 4.2.1. *The minimal principal H-distinguished series is the set of representations $I_P^G(\chi)$ where χ is a character of L trivial on L_H such that $I_P^G(\chi)$ is $\mathcal{H}_p$-distinguished for any $p \in \{1, \dots, r_0\}$.*

Fix a unitary character $\delta = (\delta_0, \dots, \delta_k)$ of $(F^*)^{k+1}$ and fix also $\mu = \mu_0\lambda_0 + \dots + \mu_k\lambda_k \in (a^{0\,*})_{\mathbb{C}}$. This defines a character δ_μ of L which is trivial on $L \cap \mathcal{H}_p$ by

$$\delta_\mu(l) = \prod_{j=0}^k \delta_j(x_j(l))|x_j(l)|^{\mu_j}, \quad l \in L.$$

We set $\pi_{\delta,\mu} = I_P^G(\delta_\mu)$ and denote by $I_{\delta,\mu} = I_{\delta_\mu}$ its space.
Let K be a maximal compact subgroup of G which is the stabilizer of a special point in the appartment associated to A in the building of G. Let $I(\delta)$ be the space of functions $v : K \longmapsto \mathbb{C}$ which are right invariant under an open compact subgroup of K and satisfy the following condition:

$$v(nlk) = \delta(l)v(k) \text{ for } nl \in P \cap K \text{ and } k \in K.$$

Then the restriction of functions in $I_{\delta,\mu}$ to K gives an isomorphism of K-modules from $I_{\delta,\mu}$ onto $I(\delta)$ and we denote by $\overline{\pi_{\delta,\mu}}$ the representation of G on $I(\delta)$ obtained from $\pi_{\delta,\mu}$ through this isomorphism (this is the so called compact picture of the induced representation).

We identify the spaces $(a^{0*})_{\mathbb{C}}$ and $\mathbb{C}^{k+1}$ by the isomorphism $\mu = \sum_{j=0}^k \mu_j\lambda_j \mapsto (\mu_0, \dots, \mu_k)$.
From ([1] Théorème 2.8), we know that there exists a rational function on $\mathcal{R}_p$ on $X(L)_\sigma \simeq (a^{0*})_{\mathbb{C}}$, which is a product of functions of the form $(1 - cq^{\sum_{j=0}^k a_j\mu_j})$, where $c \in \mathbb{C}^*$ and $a_j \in \mathbb{Z}$, such that if $\mathcal{R}_p(\mu) \neq 0$, the representation $(\pi_{\delta,\mu}, I_{\delta,\mu})$ is $\mathcal{H}_p$-distinguished. Moreover, in that case, $\dim(I_{\delta,\mu}^*)^{\mathcal{H}_p}$ is equal to the number of open P-orbits in $G/\mathcal{H}_p$ (or equivalently in Ω_p) and a basis of $(I_{\delta,\mu}^*)^{\mathcal{H}_p}$ is explicitly described in [1] (see also below for our case). Differences between our notations and the results of [1] come from the fact that there the authors consider non normalized induction.

Definition 4.2.2. *Set $\mathcal{R} = \prod_{p=1}^{r_0} \mathcal{R}_p$ where $\mathcal{R}_p$ are the rational functions mentioned above. Let $\mathbf{U}$ be the open dense subset of $(a^{0*})_{\mathbb{C}}$ defined by $\mathbf{U} = \{\mu \in (a^{0*})_{\mathbb{C}}, \mathcal{R}(\mu) \neq 0\}$. Hence for all $\mu \in \mathbf{U}$ and $p = 1, \dots, k$, the representation $(\pi_{\delta,\mu}, I_{\delta,\mu})$ is $\mathcal{H}_p$-distinguished.*

We will now describe, in our case, the basis of$(I^*_{\delta,\mu})^{\mathcal{H}_p}$ given in ([1] Théorème 2.8 and Théorème 2.16, [10] §4.2).

Let $p \in \{1, \dots, r_0\}$. The open $(P, \mathcal{H}_p)$-orbits in G can be view as the open P-orbits in V^+ contained in $\Omega_p^+ \simeq G/\mathcal{H}_p$. Remember that the open P-orbits in V^+ are the sets $P.I^+(a)$ where $I^+(a) = a_0X_0 + \ldots + a_{k-1}X_{k-1} + X_k$ for $a = (a_0, \dots, a_{k-1}, 1)$ in $\mathscr{S}_e^k \times \{1\}$ (see Lemma 1.2.4). For $\underline{a} = (a_0, \dots, a_{k-1}) \in \mathscr{S}_e^k$, we set $(\underline{a}, 1) = (a_0, \dots, a_{k-1}, 1)$.

Definition 4.2.3. *For $p \in \{1, \dots, r_0\}$, we denote by $\mathcal{S}_p$ the set of $\underline{a} \in \mathscr{S}_e^k$ such that $I^+(\underline{a}, 1) \in \Omega_p^+$. Let us fix $\gamma_{\underline{a}} \in G$ such that $\gamma_{\underline{a}} I_p^+ = I^+(\underline{a}, 1)$. Hence $\cup_{\underline{a} \in \mathcal{S}_p} P\gamma_{\underline{a}}\mathcal{H}_p$ is the union of the open $(P, \mathcal{H}_p)$-orbits in G.*

Definition 4.2.4. *Let $\mu \in (a^{0*})_{\mathbb{C}}$, $p \in \{1, \dots, r_0\}$ and $\underline{a} \in \mathcal{S}_p$. The Poisson kernel $\mathbb{P}^{\underline{a}}_{\delta,\mu}$ is the function on G defined by:*

$$\mathbb{P}^{\underline{a}}_{\delta,\mu}(g) = \begin{cases} \delta_P(l)^{1/2}(\delta_\mu)^{-1}(l) & \text{if} \quad g = nl\gamma_{\underline{a}}h \in NL\gamma_{\underline{a}}\mathcal{H}_p, \\ 0 & \text{if} \qquad g \notin P\gamma_{\underline{a}}\mathcal{H}_p. \end{cases}$$

Let $C(G, P, \delta^*, \mu)$ be the space of continuous functions $w : G \longmapsto \mathbb{C}$ such that

$$w(nlg) = \delta_P^{1/2}(l)\delta_\mu(l)^{-1}w(g) \quad n \in N, l \in L, g \in G. \tag{4.2}$$

We will say that $\mathrm{Re}(\delta_P^{-1/2}\delta_\mu)$ is strictly P-dominant if $\langle \mathrm{Re}(\mu) - \rho_P, \alpha\rangle > 0$ for all roots α of A in N. From ([1] Théorème 2.16) and ([10] Théorème 4 (ii)), we know that if $\mu \in (a^{0*})_{\mathbb{C}}$ is such that $\mathrm{Re}(\delta_P^{-1/2}\delta_\mu)$ is strictly P-dominant and if $\underline{a} \in \mathscr{S}_e^k$, the function $\mathbb{P}^{\underline{a}}_{\delta,\mu}$ is a right $\mathcal{H}_p$-invariant element of $C(G, P, \delta^*, \mu)$.
Moreover by ([1] (2.29)), the map

$$(w, v) \in C(G, P, \delta^*, \mu) \times I_{\delta,\mu} \mapsto \langle w, v\rangle = \int_K w(k)v(k)dk \tag{4.3}$$

defines a G-invariant duality between $C(G, P, \delta^*, \mu)$ and $I_{\delta,\mu}$. Hence $C(G, P, \delta^*, \mu)$ can be viewed as a subspace of $I^*_{\delta,\mu}$ and $\mathbb{P}^{\underline{a}}_{\delta,\mu}$ is identified to a $\mathcal{H}_p$-invariant linear form on $I_{\delta,\mu}$. Let us make this more precise in the following definition.

Definition 4.2.5. *Let $\mathbf{C}^+$ be the set of $\mu \in (a^{0*})_{\mathbb{C}}$ such that Re$(\delta_P^{-1/2}\delta_\mu)$ is strictly P-dominant.*
Let $\mu \in \mathbf{U} \cap \mathbf{C}^+$, $p \in \{1, \dots, N\}$ and $\underline{a} \in \mathcal{S}_p$. We denote by $\xi^{\underline{a}}_{\delta,\mu} \in (I^*_{\delta,\mu})^{\mathcal{H}_p}$ the $\mathcal{H}_p$-invariant linear form which corresponds to $\mathbb{P}^{\underline{a}}_{\delta,\mu}$ via the preceding duality between $C(G, P, \delta^*, \mu)$ and $I_{\delta,\mu}$.

The isomorphism $I_{\delta,\mu} \simeq I(\delta)$ (given by restriction to K) induces an isomorphism $\xi \mapsto \overline{\xi}$ from $I^*_{\delta,\mu}$ onto $I(\delta)^*$. From Théorème 2.8 in [1], the map $\mu \in \mathbf{U} \cap \mathbf{C}^+ \mapsto \overline{\xi^{\underline{a}}_{\delta,\mu}}$ has a meromorphic continuation to $(a^{0*})_{\mathbb{C}}$ such that $\mathcal{R}(\mu)\overline{\xi^{\underline{a}}_{\delta,\mu}}$ is a polynomial function in the variables $q^{\pm\mu_j}$. Moreover, for $p \in \{1, \dots, N\}$, $\underline{a} \in \mathcal{S}_p$ and $\mu \in \mathbf{U}$, the set of $\xi^{\underline{a}}_{\delta,\mu}$ is a basis of $(I^*_{\delta,\mu})^{\mathcal{H}_p}$ (loc.cit) .

In the rest of the paper we fix Haar measures dg and dk on G and K respectively and a right Haar measure $d_r p$ on P such that

$$\int_G f(g)dg = \int_P \int_K f(pk)\delta_P(p)^{-1} d_r p\, dk, \quad f \in L^1(G).$$

The character δ_μ is extended to P by setting $\delta_\mu(ln) = \delta_\mu(l)$ for $l \in L$ and $n \in N$.
Let $C_c^\infty(G)$ be the space of locally constant functions on G with compact support and define $L_{\delta,\mu} : C_c^\infty(G) \longmapsto I_{\delta,\mu}$ by

$$L_{\delta,\mu}(\varphi)(g) = \int_P \delta_P(p)^{-1/2}\delta_\mu(p)^{-1}\varphi(pg)d_r p, \quad g \in G.$$

It is well known that the map $L_{\delta,\mu}$ is surjective G-equivariant (here G acts on $C_c^\infty(G)$ by the right regular representation).

Lemma 4.2.6. *For $\psi \in C(G, P, \delta^*, \mu)$ and $\varphi \in C_c^\infty(G)$, one has*

$$\int_G \psi(g)\varphi(g)dg = \int_K \psi(k) L_{\delta,\mu}(\varphi)(k)dk$$

Proof. From the above given decomposition of dg and from and the definition of $C(G, P, \delta^*, \mu)$, we have

$$\begin{aligned}\int_G \psi(g)\varphi(g)dg &= \int_P \int_K \psi(pk)\varphi(pk)\delta_P(p)^{-1} d_r p\, dk \\ &= \int_P \int_K \delta_P(p)^{-1/2}\psi(k)\delta_\mu(p)^{-1}\varphi(pk)d_r p\, dk \\ = \int_K \psi(k) \int_P \delta_P(p)^{-1/2}\delta_\mu(p)^{-1}\varphi(pk)d_r p dk &= \int_K \psi(k) L_{\delta,\mu}(\varphi)(k)dk.\end{aligned}$$

□

Corollary 4.2.7. *Let $\mu \in \mathbf{U} \cap \mathbf{C}^+$ and $v \in I_{\delta,\mu}$. Then, for $x \in G$, one has*

$$\langle \pi^*_{\delta,\mu}(x)\xi^a_{\delta,\mu}, v\rangle = \int_K \mathbb{P}^a_{\delta,\mu}(k)v(kx^{-1})dk = \int_K \mathbb{P}^a_{\delta,\mu}(kx)v(k)dk.$$

Proof. One has $\langle \pi^*_{\delta,\mu}(x)\xi^a_{\delta,\mu}, v\rangle = \langle \xi^a_{\delta,\mu}, \pi_{\delta,\mu}(x^{-1})v\rangle$. As $\mu \in \mathbf{U} \cap \mathbf{C}^+$, the linear form $\xi^a_{\delta,\mu}$ is given by the Poisson kernel. Hence

$$\langle \pi^*_{\delta,\mu}(x)\xi^a_{\delta,\mu}, v\rangle = \int_K \mathbb{P}^a_{\delta,\mu}(k)\big(\pi_{\delta,\mu}(x^{-1})v\big)(k)dk = \int_K \mathbb{P}^a_{\delta,\mu}(k)v(kx^{-1})dk.$$

This gives the first equality.

Take $\varphi \in C_c^\infty(G)$ such that $L_{\delta,\mu}(\varphi) = v$. As $\mathbb{P}^a_{\delta,\mu} \in C(G, P, \delta^*, \mu)$, Lemma 4.2.6 implies

$$\int_K \mathbb{P}^a_{\delta,\mu}(k)v(kx^{-1})dk = \int_G \mathbb{P}^a_{\delta,\mu}(g)\varphi(gx^{-1})dg = \int_G \mathbb{P}^a_{\delta,\mu}(gx)\varphi(g)dg.$$

For x fixed, the map $g \mapsto \mathbb{P}^a_{\delta,\mu}(gx)$ belongs also to $C(G, P, \delta^*, \mu)$. Again Lemma 4.2.6 implies that $\int_G \mathbb{P}^a_{\delta,\mu}(gx)\varphi(g)dg = \int_K \mathbb{P}^a_{\delta,\mu}(kx)v(k)dk$.

□

4.3. **Zeta functions associated to the minimal principal H-distinguished series.**
Let (dX, dY) be the pair of Haar measures on V^+ and V^- which were determined in Proposition 2.2.2. Recall that d^*X and d^*Y are the G-invariant measures on V^+ and V^- defined by

$$d^*X = \frac{dX}{|\Delta_0(X)|^m}, \quad dX \text{ and } \quad d^*Y = \frac{dY}{|\nabla_0(X)|^m} \quad \text{where} \quad m = 1 + \frac{kd}{2}.$$

(see (1.2)).

By Proposition 2.2.3, this determines uniquely a G-invariant measure $d_p\dot{g}$ on $G/\mathcal{H}_p$, for all $p \in \{1, \dots r_0\}$.
Then the fixed Haar measure dg on G determines a Haar measure d_ph on $\mathcal{H}_p$ such that

$$\int_G f(g)dg = \int_{G/\mathcal{H}_p}\int_{\mathcal{H}_p} f(gh)d_ph\ d_p\dot{g}, \quad f \in L^1(G).$$

Definition 4.3.1. *Let $p = 1, \dots, r_0$ and let ξ be a $\mathcal{H}_p$-invariant linear form on $I_{\delta,\mu}$ (i.e. an element of $(I^*_{\delta,\mu})^{\mathcal{H}_p}$). For $X \in G.I_p^+$ and $Y \in G.I_p^-$, we define*

$$\pi^*_{\delta,\mu}(X)\xi = \pi^*_{\delta,\mu}(g)\xi \ \text{ if } \ X = g.I_p^+, g \in G, \qquad \pi^*_{\delta,\mu}(Y)\xi = \pi^*(g)\xi \ \text{ if } \ Y = g.I_p^-, g \in G.$$

(Remember that $\mathcal{H}_p$ is the stabilizer of I_p^+ and of I_p^-).
Let $z \in \mathbb{C}$ and $w \in I_{\delta,\mu}$. The zeta fonction associated to $(\pi_{\delta,\mu}, z, \xi, w)$ are formally defined, for $\Phi \in \mathcal{S}(V^+)$, by

$$Z_p^+(\Phi, z, \xi, w) = \int_{\Omega_p^+} \Phi(X)|\Delta_0(X)|^z\langle\pi^*_{\delta,\mu}(X)\xi, w\rangle d^*X,$$

and for $\Psi \in \mathcal{S}(V^-)$, by

$$Z_p^-(\Psi, z, \xi, w) = \int_{\Omega_p^-} \Psi(Y)|\nabla_0(Y)|^z\langle\pi^*_{\delta,\mu}(Y)\xi, w\rangle d^*Y.$$

Remark 4.3.2. 1) If $k = 0$, then G and P are equal and isomorphic to F^*. Thus we have $\pi_{\delta,\mu} = \delta_\mu$ and we can take the generalized coefficient $\langle\pi^*(g)\xi, w\rangle$ to be equal to $(\delta_\mu)^{-1}(g)$, $g \in G$. The spaces $V^\pm$ are isomorphic to F and an element $a \in F^*$ acts by the multiplication by a on $V^+ = F$, and by the multiplication by a^{-1} on V^-. Thus, we have

$$Z^+(\Phi, z, \xi, w) = Z(\Phi, \delta^{-1}, z - \mu), \quad \text{and} \quad Z^-(\Psi, z, \xi, w) = Z(\Psi, \delta, z + \mu),$$

where $Z(g, \delta, s)$ is Tate's zeta function (see section 3.3).

2) Let $\mu \in \mathbf{U}$, $\xi \in (I^*_{\delta,\mu})^{\mathcal{H}_p}$ and $w \in I_{\delta,\mu}$. Then for $z \in \mathbb{C}$ and $\Phi \in \mathcal{S}(\Omega^+)$, the zeta function $Z_p^+(\Phi, z, \xi, w)$ can be seen as a generalized coefficient of $\pi_{\delta,\mu}$. More precisely if $\varphi \in C_c^\infty(G)$ is such that $\Phi(g.I_p^+) = \int_{\mathcal{H}_p}\varphi(gh)d_ph$ and if $\varphi_z \in C_c^\infty(G)$ is the function defined by $\varphi_z(g) = \varphi(g)|\Delta_0(g.I_p^+)|^z$, it is easily seen that

$$Z_p^+(\Phi, z, \xi, w) = \langle\pi^*_{\delta,\mu}(\varphi_z)\xi, w\rangle.$$

A similar result holds for the zeta functions $Z_p^-(\Psi, z, \xi, w)$, for $\Psi \in \mathcal{S}(\Omega^-)$.

We will now study the zeta functions associated to the linear forms $\xi^a_{\delta,\mu}$ by giving a connection between them and the zeta function $\mathcal{K}_c^\pm(f, \omega, s)$ ($c \in \mathscr{S}_e^{k+1}$) introduced in Definition 3.2.2.

Definition 4.3.3. *For* $\mu = \sum_{j=0}^{k} \mu_j \lambda_j \in (a^{0*})_{\mathbb{C}}$, *we define* $s(\mu) = (s_0, \ldots, s_k) \in (\mathbb{C}^*)^{k+1}$ *by the relation*

$$\rho_P - \mu = s_0\lambda_0 + (s_0 + s_1)\lambda_1 + \ldots + (s_0 + \ldots + s_k)\lambda_k.$$

If $\delta = (\delta_0, \ldots, \delta_k)$ *is a unitary character of* $(F^*)^{k+1}$, *we define* $\omega(\delta) \in \widehat{(F^*)^{k+1}}$ *by*

$$\omega(\delta) = (\omega_0, \ldots, \omega_k) = (\delta_0^{-1}, \delta_0\delta_1^{-1}, \ldots, \delta_{k-1}\delta_k^{-1}).$$

Then

$$\omega_0 \ldots \omega_j = \delta_j^{-1}, \ (\textit{for } j = 0, \ldots, k)$$

and, using Lemma 4.1.1:

$$s_0 = \frac{kd}{4} - \mu_0, \quad s_j = \mu_{j-1} - \mu_j - \frac{d}{2}, \quad (\textit{for } j = 1, \ldots, k).$$

Also if $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$ *and* $z \in \mathbb{C}$, *we set* $s - z = (s_0 - z, s_1, \ldots, s_k)$.

Lemma 4.3.4. *Let* $\mu \in (a_0^*)_{\mathbb{C}}$. *Then* $\delta_P^{-1/2}\delta_\mu$ *is strictly* P*-dominant (or equivalently* $\mu \in \mathbf{C}^+$*) if and only if* $s(\mu) = (s_0, \ldots, s_k)$ *satisfies the condition* $Re(s_j) > 0$ *for* $j = 1, \ldots, k$.

Proof. By definition $\mu \in \mathbf{C}^+$ if and only if $\langle \mathrm{Re}(\mu) - \rho_P, \alpha \rangle > 0$ for any root α of A in N. From the definition of N, for such a root, there exists a pair $i < j$ such that $\tilde{\mathfrak{g}}^\alpha \subset E_{i,j}(1, -1)$. Let $\rho_P = \sum_{s=0}^{k} \rho_s\lambda_s$ and $\mu = \sum_{s=0}^{k} \mu_s\lambda_s$.
Then the condition $\mu \in \mathbf{C}^+$ is equivalent to $\mathrm{Re}(\mu_i - \mu_j) - (\rho_i - \rho_j) > 0$ for any pair $i < j$.
From Lemma 4.1.1, we get $\rho_{i-1} - \rho_i = \frac{d}{2}$, and therefore $\mu \in \mathbf{C}^+$ if and only if $\mathrm{Re}(\mu_{i-1} - \mu_i) > \frac{d}{2}$ for $i = 1, \ldots, k$.
As we have seen in the preceding definition that $s_i = \mu_{i-1} - \mu_i - \frac{d}{2}$ for $i = 1, \ldots, k$, the Lemma is proved.

□

Definition 4.3.5. *Let* $p \in \{1, \ldots r_0\}$, $\underline{a} \in \mathcal{S}_p$, $\omega = (\omega_0, \ldots, \omega_k) \in \widehat{(F^*)}^{k+1}$ *and* $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$. *Define the functions* $\mathbf{P}_{\omega,s}^{\underline{a},\pm}$ *on* $V^\pm$ *by*

$$\mathbf{P}_{\omega,s}^{\underline{a},+}(X) = \begin{cases} (\omega, s)(\Delta(X)) & \text{if} \quad X \in P.I^+(\underline{a}, 1), \\ 0 & \text{if} \quad X \in V^+ \setminus P.I^+(\underline{a}, 1). \end{cases}$$

and

$$\mathbf{P}_{\omega,s}^{\underline{a},-}(Y) = \begin{cases} (\omega, s)(\nabla(Y)) & \text{if} \quad Y \in P.I^-(\underline{a}, 1), \\ 0 & \text{if} \quad Y \in V^- \setminus P.I^-(\underline{a}, 1). \end{cases}$$

Remark 4.3.6. The union of the open P-orbits in V^+ is the set $\{X \in V^+, \Delta_j(X) \neq 0 \text{ for } j = 0, \ldots, k\}$. On the other hand we have $\Omega_p^+ \subset \Omega^+ = \{X \in V^+; \Delta_0(X) \neq 0\}$. Hence, if $\mathrm{Re}(s_j) > 0$ for $j = 1, \ldots k$, the function $\mathbf{P}_{\omega,s}^{\underline{a},+}$ is continuous on Ω_p^+. If moreover $\mathrm{Re}(s_0) > 0$, $\mathbf{P}_{\omega,s}^{\underline{a},+}$ is continuous on V^+.

Similarly the union of the open P-orbits in V^- is the set $\{Y \in V^-, \nabla_j(Y) \neq 0 \text{ for } j = 0, \dots, k\}$ and we have $\Omega_p^- \subset \Omega^- = \{Y \in V^-; \nabla_0(Y) \neq 0\}$. Hence if $\mathrm{Re}(s_j) > 0$ for $j = 1, \dots k$, the function $\mathbf{P}^{\underline{a},-}_{\omega,s}$ is continuous on Ω_p^- and if moreover $\mathrm{Re}(s_0) > 0$, then $\mathbf{P}^{\underline{a},-}_{\omega,s}$ is continuous on V^-.

Lemma 4.3.7. *Let $p \in \{1, \dots, r_0\}$ and $\underline{a} \in \mathcal{S}_p$. For $g \in G$, we have*

$$\mathbb{P}^{\underline{a}}_{\delta,\mu}(g) = c_{\delta,\mu}(\underline{a})\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(g.I_p^+) = c_{\delta,\mu}(\underline{a})\mathbf{P}^{\underline{a},-}_{\omega(\delta)^\sharp,s(\mu)^\sharp}(g.I_p^-),$$

where $\omega(\delta)^\sharp$ and $s(\mu)^\sharp$ are given in definition 3.2.1 and $c_{\delta,\mu}(\underline{a}) = \prod_{j=0}^{k-1} \delta_j(a_j)|a_j|^{-(\rho_j-\mu_j)}$.

Proof. Let us prove the first equality. Remember that $I(\underline{a}, 1)^+ = \gamma_{\underline{a}}.I_p^+$ (cf. Definition 4.2.3) . If $g \notin P\gamma_{\underline{a}}\mathcal{H}_p$, then $g.I_p^+ \notin P.I^+(\underline{a}, 1)$, and hence the two members of the first equality are equal to zero. These two members are also left invariant by N and right invariant by $\mathcal{H}_p$. Therefore it is enough to show the results for $g = l\gamma_{\underline{a}}$ where $l \in L$. Define then $X = g.I_p^+ = l.I^+(\underline{a}, 1)$. From Definition 4.3.3 we have

$$\omega(\delta)(\Delta(X)) = \prod_{j=0}^{k} \omega_j(\Delta_j(X)) = \delta_0^{-1}(\Delta_0(X))(\delta_1^{-1}\delta_0)(\Delta_1(X)) \dots (\delta_k^{-1}\delta_{k-1})(\Delta_k(X))$$

$$= \delta_0^{-1}(\frac{\Delta_0(X)}{\Delta_1(X)})\ \delta_1^{-1}(\frac{\Delta_1(X)}{\Delta_2(X)}) \dots \delta_{k-1}^{-1}(\frac{\Delta_{k-1}(X)}{\Delta_k(X)})\delta_k^{-1}(\Delta_k(X)).$$

As $X = l.I^+(\underline{a}, 1) = a_0x_0(l)X_0 + a_1x_1(l)X_1 + \dots + a_{k-1}x_{k-1}(l)X_{k-1} + x_k(l)X_k$, we obtain:

$$\omega(\delta)(\Delta(X)) = \delta_k(x_k(l))^{-1} \prod_{j=0}^{k-1} \delta_j(x_j(l)a_j)^{-1} = \delta(l)^{-1} \prod_{j=0}^{k-1} \delta_j(a_j)^{-1}.$$

Also, as $s(\mu) = (s_0, \dots, s_k)$ is such that $s_0 + \dots s_j = \rho_j - \mu_j$, we obtain

$$|\Delta(X)|^{s(\mu)} = |\Delta_0(X)|^{s_0}|\Delta_1(X)|^{s_1}|\Delta_2(X)|^{s_2} \dots |\Delta_k(X)|^{s_k}$$

$$= \left(\frac{|\Delta_0(X)|}{|\Delta_1(X)|}\right)^{s_0} \left(\frac{|\Delta_1(X)|}{|\Delta_2(X)|}\right)^{s_0+s_1} \dots \left(\frac{|\Delta_{k-1}(X)|}{|\Delta_k(X)|}\right)^{s_0+\dots+s_{k-1}} |\Delta_k(X)|^{s_0+\dots+s_k}$$

$$= |x_k(l)|^{\rho_k-\mu_k} \prod_{j=0}^{k-1} (a_j|x_j(l)|)^{\rho_j-\mu_j}.$$

Using Lemma 4.1.1, we get

$$|\Delta(X)|^{s(\mu)} = \big(\prod_{j=0}^{k-1} |a_j|^{\rho_j-\mu_j}\big)\delta_P(l)^{1/2} \prod_{j=0}^{k} |x_j(l)|^{-\mu_j}.$$

Therefore

$$\begin{aligned}
\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(g.I_p^+) &= \omega(\delta)(\Delta(X))|\Delta(X)|^{s(\mu)} \\
&= \delta(l)^{-1} \prod_{j=0}^{k-1} \delta_j(a_j)^{-1} \big(\prod_{j=0}^{k-1} |a_j|^{\rho_j-\mu_j}\big)\delta_P(l)^{1/2} \prod_{j=0}^{k} |x_j(l)|^{-\mu_j} \\
&= (c_{\delta,\mu}(\underline{a}))^{-1}\mathbb{P}^{\underline{a}}_{\delta,\mu}(g).
\end{aligned}$$

The first equality is proved.

If $l \in L$ then $\{l.I^-(\underline{a}, 1), H_0, l.I^+(\underline{a}, 1)\}$ is an $\mathfrak{sl}_2$-triple. From ([8] corollaire 4.5.9) we know that $|\nabla(l.I^-(\underline{a}, 1))|^{s(\mu)^\sharp} = |\Delta(l.I^+(\underline{a}, 1))|^{s(\mu)}$, similarly it is easy to see that $\omega(\delta)^\sharp(\nabla(l.I^-(\underline{a}, 1))) = \omega(\delta)(\Delta(l.I^+(\underline{a}, 1)))$. Hence $\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(g.I_p^+) = \mathbf{P}^{\underline{a},-}_{\omega(\delta)^\sharp,s(\mu)^\sharp}(g.I_p^-)$. This gives the second equality.

□

Theorem 4.3.8. *Let $p \in \{1, \dots, r_0\}$, $\underline{a} \in \mathcal{S}_p$ and $z \in \mathbb{C}$. Let also $\delta = (\delta_0, \delta_1, \dots, \delta_k)$ be a unitary character of $(F^*)^{k+1}$.*

(1) (a) *Let $\mu = \mu_0\lambda_0 + \dots \mu_k\lambda_k \in \mathbf{U} \cap \mathbf{C}^+$. Suppose Re$(z - \mu_0) > m - \dfrac{kd}{4} = 1 + \dfrac{kd}{4}$. Then, for all $\Phi \in \mathcal{S}(V^+)$, $\xi \in (I^*_{\delta,\mu})^{\mathcal{H}_p}$ and $w \in I_{\delta,\mu}$, the integral defining the zeta function $Z_p^+(\Phi, z, \xi, w)$ (cf. Definition 4.3.1) is absolutely convergent and the following relation holds:*

$$Z_p^+(\Phi, z, \xi^{\underline{a}}_{\delta,\mu}, w) = c_{\delta,\mu}(\underline{a}) \sum_{x \in \mathscr{S}_e} \int_K \mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(k)\Phi, \omega(\delta), s(\mu) + z - m)w(k)dk,$$

where $\mathcal{L}$ is the left regular representation.

(b) *The function $Z_p^+(\Phi, z, \xi^{\underline{a}}_{\delta,\mu}, w)$ is a rational function in the variables $q^{\pm\mu_j}$ and $q^{\pm z}$ and hence admits a meromorphic continuation. Moreover there exists a polynomial $\mathcal{R}^+(\delta, \mu, z)$ in the variables q^{μ_j}, $q^{-\mu_j}$ and q^{-z}, which is a product of polynomials of type $(1 - d_j q^{-Nz - \sum_{j=0}^k m_j\mu_j})$, where $N \in \mathbb{N}$, $m_j \in \mathbb{Z}$ and $d_j \in \mathbb{C}$ are independent of z and of the μ_j's, such that for all $\Phi \in \mathcal{S}(V^+)$, $\xi \in (I^*_{\delta,\mu})^{\mathcal{H}_p}$ and $w \in I_{\delta,\mu}$, the function $\mathcal{R}^+(\delta, \mu, z)Z_p^+(\Phi, z, \xi, w)$ is a polynomial in $q^{\pm\mu_j}$ and $q^{\pm z}$.*

(2) (a) *Let $\mu = \mu_0\lambda_0 + \dots \mu_k\lambda_k \in \mathbf{U} \cap \mathbf{C}^+$. Suppose that Re$(z + \mu_k) > m - \dfrac{kd}{4} = 1 + \dfrac{kd}{4}$. Then for all $\Psi \in \mathcal{S}(V^-)$, $\xi \in (I^*_{\delta,\mu})^{\mathcal{H}_p}$ and $w \in I_{\delta,\mu}$, the integral defining the zeta function $Z_p^-(\Psi, z, \xi, w)$ is absolutely convergent and the following relation holds:*

$$Z_p^-(\Psi, z, \xi^{\underline{a}}_{\delta,\mu}, w) = c_{\delta,\mu}(\underline{a}) \sum_{x \in \mathscr{S}_e} \int_K \mathcal{K}^-_{(x\underline{a},x)}(\mathcal{L}(k)\Psi, \omega(\delta)^\sharp, s(\mu)^\sharp + z - m)w(k)dk,$$

(b) *The function $Z_p^-(\Psi, z, \xi^{\underline{a}}_{\delta,\mu}, w)$ is a rational function in the variables $q^{\pm\mu_j}$ and $q^{\pm z}$ and hence admits a meromorphic continuation. Moreover there exists a polynomial $\mathcal{R}^-(\delta, \mu, z)$ in the variables q^{μ_j}, $q^{-\mu_j}$ and q^{-z}, which is a product of polynomials of type $(1 - d_j q^{-Nz - \sum_{j=0}^k m_j\mu_j})$ where $N \in \mathbb{N}$, $m_j \in \mathbb{Z}$ and $d_j \in \mathbb{C}$ are independent of z and of the μ_j's, such that for all $\Psi \in \mathcal{S}(V^-)$, $\xi \in (I^*_{\delta,\mu})^{\mathcal{H}_p}$ and $w \in I_{\delta,\mu}$, the function $\mathcal{R}^-(\delta, \mu, z)Z_p^-(\Psi, z, \xi, w)$ is polynomial in $q^{\pm\mu_j}$ and $q^{\pm z}$.*

Proof. For $k = 0$, the zeta functions considered here coincide with Tate's zeta functions (see Remark 4.3.2, 1)), for which the results are well-known.

From now on , we suppose that $k \geq 1$.

We have already mentioned that the set of $\xi^{\underline{a}}_{\delta,\mu}$ for $\underline{a} \in \mathscr{S}_e^k$ is a basis of $(I^*_{\delta,\mu})^{\mathcal{H}_p}$. Therefore it is enough to prove the results for $\xi = \xi^{\underline{a}}_{\delta,\mu}$. Let us prove assertion (1).

Let $\mu = \mu_0\lambda_0 + \dots + \mu_k\lambda_k \in \mathbf{U} \cap \mathbf{C}^+$. To simplify the notations we set $s = s(\mu) = (s_0, \dots, s_k)$ and $\omega = \omega(\delta)$. From Definition 4.3.3 and Lemma 4.3.4, one has

a) Re$(s_j) =$ Re$(\mu_{j-1} - \mu_j) - \dfrac{d}{2} > 0$ if $j = 1, \dots k$ and $s_0 = \dfrac{kd}{4} - \mu_0$.

b) Also from our condition on z, we have Re$(z + s_0) - m =$ Re$(z - \mu_0) + \dfrac{kd}{4} - m > 0$.

Taking into account that δ is here a unitary character, we know from Definition 3.2.2 that these conditions imply that the functions $\mathcal{K}_c^+(f,\omega(\delta),z+s(\mu)-m)$, $c\in\mathscr{S}_e^{k+1}$, are defined by an absolutely convergent integral for all $f\in\mathcal{S}(V^+)$.
Using Lemma 1.2.4, we obtain for $f\in\mathcal{S}(V^+)$:

$$\sum_{x\in\mathscr{S}_e}\mathcal{K}^+_{(x\underline{a},x)}(f,\omega(\delta),z+s(\mu)-m)=\sum_{x\in\mathscr{S}_e}\int_{\mathcal{O}(x\underline{a},x)^+}f(X)\omega(\delta)(\Delta(X))|\Delta(X)|^{s(\mu)+z-m}dX$$

$$=\int_{P.I(\underline{a},1)^+}f(X)|\Delta_0(X)|^{z-m}\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X)dX=\int_{V^+}f(X)|\Delta_0(X)|^{z-m}\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X)dX.$$

Remark 4.3.6 and the conditions on μ, z and Φ, imply that the the function

$$(k,X)\mapsto(\mathcal{L}(k)\Phi)(X)|\Delta_0(X)|^{z-m}\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X)$$

is continuous with compact support on $K\times V^+$.
Then, from Lemma 4.3.7 and the Theorem of Fubini, we get

$$c_{\delta,\mu}(\underline{a})\sum_{x\in\mathscr{S}_e}\int_K\mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(k)\Phi,\omega(\delta),s(\mu)+z-m)w(k)dk$$

$$=c_{\delta,\mu}(\underline{a})\int_K\left(\int_{\Omega_p^+}(\mathcal{L}(k)\Phi)(X)|\Delta_0(X)|^z\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X)d^*X\right)w(k)dk$$

$$=\int_K\left(\int_{G/\mathcal{H}_p}\Phi(k^{-1}g.I_p^+)|\Delta_0(g.I_p^+)|^z\mathbb{P}^{\underline{a},+}_{\delta,\mu}(g)d_p\dot{g}\right)w(k)dk$$

$$=\int_K\left(\int_{G/\mathcal{H}_p}\Phi(g.I_p^+)|\Delta_0(g.I_p^+)|^z\mathbb{P}^{\underline{a},+}_{\delta,\mu}(kg)d_p\dot{g}\right)w(k)dk$$

(because $|\chi_0|$ is trivial on K)

$$=\int_{G/\mathcal{H}_p}\Phi(g.I_p^+)|\Delta_0(g.I_p^+)|^z\left(\int_K\mathbb{P}^{\underline{a},+}_{\delta,\mu}(kg)w(k)dk\right)d_p\dot{g}.$$

Then from Corollary 4.2.7, we obtain:

$$c_{\delta,\mu}(\underline{a})\sum_{x\in\mathscr{S}_e}\int_K\mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(k)\Phi,\omega(\delta),z+s(\mu)-m)w(k)dk$$

$$=\int_{G/\mathcal{H}_p}\Phi(g.I_p^+)|\Delta_0(g.I_p^+)|^z\langle\pi^*_{\delta,\mu}(g)\xi^{\underline{a}}_{\delta,\mu},w\rangle d_p\dot{g}$$

$$=Z_p^+(\Phi,z,\xi^{\underline{a}}_{\delta,\mu},w),$$

which is assertion (1)(a).

From Theorem 3.2.3 there exists a polynomial $R^+(\omega,s)\in\mathbb{C}[q^{-s_0},q^{-s_1},\dots,q^{-s_k}]$ in the variables q^{-s_j}, product of polynomials of type $(1-b_jq^{-\sum_{j=0}^k N_js_j)}$ with $N_j\in\mathbb{N}$ and $b_j\in\mathbb{C}$, such that $R^+(\omega,s)\mathcal{K}_c^+(f,\omega,s)\in\mathbb{C}[q^{\pm s_0},q^{\pm s_1},\dots,q^{\pm s_k}]$ for all $f\in\mathcal{S}(V^+)$ and $c\in\mathscr{S}_e^{k+1}$. But here we must consider integrals of the form $\mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(k)\Phi,\omega(\delta),z+s(\mu)-m)$, where by Definition 4.3.3,

$$(z+s(\mu)-m)=(\frac{kd}{4}-\mu_0+z-m,\mu_0-\mu_1-\frac{d}{2},\dots,\mu_{k-1}-\mu_k-\frac{d}{2}).$$

This implies immediately assertion (1)(b).

Remember that $s(\mu)^\sharp = (-(s_0+\ldots+s_k), s_k, \ldots, s_1)$ with $s_0+\ldots+s_k = -\frac{kd}{4} - \mu_k$. Therefore $\mathrm{Re}(z-(s_0+\ldots+s_k)) - m = \mathrm{Re}(z+\mu_k) + \frac{kd}{4} - m > 0$.
A similar computation to the one in the proof of 1) b) above shows thats for $g \in \mathcal{S}(V^-)$ one has

$$c_{\delta,\mu}(\underline{a}) \sum_{x\in\mathscr{S}_e} \mathcal{K}^-_{(x\underline{a},x)}(g, \omega(\delta)^\sharp, z+s(\mu)^\sharp - m) = \int_{V^-} g(Y)|\nabla_0(Y)|^{z-m} \mathbf{P}^{\underline{a},-}_{\omega(\delta)^\sharp, s(\mu)^\sharp}(Y)dY.$$

Then the same arguments prove assertion 2)a) and 2)b).

□

Theorem 4.3.9. (Main Theorem, version 1)
Let $k \geq 1$ (for $k=0$, the zeta functions we consider coincide with Tate's zeta functions by Remark 4.3.2). Let δ be a unitary character of $(F^*)^{k+1}$ and $\mu \in \mathbf{U}$. If $\underline{a} \in \mathscr{S}_e^k$, we denote by $p_{\underline{a}}$ the unique integer in $\{1,\ldots,r_0\}$ such that $P.I(\underline{a},1)^+ \subset \Omega^+_{p_{\underline{a}}}$. Let also γ_j ($j \in \mathbb{N}^*$) be the function defined in Proposition 2.5.5.
Then for $\Phi \in \mathcal{S}(V^+)$ and $w \in I_{\delta,\mu}$, the zeta functions satisfy the following functional equation:

$$Z^-_{p_{\underline{a}}}(\mathcal{F}\Phi, \frac{m+1}{2} - z, \xi^{\underline{a}}_{\delta,\mu}, w) = \sum_{\underline{c}\in\mathscr{S}_e^k} \mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z) Z^+_{p_{\underline{c}}}(\Phi, z+\frac{m-1}{2}, \xi^{\underline{c}}_{\delta,\mu}, w),$$

where the functions $\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)$ are given by

$$\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z) = c_{\delta,\mu}(\underline{a}) c_{\delta,\mu}(\underline{c})^{-1} \sum_{y\in\mathscr{S}_e} \Big(\gamma_k((a_0,\ldots,a_{k-1}),y)\delta_k(-1)\tilde{\rho}(\delta_k, \mu_k - z + 1; -y)$$
$$\times \prod_{j=1}^{k-1} \gamma_j((a_0,\ldots,a_{j-1}), yc_j) \prod_{j=0}^{k-1} \delta_j(-1)\tilde{\rho}(\delta_j, \mu_j - z + 1; -a_jc_jy)\Big).$$

Moreover these functions $\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)$ do not depend on the character of $(F^*)^{k+1}$ which defines δ_μ (see Lemma 4.1.2).

Proof. Let us first show that for $k \in K$, one has $\mathcal{F}(\mathcal{L}(k)\Phi) = \mathcal{L}(k)\mathcal{F}(\Phi)$.
By definition we have

$$\mathcal{F}(\Phi)(Y) = \int_{V^+} \Phi(X)\tau\circ b(X,Y)dX, \quad Y \in V^-.$$

As the measure $d^*X = \frac{dX}{|\Delta_0(X)|^m}$ is G-invariant and as $|\Delta_0(k.X)| = |\Delta_0(X)|$ for $k \in K$ and $X \in V^+$, the measure dX is K-invariant.
As the bilinear form $b(\cdot,\cdot)$ is G-invariant, we see that the Fourier transform on V^+ commutes with the action of K:

$$\mathcal{L}(k)\mathcal{F}(\Phi) = \mathcal{F}\big(\mathcal{L}(k)\Phi\big), \text{ for } k \in K.$$

From Theorem 4.3.8 (2) (a), we obtain

$$Z^-_{p_{\underline{a}}}(\mathcal{F}\Phi, \frac{m+1}{2} - z, \xi^{\underline{a}}_{\delta,\mu}, w) = c_{\delta,\mu}(\underline{a}) \sum_{x\in\mathscr{S}_e} \int_K \mathcal{K}^-_{(x\underline{a},x)}(\mathcal{F}\big(\mathcal{L}(k)\Phi\big), \omega(\delta)^\sharp, s(\mu)^\sharp - z + \frac{m+1}{2} - m) w(k)dk.$$

As $s(\mu)^\sharp - z + \frac{m+1}{2} = (s(\mu) + z - \frac{m+1}{2})^\sharp$ (see Definition 3.2.1), Theorem 3.3.3 implies that

$$Z^-_{p_{\underline{a}}}(\mathcal{F}\Phi, \frac{m+1}{2} - z, \xi^{\underline{a}}_{\delta,\mu}, w) = c_{\delta,\mu}(\underline{a}) \sum_{x\in\mathscr{S}_e} \sum_{c\in\mathscr{S}_e^{k+1}} D^k_{((x\underline{a},x),c)}(\omega(\delta), s(\mu) + z - \frac{m+1}{2})$$

$$\times \int_K \mathcal{K}^+_c(\mathcal{L}(k)\Phi, \omega(\delta), s(\mu) + z - \frac{m+1}{2}) w(k)dk, \tag{4.4}$$

where the functions D^k satisfy the following induction relation:

$$D^k_{(a,c)}(\omega', s')$$

$$= \gamma_k(\underline{a}, c_k))(\omega'_0 \dots \omega'_k)(-1)\tilde{\rho}((\omega'_0 \dots \omega'_k)^{-1}, -(s'_0 + \dots + s'_k + \frac{kd}{2}); -a_k c_k) D^{k-1}_{(\underline{a},\underline{c})}(\underline{\omega'}, \underline{s'}).$$

Proposition 2.5.5 implies that $\gamma_k(x\underline{a}, c_k) = \gamma_k(\underline{a}, xc_k)$ for $x \in \mathscr{S}_e$. Then by an easy induction from Theorem 3.3.3 we see that $D^k_{((x\underline{a},x),c)}(\omega', s') = D^k_{((\underline{a},1),xc)}(\omega', s')$.
It follows that if we define $A(\omega', s')(\underline{a}, c) = \sum_{x\in\mathscr{S}_e} D^k_{((x\underline{a},x),c)}(\omega', s')$ then $A(\omega', s')(\underline{a}, yc) = A(\omega', s')(\underline{a}, c)$ for all $y \in \mathscr{S}_e$. Then, for $f \in \mathcal{S}(V^+)$,

$$\sum_{x\in\mathscr{S}_e} \sum_{c\in\mathscr{S}_e^{k+1}} D^k_{((x\underline{a},x),c)}(\omega', s')\mathcal{K}^+_c(f, \omega', s') = \sum_{c\in\mathscr{S}_e^{k+1}} A_k(\omega', s')(\underline{a}, c)\mathcal{K}^+_c(f, \omega', s')$$

$$= \sum_{\underline{c}\in\mathscr{S}_e^k} \sum_{x\in\mathscr{S}_e} A_k(\omega', s')(\underline{a}, (x\underline{c}, x))\mathcal{K}^+_{(x\underline{c},x)}(f, \omega', s')$$

$$= \sum_{\underline{c}\in\mathscr{S}_e^k} A_k(\omega', s')(\underline{a}, (\underline{c}, 1)) \sum_{x\in\mathscr{S}_e} \mathcal{K}^+_{(x\underline{c},x)}(f, \omega', s').$$

Then by Theorem 4.3.8 and the expression of $Z^-_{p_{\underline{a}}}(\mathcal{F}\Phi, \frac{m+1}{2} - z, \xi^{\underline{a}}_{\delta,\mu}, w)$ given in (4.4), we obtain that

$$Z^-_{p_{\underline{a}}}(\mathcal{F}\Phi, \frac{m+1}{2} - z, \xi^{\underline{a}}_{\delta,\mu}, w)$$

$$= c_{\delta,\mu}(\underline{a}) \sum_{\underline{c}\in\mathscr{S}_e^k} A_k(\omega(\delta), s(\mu) + z - \frac{m+1}{2})(\underline{a}, (\underline{c}, 1))c_{\delta,\mu}(\underline{c})^{-1} Z^+_{p_{\underline{c}}}(\Phi, z + \frac{m-1}{2}, \xi^{\underline{c}}_{\delta,\mu}, w).$$

In order to prove the functional equation it remains to show that the functions

$$\mathcal{B}_{\underline{a},\underline{c}}(\delta, \mu)(z) = c_{\delta,\mu}(\underline{a})c_{\delta,\mu}(\underline{c})^{-1} A_k(\omega(\delta), s(\mu) + z - \frac{m+1}{2})(\underline{a}, (\underline{c}, 1))$$

have the form requested in the statement.

From Theorem 3.3.3, we obtain

$$D^k_{((x\underline{a},x),(\underline{c},1))}(\omega(\delta), s(\mu) + z - \frac{m+1}{2}) = D^k_{((\underline{a},1),(x\underline{c},x))}(\omega(\delta), s(\mu) + z - \frac{m+1}{2})$$

$$= \gamma_k(\underline{a}, x)(\omega_0 \dots \omega_k)(-1)\tilde{\rho}((\omega_0 \dots \omega_k)^{-1}, -(s_0 + \dots + s_k + z - \frac{m+1}{2} + \frac{jd}{2}); -x)$$

$$\times \prod_{j=1}^{k-1} \gamma_j((a_0, \dots, a_{j-1}), xc_j) \prod_{j=0}^{k} (\omega_0 \dots \omega_j)(-1)\tilde{\rho}((\omega_0 \dots \omega_j)^{-1}, -(s_0 + \dots + s_j + z - \frac{m+1}{2} + \frac{jd}{2}); -xa_j c_j).$$

From Definition 4.3.3 and the definition of $s(\mu)$ we have $\omega_0 \dots \omega_j = \delta_j^{-1}$ (if $\omega(\delta) = (\omega_0, \dots, \omega_k)$) , and $s_0 + \dots + s_j = \rho_j - \mu_j$. Hence

$$-(s_0 + \dots + s_j + \frac{jd}{2} + z - \frac{m+1}{2}) = -(\frac{(k-2j)d}{4} - \mu_j + \frac{jd}{2} + z - \frac{kd}{4} - 1)$$

$$= \mu_j - z + 1.$$

Hence

$$D^k_{((x\underline{a},x),(\underline{c},1))}(\omega(\delta), s(\mu)+z-\frac{m+1}{2})$$
$$=\gamma_k(\underline{a},x)\delta_k(-1)\tilde{\rho}(\delta_k,\mu_k-z+1;-x)\prod_{j=1}^{k-1}\gamma_j((a_0,\dots,a_{j-1}),xc_j)\prod_{j=0}^{k}\delta_j(-1)\tilde{\rho}(\delta_j,\mu_j-z+1;-xa_jc_j).$$

This implies the result concerning the functional equation.

It remains to show that the functions $\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)$ do not depend on the character of $(F^*)^{k+1}$ which defines δ_μ.

To see this we take $\Phi\in\mathcal{S}(\Omega^+_{p_{\underline{a}}})$. Then from the functional equation we have

$$\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)=\frac{Z^-_{p_{\underline{c}}}(\mathcal{F}\Phi,\frac{m+1}{2}-z,\xi^{\underline{c}}_{\delta,\mu},w)}{Z^+_{p_{\underline{a}}}(\Phi,z+\frac{m-1}{2},\xi^{\underline{a}}_{\delta,\mu},w)}.$$

By definition (4.3.1) the zeta functions do only depend on the induction parameters.

□

Remark 4.3.10. Of course one can also check directly that the functions $\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)$ do not depend on the choice of the character δ of $(F^*)^{k+1}$ defining the character δ_μ of L.

If δ' is another character $(F^*)^{k+1}$ which defines the same character δ_μ then, by Lemma 4.1.2, for $j\in\{0,\dots,k\}$, there exists a character χ^j of F^* with values in $\{\pm1\}$ and trivial on S_e, such that $\delta_j(x)=\chi^j(x)\delta'_j(x)$ and $\prod_{j=0}^k\chi^j(x)=1$ for $x\in F^*$. By definition (cf. Lemma 4.3.7), we have $c_{\delta,\mu}(\underline{a})=\prod_{j=0}^{k-1}\delta_j(a_j)|a_j|^{-(\rho_j-\mu_j)}$. Hence

$$c_{\delta',\mu}(\underline{a})c_{\delta',\mu}(\underline{c})^{-1}=\prod_{j=0}^{k-1}\chi^j(c_j)\chi^j(a_j)c_{\delta,\mu}(\underline{a})c_{\delta,\mu}(\underline{c})^{-1}. \tag{4.5}$$

From the definition of $\tilde{\rho}$ (cf. Definition 3.3.1), for a character $\tilde{\delta}$ of F^*, $s\in\mathbb{C}$ and $b\in\mathscr{S}_e$, we have

$$\tilde{\rho}(\tilde{\delta},s;b)=\frac{1}{|\mathscr{S}_e|}\sum_{\chi\in\widehat{\mathscr{S}_e}}\chi(b)\rho(\tilde{\delta}\chi,s).$$

Then, for $y\in\mathscr{S}_e$:

$$\tilde{\rho}(\delta_k\chi^k,\mu_k-z+1;-y)=\chi^k(-y)\tilde{\rho}(\delta_k,\mu_k-z+1;-y)$$
$$\text{and }\tilde{\rho}(\delta_j\chi^j,\mu_j-z+1;-a_jc_jy)=\chi^j(-ya_jc_j)\tilde{\rho}(\delta_j,\mu_j-z+1;-a_jc_jy).$$

As $\prod_{j=0}^k\chi^j(-y)=1$, relation (4.5) implies that $\mathcal{B}_{\underline{a},\underline{c}}(\delta',\mu)(z)=\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)$.

In the case where $e=0$ or 4 we have $\mathscr{S}_e=\{1\}$ and the groups G and P have a unique open orbit in V^+. We therefore omit the indices $p\in\{1\}$ and $a\in\mathscr{S}_e^k$. Moreover, $L/L\cap\mathcal{H}$ is isomorphic to $(F^*)^{k+1}$, therefore there exists a unique character δ of $(F^*)^{k+1}$ such that $\delta(l)=\delta(x_0(l),\dots,\delta_k(l))$ for $l\in L$. Then the functional equation is scalar and has the following form.

Corollary 4.3.11. *Let $e=0$ or 4. Then for $\Phi\in\mathcal{S}(V^+)$, $\xi\in(I^*_{\delta,\mu})^{\mathcal{H}}$ and $w\in I_{\delta,\mu}$, we have*

$$Z^-(\mathcal{F}\Phi,\frac{(m+1)}{2}-z,\xi,w)=d(\delta,\mu,z)\;\;Z^+(\Phi,z+\frac{(m-1)}{2},\xi,w))$$

where $d(\delta,\mu,z)=\gamma_\tau(q)^{\frac{k(k+1)}{2}}\prod_{j=0}^k\delta_j(-1)\rho(\delta_j,\mu_j+1-z)$.

Proof. As $\mathscr{S}_e = \{1\}$, we have $\tilde{\rho}(\chi, s; 1) = \rho(\chi, s)$ for all $\chi \in \widehat{F^*}$ and $s \in \mathbb{C}$. Also $\gamma_j(1, \dots, 1) = \gamma_\tau(q)^j$ from Proposition 2.5.5. The Corollary is then an immediate consequence of Theorem 4.3.9.

□

4.4. **Another version of the functional equation.**

Let δ be a unitary character of $(F^*)^{k+1}$ and $\mu \in \mathbf{U} \cap \mathbf{C}^+$.
It is known from Theorem 4.3.8 that if $\mathrm{Re}(\mu_0) < -1 - \frac{kd}{4}$ then $Z_p^+(\Phi, 0, \xi^{\underline{a}}_{\delta,\mu}, w)$ is defined by an absolutely convergent integral for any $\Phi \in \mathcal{S}(V^+)$ and that if $\mathrm{Re}(\mu_k) > 1 + \frac{kd}{4}$, then $Z_p^-(\Psi, 0, \xi^{\underline{a}}_{\delta,\mu}, w)$ is also defined by an absolutely convergent integral for any $\Psi \in \mathcal{S}(V^-)$.
We will first describe the linear form $w \longmapsto Z_p^\pm(\Phi, 0, \xi^{\underline{a}}_{\delta,\mu}, w)$ in terms of the duality (4.3).

Definition 4.4.1. *Let δ be a unitary character of $(F^*)^{k+1}$ and let $\mu \in \mathbf{U} \cap \mathbf{C}^+$. Let also $p \in \{1, \dots, r_0\}$ and $\underline{a} \in \mathcal{S}_p$. If $\Phi \in \mathcal{S}(V^+)$ and $\Psi \in \mathcal{S}(V^-)$, we define the two following functions of $C(G, P, \delta^*, \mu)$ (cf. (4.2)).*

- for $\mathrm{Re}(\mu_0) < -1 - \frac{kd}{4}$:

$$\tilde{Z}_p^+(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})(g) = c_{\delta,\mu}(\underline{a}) \int_{V^+} \Phi(g^{-1}X) \mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X) d^*X = \int_{G/\mathcal{H}_p} \Phi(yI_p^+) \mathbb{P}^{\underline{a}}_{\delta,\mu}(gy) dy \quad (g \in G)$$

- for $\mathrm{Re}(\mu_k) > 1 + \frac{kd}{4}$:

$$\tilde{Z}_p^-(\Psi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})(g) = c_{\delta,\mu}(\underline{a}) \int_{V^-} \Psi(g^{-1}X) \mathbf{P}^{\underline{a},-}_{\omega(\delta)^\sharp,s(\mu)^\sharp}(X) d^*X = \int_{G/\mathcal{H}_p} \psi(yI_p^-) \mathbb{P}^{\underline{a}}_{\delta,\mu}(gy) dy \quad (g \in G).$$

(The functions $\mathbb{P}^{\underline{a}}_{\delta,\mu}$, $\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}$ et $\mathbf{P}^{\underline{a},-}_{\omega(\delta)^\sharp,s(\mu)^\sharp}$ were given in Definition 4.2.4 and 4.3.5 and the second equalities are a consequence of Lemma 4.3.7).
Note also that these functions are well defined because the Poisson kernels $\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}$ and $\mathbf{P}^{\underline{a},-}_{\omega(\delta)^\sharp,s(\mu)^\sharp}$ are continuous for $\mu \in \mathbf{U} \cap \mathbf{C}^+$ satisfying the additional conditions $\mathrm{Re}(\mu_0) < -1 - \frac{kd}{4}$ and $\mathrm{Re}(\mu_k) > 1 + \frac{kd}{4}$ respectively (see Remark 4.3.6).

Let $C(K, P \cap K, \delta^*)$ be the space of continuous functions $v : K \longrightarrow \mathbb{C}$ such that $v(pk) = \delta(p)^{-1} v(k)$ for $k \in K$ and $p \in P \cap K$. The restriction to K is an isomorphism from $C(G, P, \delta^*, \mu)$ onto $C(K, P \cap K, \delta^*)$.

Proposition 4.4.2. *Let δ be a unitary character of $(F^*)^{k+1}$ and let $\mu \in \mathbf{U} \cap \mathbf{C}^+$. Let also $p \in \{1, \dots, r_0\}$ and $\underline{a} \in \mathcal{S}_p$*
1) Let $\varphi \in C_c^\infty(G)$, the linear form $\pi^*_{\delta,\mu}(\varphi)\xi^{\underline{a}}_{\delta,\mu}$ is given (via the duality (4.3)) by the following function in $C(G, P, \delta^*, \mu)$:

$$\pi^*_{\delta,\mu}(\varphi)\xi^{\underline{a}}_{\delta,\mu}(g) = \int_G \varphi(y) \mathbb{P}^{\underline{a}}_{\delta,\mu}(gy) dy, \quad g \in G.$$

2) If $\Phi^\pm \in C_c^\infty(\Omega_p^\pm)$ we fix $\varphi^\pm \in C_c^\infty(G)$ such that $\Phi^\pm(g.I_p^\pm) = \int_{\mathcal{H}_p} \varphi^\pm(gh) d_p h$ for $g \in G$. Then

$$\tilde{Z}_p^\pm(\Phi^\pm, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu}) = \pi^*_{\delta,\mu}(\varphi^\pm)\xi^{\underline{a}}_{\delta,\mu}.$$

Proof.
1) Using Corollary 4.2.7, we obtain for $v \in I_{\delta,\mu}$:

$$\langle \pi^*_{\delta,\mu}(\varphi)\xi^{\underline{a}}_{\delta,\mu}, v\rangle = \int_G \varphi(y)\langle \pi^*_{\delta,\mu}(y)\xi^{\underline{a}}_{\delta,\mu}, v\rangle dy = \int_K (\int_G \varphi(y)\mathbb{P}^{\underline{a}}_{\delta,\mu}(ky)dy)v(k)dk.$$

Hence the linear form $\pi^*_{\delta,\mu}(\varphi)\xi^{\underline{a}}_{\delta,\mu}$ is given via the duality (4.3)) by the the function $g \mapsto \int_G \varphi(y)\mathbb{P}^{\underline{a}}_{\delta,\mu}(gy)dy$.
2) We only prove the assertion for $\tilde{Z}^+_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$, the proof for $\tilde{Z}^-_p(\Psi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$ is similar.

From the definition of $\tilde{Z}^+_p(\Phi^+, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$ and from 1) we get

$$\tilde{Z}^+_p(\Phi^+, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})(g) = \int_{G/\mathcal{H}_p} \Phi^+(yI_p^+)\mathbb{P}^{\underline{a}}_{\delta,\mu}(gy)dy = \int_{G/\mathcal{H}_p}(\int_{\mathcal{H}_p} \varphi^+(yh)d_ph)\mathbb{P}^{\underline{a}}_{\delta,\mu}(gy)dy$$
$$= \int_G \varphi^+(y)\mathbb{P}^{\underline{a}}_{\delta,\mu}(gy)dy = \pi^*_{\delta,\mu}(\varphi^+)\xi^{\underline{a}}_{\delta,\mu}(g).$$

□

Proposition 4.4.3. *Let δ be a unitary character of $(F^*)^{k+1}$ and let $\mu \in \mathbf{U} \cap \mathbf{C}^+$. Let also $p \in \{1, \dots, r_0\}$ and $\underline{a} \in \mathcal{S}_p$*
1) Let $\Phi^\pm \in \mathcal{S}(V^\pm)$. Then the map

$$\mu \mapsto \tilde{Z}^+_p(\Phi^+, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})_{|K} \in C(K, P \cap K, \delta^*)$$

extends meromorphically from the set $\{\mu \in \mathbf{U} \cap \mathbf{C}^+; Re(\mu_0) < -1 - \frac{kd}{4}\}$ to $\mathbf{U}$,
and the map

$$\mu \mapsto \tilde{Z}^-_p(\Phi^-, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})_{|K} \in C(K, P \cap K, \delta^*)$$

extends meromorphically from the set $\{\mu \in \mathbf{U} \cap \mathbf{C}^+; Re(\mu_k) > 1 + \frac{kd}{4}\}$ to $\mathbf{U}$.

2) Using the duality bracket (4.3), the linear form on $I_{\delta,\mu}$ represented by the functions $\tilde{Z}^\pm_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$ are given for $w \in I_{\delta,\mu}$, by

$$\langle \tilde{Z}^+_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu}), w\rangle = Z^+_p(\Phi, 0, \xi^{\underline{a}}_{\delta,\mu}, w), \quad \text{and} \quad \langle \tilde{Z}^-_p(\Psi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu}), w\rangle = Z^-_p(\Psi, 0, \xi^{\underline{a}}_{\delta,\mu}, w).$$

Proof. We only prove the two assertions for $\tilde{Z}^+_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$, the proof for $\tilde{Z}^-_p(\Psi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$ is similar.

1) The proof of Theorem 4.3.8 shows that

$$\tilde{Z}^+_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})(g) = c_{\delta,\mu}(\underline{a}) \int_{V^+} \Phi(g^{-1}X)\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X)d^*X$$
$$= c_{\delta,\mu}(\underline{a}) \sum_{x \in \mathscr{S}_e} \mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(g)\Phi, \omega(\delta), s(\mu) - m).$$

Considering the right hand side of the preceding equality, Theorem 3.2.3 implies that, for a fixed $g \in G$, the map

$$\mu \quad \longmapsto \tilde{Z}^+_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})(g)$$

is a rational function in the variables $q^{\pm\mu_j}$ and hence extends to $\mathbf{U}$ as a meromorphic function. As the map $g \mapsto \mathcal{L}(g)\Phi$ is locally constant (because the representation $\mathcal{L}$ on $\mathcal{S}(V^+)$ is smooth), the function $\tilde{Z}^+_p(\Phi, \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu})$ is continuous for any value of the parameter μ for which it is defined.

Moreover as $\tilde{Z}_p^+(\Phi,\pi_{\delta,\mu},\xi_{\delta,\mu}^{\underline{a}})(g)=c_{\delta,\mu}(\underline{a})\int_{V^+}\Phi(X)\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(gX)d^*X$ for $\mu\in\mathbf{U}\cap\mathbf{C}^+$ such that $\mathrm{Re}(\mu_0)<-1-\frac{kd}{4}$, Lemma 4.3.7 implies that this function belongs to $C(G,P,\delta^*,\mu)$. This remains true by analytic continuation.

2) Again from the proof of Theorem 4.3.8 we get:

$$\begin{aligned}Z_p^+(\Phi,z,\xi_{\delta,\mu}^{\underline{a}},w)&=c_{\delta,\mu}(\underline{a})\sum_{x\in\mathscr{S}_e}\int_K\mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(k)\Phi,\omega(\delta),s(\mu)-m)w(k)dk\\&=\int_K(c_{\delta,\mu}(\underline{a})\sum_{x\in\mathscr{S}_e}\mathcal{K}^+_{(x\underline{a},x)}(\mathcal{L}(k)\Phi,\omega(\delta),s(\mu)-m))w(k)dk\\&=\int_K\tilde{Z}_p^+(\Phi,\pi_{\delta,\mu},\xi_{\delta,\mu}^{\underline{a}})(k)w(k)dk\\&=\langle\tilde{Z}_p^+(\Phi,\pi_{\delta,\mu},\xi_{\delta,\mu}^{\underline{a}}),w\rangle\end{aligned}$$

□

Consider the representation $|\chi_0|^{-m}\otimes\pi_{\delta,\mu}$ of G on $I_{\delta,\mu}$ given by $|\chi_0|^{-m}\otimes\pi_{\delta,\mu}(g)w=|\chi_0(g)|^{-m}\pi_{\delta,\mu}(g)w$. If $p\in\{1,\dots,r_0\}$, the $\mathcal{H}_p$-invariant linear forms on $I_{\delta,\mu}$ are the same for $|\chi_0|^{-m}\otimes\pi_{\delta,\mu}$ and for $\pi_{\delta,\mu}$ and for $\underline{a}\in\mathcal{S}_p$, we have

$$(|\chi_0|^{-m}\otimes\pi_{\delta,\mu})^*(g)\xi_{\delta,\mu}^{\underline{a}}=|\chi_0(g)|^m\pi^*_{\delta,\mu}(g)\xi_{\delta,\mu}^{\underline{a}}.$$

From the definition, if $\Phi\in\mathcal{S}(V^+)$ and $w\in I_{\delta,\mu}$, we have

$$\begin{aligned}Z_p^+(\Phi,m,\xi_{\delta,\mu}^{\underline{a}},w)&=\int_{\Omega_p^+}\Phi(X)|\Delta_0(X)|^m\langle\pi^*_{\delta,\mu}(X)\xi_{\delta,\mu}^{\underline{a}},w\rangle d^*X\\&=\int_{\Omega_p^+}\Phi(X)\langle(|\chi_0|^{-m}\otimes\pi_{\delta,\mu})^*(X)\xi_{\delta,\mu}^{\underline{a}},w\rangle d^*X.\end{aligned}$$

As in Definition 4.4.1, for $\mu\in\mathbf{U}\cap\mathbf{C}^+$ such that $\mathrm{Re}(\mu_0)<\frac{kd}{4}$ and $\Phi\in\mathcal{S}(V^+)$, we define

$$\tilde{Z}_p^+(\Phi,|\chi_0|^{-m}\otimes\pi_{\delta,\mu},\xi_{\delta,\mu}^{\underline{a}})(g)=c_{\delta,\mu}(\underline{a})\int_{V^+}\Phi(g^{-1}X)|\Delta_0(g^{-1}X)|^m\mathbf{P}^{\underline{a},+}_{\omega(\delta),s(\mu)}(X)d^*X.\tag{4.6}$$

Then the same arguments as in the proof of Proposition 4.4.3 show that

$$\langle\tilde{Z}_p^+(\Phi,|\chi_0|^{-m}\otimes\pi_{\delta,\mu},\xi_{\delta,\mu}^{\underline{a}}),w\rangle=Z_p^+(\Phi,m,\xi_{\delta,\mu}^{\underline{a}},w).\tag{4.7}$$

Definition 4.4.4. *Let δ be a unitary character of $(F^*)^{k+1}$ and $\mu\in\mathbf{U}$. Let $\xi=(\xi_1,\dots,\xi_{r_0})\in\prod_{p=1}^{r_0}(I^*_{\delta,\mu})^{\mathcal{H}_p}$. For $\Phi\in\mathcal{S}(V^+)$, $\Psi\in\mathcal{S}(V^-)$, we define the following linear forms on $I_{\delta,\mu}$:*

$$\tilde{Z}^+(\Phi,|\chi_0|^{-m}\otimes\pi_{\delta,\mu},\xi)=\sum_{p=1}^{r_0}\tilde{Z}_p^+(\Phi,|\chi_0|^{-m}\otimes\pi_{\delta,\mu},\xi_p)$$

and also

$$\tilde{Z}^-(\Psi,\pi_{\delta,\mu},\xi)=\sum_{p=1}^{r_0}\tilde{Z}_p^-(\Psi,\pi_{\delta,\mu},\xi_p).$$

Theorem 4.4.5. (Main Theorem, version 2)

Let δ be a unitary character of $(F^*)^{k+1}$ and $\mu \in \mathbf{U}$. There exists an operator $A^{\delta,\mu} \in \mathrm{End}\Big(\prod_{p=1}^{r_0}(I^*_{\delta,\mu})^{\mathcal{H}_p}\Big)$ such that for all $\Phi \in \mathcal{S}(V^+)$ and all $\xi \in \prod_{p=1}^{r_0}(I^*_{\delta,\mu})^{\mathcal{H}_p}$ the following functional equation is satisfied:

$$\tilde{Z}^-(\mathcal{F}(\Phi), \pi_{\delta,\mu}, \xi) = \tilde{Z}^+(\Phi, |\chi_0|^{-m} \otimes \pi_{\delta,\mu}, A^{\delta,\mu}\xi).$$

The operator $A^{\delta,\mu}$ is represented by a square matrix of size r_0 whose coefficients $A^{\delta,\mu}_{p,q}$ belong to $\mathrm{Hom}\Big((I^*_{\delta,\mu})^{\mathcal{H}_q}, (I^*_{\delta,\mu})^{\mathcal{H}_p}\Big)$. In the bases $(\xi^{\underline{a}}_{\delta,\mu})_{\underline{a}\in\mathcal{S}_p}$ and $(\xi^{\underline{c}}_{\delta,\mu})_{\underline{c}\in\mathcal{S}_q}$ of $(I^*_{\delta,\mu})^{\mathcal{H}_p}$ and $(I^*_{\delta,\mu})^{\mathcal{H}_q}$ respectively, the matrix of of $A^{\delta,\mu}_{p,q}$ is given by:

$$\Big(A^{\delta,\mu}_{p,q}\Big)_{\underline{a},\underline{c}} = \mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(\frac{m+1}{2}),$$

where $\mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(z)$ has been defined in Theorem 4.3.9.

Proof.
Soit $p \in \{1,\dots,r_0\}$ et $\underline{a} \in \mathcal{S}_p$.
Using Proposition 4.4.3, Theorem 4.3.9 and (4.6), we obtain for $w \in I_{\delta,\mu}$:

$$\langle \tilde{Z}^-_p(\mathcal{F}(\Phi), \pi_{\delta,\mu}, \xi^{\underline{a}}_{\delta,\mu}), w\rangle = Z^-_p(\mathcal{F}(\Phi), 0, \xi^{\underline{a}}_{\delta,\mu}, w) = \sum_{\underline{c}\in\mathscr{S}^k_e} \mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(\frac{m+1}{2}) Z^+_{p_{\underline{c}}}(\Phi, m, \xi^{\underline{c}}_{\delta,\mu}, w)$$

$$= \sum_{q=1}^{r_0}\sum_{\underline{c}\in\mathcal{S}_q} \mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(\frac{m+1}{2}) Z^+_q(\Phi, m, \xi^{\underline{c}}_{\delta,\mu}, w)$$

$$= \sum_{q=1}^{r_0}\sum_{\underline{c}\in\mathcal{S}_q} \mathcal{B}_{\underline{a},\underline{c}}(\delta,\mu)(\frac{m+1}{2}) \langle \tilde{Z}^+_q(\Phi, \chi_0^{-m} \otimes \pi_{\delta,\mu}, \xi^{\underline{c}}_{\delta,\mu}), w\rangle$$

$$= \langle \tilde{Z}^+(\Phi, |\chi_0|^{-m} \otimes \pi_{\delta,\mu}, A^{\delta,\mu}\xi), w\rangle.$$

□

4.5. L-functions and ε-factors.

In this section we will always suppose that $e = 0$ or 4. This implies that the groups G and P have both a unique open orbit in $V^\pm$ which we denote by $\Omega^\pm$ and $\mathcal{O}^\pm$ respectively.
Then $\Omega^\pm \simeq G/H$ where H is the centralizer of $I^\pm$ in G.

We fix a unitary character $\delta = (\delta_0,\dots,\delta_k) \in \widehat{F^*}^{k+1}$ and $\mu \in \mathbf{U}$. As before we denote by $(\pi_{\delta,\mu}, I_{\delta,\mu})$ the minimal spherical principal series. As $e = 0$ or 4, the group S_e is equal to F^* and hence $\mathscr{S}_e = \{1\}$. Therefore, in what follows, we will omit the indexes in this space. The space $(I^*_{\delta,\mu})^H$ is 1-dimensional and we fix a non zero linear form ξ in this space.

The aim of this section is to prove an analogue of ([7] Theorem 3.3.3) for the minimal spherical principal series.

Definition 4.5.1.

(1) *An Euler factor is a function L on $\mathbb{C}$ of the form $L(s) = \dfrac{1}{P(q^{-s})}$ where $P \in \mathbb{C}[T]$ is a polynomial such that $P(0) = 1$.*

(2) *We denote by* $\mathcal{E}^{\pm}$ *the set of Eulor factors* $L^{\pm}(z)$ *such that for all* $\Phi^{\pm} \in \mathcal{S}(V^{\pm})$ *and all* $w \in I_{\delta,\mu}$ *the quotient* $\dfrac{Z^{\pm}(\Phi^{\pm}, z+\frac{1}{2}(m-1), \xi, w)}{L^{\pm}(z)}$ *is a polynomial in the variables* q^{-z} *and* q^{z}.

We know from Theorem 4.3.8, that there exist polynomials $\mathcal{R}^{\pm}(\delta,\mu,z)$ in the variable q^{-z}, which are products of polynomials of the form $(1-cq^{-Nz})$ ($c \in \mathbb{C}$ and $N \in \mathbb{N}^*$) such that, for all $w \in I_{\delta,\mu}$ and for all $\Phi^{\pm} \in \mathcal{S}(V^{\pm})$, the product $\mathcal{R}^{\pm}(\delta,\mu,z)Z^{\pm}(\Phi^{\pm},z,\xi,w)$ is a polynomial in the variables q^{-z} and q^{z}.

Hence $\mathcal{E}^{\pm} \neq \emptyset$.

Lemma 4.5.2.
(1) *Let* $L_0^+(z) = P_0(q^{-z})^{-1} \in \mathcal{E}^+$. *We denote by* $\mathcal{J}_{P_0}$ *the set of Laurent polynomials* $P \in \mathbb{C}[T,T^{-1}]$ *such that there exists finite families* $(\Phi_j)_{j\in J}$ *in* $\mathcal{S}(V^+)$ *and* $(w_j)_{j\in J}$ *in* $I_{\delta,\mu}$ *such that*

$$\sum_{j\in J} Z^+(\Phi_j, z+\frac{1}{2}(m-1),\xi,w_j) = \frac{P(q^{-z},q^{z})}{P_0(q^{-z})}.$$

Then $\mathcal{J}_{P_0}$ *is an ideal of* $\mathbb{C}[T,T^{-1}]$.
(2) *Let* $L_0^-(z) = Q_0(q^{-z})^{-1} \in \mathcal{E}^-$. *We denote by* $\mathcal{J}_{Q_0}$ *the set of Laurent polynomials* $Q \in \mathbb{C}[T,T^{-1}]$ *such that there exists finite families* $(\Psi_j)_{j\in J}$ *in* $\mathcal{S}(V^+)$ *and* $(w_j)_{j\in J}$ *in* $I_{\delta,\mu}$ *such that*

$$\sum_{j\in J} Z^-(\Psi_j, z+\frac{1}{2}(m-1),\xi,w_j) = \frac{Q(q^{-z},q^{z})}{Q_0(q^{-z})}.$$

Then $\mathcal{J}_{Q_0}$ *is an ideal of* $\mathbb{C}[T,T^{-1}]$.

Proof. We only prove assertion (1), the proof of (2) is similar. As $e=0$ or 4, one has $\chi_0(G) = F^*$ ([8] Theorem 3.8.8), and therefore there exists $g \in G$ such $|\chi_0(g)| = q^{-r}$ (for any $r \in \mathbb{Z}$).
Let $\Phi \in \mathcal{S}(V^+)$ and $w \in I_{\delta,\mu}$. Define:

$$\Phi_r(X) = \Phi(g^{-1}X)|\chi_0(g)|^{-\frac{1}{2}(m-1)} \text{ and } w_r = \pi_{\delta,\mu}(g)w$$

.
As $m = 1+\frac{kd}{2}$ in our case, we know from Theorem 4.3.8 (1)(a), that for $\mu = \mu_0\lambda_0 + \ldots + \mu_k\lambda_k \in \mathbf{U}\cap\mathbf{C}^+$ such that $\mathrm{Re}(z-\mu_0) > 1$, the zeta function $Z^+(\Phi_r, z+\frac{1}{2}(m-1),\xi,w_r)$ is defined by an absolutely convergent integral. More precisely we have:

$$\begin{aligned}
Z^+(\Phi_r, z+\frac{1}{2}(m-1),\xi,w_r) &= \int_{V^+} \Phi_r(X)|\Delta_0(X)|^{z+\frac{1}{2}(m-1)}\langle \pi^*_{\delta,\mu}(X)\xi, w_r\rangle d^*X \\
&= \int_{V^+} \Phi(g^{-1}X)|\chi_0(g)|^{-\frac{1}{2}(m-1)}|\Delta_0(X)|^{z+\frac{1}{2}(m-1)}\langle \pi^*_{\delta,\mu}(X)\xi, \pi_{\delta,\mu}(g)w\rangle d^*X \\
&= \int_{V^+} \Phi(g^{-1}X)|\chi_0(g)|^{-\frac{1}{2}(m-1)}|\Delta_0(X)|^{z+\frac{1}{2}(m-1)}\langle \pi^*_{\delta,\mu}(g^{-1}X)\xi, w\rangle d^*X \\
&= \int_{V^+} \Phi(X)|\chi_0(g)|^{z}|\Delta_0(X)|^{z+\frac{1}{2}(m-1)}\langle \pi^*_{\delta,\mu}(X)\xi, w\rangle d^*X \\
&= q^{-zr}Z^+(\Phi, z+\frac{1}{2}(m-1),\xi,w).
\end{aligned}$$

By analytic continuation we obtain the following equality of rational functions:

$$Z^+(\Phi_r, z+\frac{1}{2}(m-1),\xi,w_r) = q^{-rz}Z^+(\Phi, z+\frac{1}{2}(m-1),\xi,w)$$

This shows that if $P(T,T^{-1}) \in \mathcal{J}_{P_0}$ then, for all $r \in \mathbb{Z}$, $T^r P(T,T^{-1}) \in \mathcal{J}_{P_0}$. Hence $\mathcal{J}_{P_0}$ is an ideal (the fact that $\mathcal{J}_{P_0}$ is stable under addition is obvious from the definition).

□

Proposition 4.5.3.

(1) *There exist unique Euler factors $L^{\pm}(\pi_{\delta,\mu}, z)$ (called L-functions) satisfying the two following conditions:*

(a) For all $\Phi^{\pm} \in \mathcal{S}(V^{\pm})$ and $w \in I_{\delta,\mu}$, the quotients

$$\frac{Z^{\pm}(\Phi^{\pm}, z+\frac{m-1}{2}, \xi, w)}{L^{\pm}(\pi_{\delta,\mu}, z)}$$

are polynomials in the variables q^z and q^{-z},

(b) There exists two finite families $(\Phi_i^{\pm}, v_i)_{i\in I}$ in $\mathcal{S}(V^{\pm}) \times I_{\delta,\mu}$ such that

$$\sum_{i\in I} \frac{Z^{\pm}(\Phi_i^{\pm}, z+\frac{m-1}{2}, \xi, v_i)}{L^{\pm}(\pi_{\delta,\mu}, z)} = 1.$$

(2) *Moreover, if $L^{\pm} \in \mathcal{E}^{\pm}$, then the quotients $\dfrac{L^{\pm}(\pi_{\delta,\mu}, z)}{L^{\pm}(z)}$ are polynomials in q^{-z}.*

Proof.
We only prove the result for $L^{+}(\pi_{\delta,\mu}, z)$. The proof for $L^{-}(\pi_{\delta,\mu}, z)$ is similar.

Let $L_0(z) = P_0(q^{-z})^{-1} \in \mathcal{E}^{+}$. From Lemma 4.5.2, the set $\mathcal{J}_{P_0}$ is an ideal of the principal ideal domain $\mathbb{C}[T,T^{-1}]$. Hence there exists a polynomial $R_0 \in \mathbb{C}[T]$ such that $\mathcal{J}_{P_0} = R_0(T)\mathbb{C}[T,T^{-1}]$ and we can suppose that either $R_0 = 0$ or $R_0(0) = 1$.

We show first that $R_0 \neq 0$.
Let $w \in I_{\delta,\mu}$ such that $\langle \xi, w\rangle \neq 0$. As w is right invariant under an open compact subgroup, there exists an open compact neighborhood $\mathcal{V}$ of I^{+} in Ω^{+} such that for all $X \in \mathcal{V}$, $\langle \pi^{*}_{\delta,\mu}(X)\xi, w\rangle = \langle \xi, w\rangle \neq 0$. Let $\Phi \in \mathcal{S}(V^{+})$ be the characteristic function of $\mathcal{V}$. Then for all $z \in \mathbb{C}$, we have $Z^{+}(\Phi, z+\frac{1}{2}(m-1), \xi, w) = \langle \xi, w\rangle \int_{\mathcal{V}} |\Delta_0(X)|^{z+\frac{1}{2}(m-1)} d^{*}X \neq 0$. Hence $\mathcal{J}_{P_0} \neq \{0\}$ and $R_0 \neq 0$.

Therefore we can take $R_0(0) = 1$.

Let $Q(T)$ be the HCF of R_0 and P_0 normalized by the condition $Q(0) = 1$. There exist coprime polynomials R_1 and P_1 de $\mathbb{C}[T]$ such that $R_0 = R_1 Q$, $P_0 = P_1 Q$ and $P_1(0) = R_1(0) = 1$.

If $R_1 \neq 1$ then R_1 has a non zero root, and it exists $z_1 \in \mathbb{C}$ such that

$$R_1(q^{-z_1}) = 0, \quad \text{and} \quad P_1(q^{-z_1}) \neq 0.$$

In that case, for all $\Phi \in \mathcal{S}(V^{+})$ and $w \in I_{\delta,\mu}$ we would have $Z^{+}(\Phi, z_1 + \frac{1}{2}(m-1), \xi, w) = 0$, and this is not possible (take for example w and Φ as above).

Therefore $R_1 = 1$, $Q = R_0$ and $P_0 = P_1 R_0$. Hence for any $P \in \mathbb{C}[T, T^{-1}]$, there exist finite families $(\Phi_i)_{i\in I}$ in $\mathcal{S}(V^+)$ and $(w_i)_{i\in I}$ in $I_{\delta,\mu}$ such that

$$\sum_{i\in I} Z^+(\Phi_i, z + \frac{1}{2}(m-1), \xi, w_i) = \frac{R_0(T)P(T,T^{-1})}{P_0(T)} = \frac{P(T,T^{-1})}{P_1(T)}.$$

Define $L^+(\pi_{\delta,\mu}, z) = \dfrac{1}{P_1(q^{-z})}$. Then from the discussion above we see that this Euler factor satisfies condition (1) (a). Moreover if we take $P = 1$ in the equation above we obtain the condition (1)(b).
If $\tilde{L}$ is another Euler factor satisfying conditions (1) (a) (1)(b), then $\dfrac{L^+(\pi_{\delta,\mu}, z)}{\tilde{L}(z)} \in \mathbb{C}[q^{-z}]$ and $\dfrac{\tilde{L}(z)}{L^+(\pi_{\delta,\mu}, z)} \in \mathbb{C}[q^{-z}]$. Hence $\tilde{L}(z) = L(\pi_{\delta,\mu}, z)$ and the uniqueness is proved.
Let us show assertion (2). From the construction of $L(\pi_{\delta,\mu}, z)$ above we see that we can take $L^+ = L_0$. Then $\dfrac{L(\pi_{\delta,\mu}, z)}{L_0(z)} = \dfrac{P_0(q^{-z})}{P_1(q^{-z})} = R_0(q^{-z})$.

□

Definition 4.5.4.
If $w \in I_{\delta,\mu}$, $\Phi \in \mathcal{S}(V^+)$ and $\Psi \in \mathcal{S}(V^-)$, we define

$$\Xi^+(\Phi, z, \xi, w) = \frac{Z^+(\Phi, z + \frac{1}{2}(m-1), \xi, w)}{L^+(\pi_{\delta,\mu}, z)},$$

and

$$\Xi^-(\Psi, z, \xi, w) = \frac{Z^-(\Psi, z + \frac{1}{2}(m-1), \xi, w)}{L^-(\pi_{\delta,\mu}, z)}.$$

From Proposition 4.5.3, these two functions are polynomials in q^{-z} and q^z.

If $k = 0$ (that is in the case of Tate's theory) we denote by $L_0(\delta, z)$ the L-function associated to the character $t \longmapsto \delta(t)|t|^z$ (remember that here δ is a character of F^*). It is given by

$$L_0(\delta, z) = \begin{cases} (1 - \delta(\pi)q^{-z})^{-1} & \text{if } \delta \text{ is unramified} \\ 1 & \text{if } \delta \text{ is ramified,} \end{cases} \tag{4.8}$$

(see [5] (23.4.1))
We define now the ε-factor $\varepsilon_0(\delta, z, \psi)$ by the equation:

$$\delta(-1)\rho(\delta^{-1}, 1-z) = \rho(\delta, z)^{-1} = \frac{L_0(\delta^{-1}, 1-z)\varepsilon_0(\delta, z, \psi)}{L_0(\delta, z)}, \tag{4.9}$$

where Tate's ρ function was defined in section 3.3. (This is the Definition given in ([5], §23.4 p.142), taking into account that our ρ is the inverse of the γ function of Bushnell and Henniart).
The function $\varepsilon_0(z, \delta, \psi)$ is explicitly known ([5] Theorem 23.4 p.144). It is always of the form $c_0 q^{-n_0 z}$ where $c_0 \in \mathbb{C}$ and $n_0 \in \mathbb{Z}$.
Moreover we know from ([5] (23.4.2), p.142) that

$$\varepsilon_0(\delta, z, \psi) = \frac{\delta(-1)}{\varepsilon_0(\delta^{-1}, 1-z, \psi)} \tag{4.10}$$

Theorem 4.5.5.
(1) *Let $\Phi \in \mathcal{S}(V^+)$ and $\Psi \in \mathcal{S}(V^-)$. The functions $\Xi^+(\Phi, z, \xi, w)$ and $\Xi^-(\Psi, z, \xi, w)$ satisfy the following functional equations:*

$$\Xi^-(\mathcal{F}(\Phi), 1-z, \xi, w) = \varepsilon^+(\pi_{\delta,\mu}, z, \psi)\Xi^+(\Phi, z, \xi, w),$$

$$\Xi^+(\mathcal{F}(\Psi), 1-z, \xi, w) = \varepsilon^-(\pi_{\delta,\mu}, z, \psi)\Xi^-(\Psi, z, \xi, w),,$$

where

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi) = \gamma_\psi(q)^{\frac{k(k+1)}{2}} \frac{L^+(\pi_{\delta,\mu}, z)}{L^-(\pi_{\delta,\mu}, 1-z)} \prod_{j=0}^{k} \frac{L_0(\delta_j, 1-(z-\mu_j))}{L_0(\delta_j^{-1}, z-\mu_j)} \varepsilon_0(\delta_j^{-1}, z-\mu_j, \psi),$$

and

$$\varepsilon^-(\pi_{\delta,\mu}, z, \psi) = \gamma_\psi(q)^{-\frac{k(k+1)}{2}} \frac{L^-(\pi_{\delta,\mu}, z)}{L^+(\pi_{\delta,\mu}, 1-z)} \prod_{j=0}^{k} \frac{L_0(\delta_j^{-1}, 1-(\mu_j+z))}{L_0(\delta_j, \mu_j+z)} \varepsilon_0(\delta_j, \mu_j+z, \psi).$$

(2) *The ε-factors $\varepsilon^+(\pi_{\delta,\mu}, z, \psi)$ and $\varepsilon^-(\pi_{\delta,\mu}, z, \psi)$ are monomials of the form cq^{-nz} where $c \in \mathbb{C}$ and $n \in \mathbb{Z}$, and they satisfy the relation:*

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi)\varepsilon^-(\pi_{\delta,\mu}, 1-z, \psi) = (\delta_0 \dots \delta_k)(-1).$$

Proof.
Let $\Phi \in \mathcal{S}(V^+)$. Corollary 4.3.11 implies that

$$\Xi^-(\mathcal{F}(\Phi), 1-z, \xi, w) = \frac{1}{L^-(\pi_{\delta,\mu}, 1-z)} Z^-(\mathcal{F}(\Phi), 1-z+\frac{1}{2}(m-1), \xi, w)$$

$$= \frac{d(\delta,\mu,z)}{L^-(\pi_{\delta,\mu}, 1-z)} Z^+(\Phi, z+\frac{1}{2}(m-1), \xi, w) = d(\delta,\mu,z)\frac{L^+(\pi_{\delta,\mu}, z)}{L^-(\pi_{\delta,\mu}, 1-z)}\Xi^+(\Phi, z, \xi, w),$$

where $d(\delta,\mu,z) = \gamma_\psi(q)^{\frac{k(k+1)}{2}} \prod_{j=0}^{k} \delta_j(-1)\rho(\delta_j, \mu_j+1-z)$. We define then

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi) = d(\delta,\mu,z)\frac{L^+(\pi_{\delta,\mu}, z)}{L^-(\pi_{\delta,\mu}, 1-z)}. \tag{4.11}$$

From (4.9) and (4.10) we obtain

$$d(\delta,\mu,z) = \gamma_\psi(q)^{\frac{k(k+1)}{2}} \prod_{j=0}^{k} \varepsilon_0(\delta_j^{-1}, z-\mu_j, \psi)\frac{L_0(\delta_j, 1-(z-\mu_j))}{L_0(\delta_j^{-1}, z-\mu_j)},$$

and this proves the first functional equation.

Let now $\Psi \in \mathcal{S}(V^-)$. Applying Corollary 4.3.11 to the function $\Phi = \mathcal{F}(\Psi)$, we get

$$Z^-(\mathcal{F}\circ\mathcal{F}(\Psi), \frac{m+1}{2} - z, \xi, w) = d(\delta,\mu,z)Z^+(\mathcal{F}(\Psi), z+\frac{m-1}{2}, \xi, w). \tag{4.12}$$

As $\mathcal{F}\circ\mathcal{F}(\Psi)(Y) = \Psi(-Y)$, Theorem 4.3.8 (2)(a) implies that for $z \in \mathbb{C}$ and $\mu \in \mathbf{U} \cap \mathbf{C}^+$ such that $\mathrm{Re}(\mu_k - z) > 0$, we have:

$$Z^-(\mathcal{F}\circ\mathcal{F}(\Psi), \frac{m+1}{2} - z, \xi, w) = \int_{V^-} \Psi(-Y)|\nabla_0(Y)|^{\frac{m+1}{2}-z}\langle\pi^*(Y)\xi, w\rangle d^*Y.$$

Remember that there exists an element $m_{-1} \in G$ which acts by -1 on V^+ and V^- and trivially on $\mathfrak{g}$ (cf. Definition 1.1.5). Hence m_{-1} is central in G and from the definition we have $\pi_{\delta,\mu}(m_{-1}) = \prod_{j=0}^{k} \delta_j(-1)\mathrm{Id}_{\mathrm{I}_{\delta,\mu}}$. Therefore

$$Z^-(\mathcal{F}\circ\mathcal{F}(\Psi), \frac{m+1}{2} - z, \xi, w) = (\prod_{j=0}^{k} \delta_j(-1)) Z^-(\Psi, \frac{m+1}{2} - z, \xi, w). \tag{4.13}$$

By analytic continuation, this equality between these two rational functions in the variables q^{-z} and q^z remains true for all $\mu \in \mathbf{U}$. From (4.12), we get

$$Z^+(\mathcal{F}(\Psi), z + \frac{m-1}{2}, \xi, w) = \frac{(\delta_0 \dots \delta_k)(-1)}{d(\delta,\mu,z)} Z^-(\Psi, \frac{m+1}{2} - z, \xi, w).$$

Making the change of variable $z \to 1 - z$, we get

$$Z^+(\mathcal{F}(\Psi), \frac{m+1}{2} - z, \xi, w) = \frac{(\delta_0 \dots \delta_k)(-1)}{d(\delta,\mu,1-z)} Z^-(\Psi, z + \frac{m-1}{2}, \xi, w),$$

and hence

$$\Xi^+(\mathcal{F}(\Psi), 1-z, \xi, w) = \frac{(\delta_0 \dots \delta_k)(-1)}{d(\delta,\mu,1-z)} \frac{L^-(\pi_{\delta,\mu}, z)}{L^+(\pi_{\delta,\mu}, 1-z)} \Xi^-(\Psi, z, \xi, w).$$

Therefore we set:

$$\varepsilon^-(\pi_{\delta,\mu}, z, \psi) = \frac{(\delta_0 \dots \delta_k)(-1)}{d(\delta,\mu,1-z)} \frac{L^-(\pi_{\delta,\mu}, z)}{L^+(\pi_{\delta,\mu}, 1-z)}. \tag{4.14}$$

From the knowledge of $d(\delta,\mu,z)$ (Corollary 4.3.11) and (4.9), we obtain

$$\begin{aligned}\frac{(\delta_0 \dots \delta_k)(-1)}{d(\delta,\mu,1-z)} &= \gamma_\psi(q)^{-\frac{k(k+1)}{2}} \prod_{j=0}^{k} \rho(\delta_j, \mu_j + z)^{-1} \\ &= \gamma_\psi(q)^{-\frac{k(k+1)}{2}} \prod_{j=0}^{k} \frac{L_0(\delta_j^{-1}, 1-(\mu_j+z))}{L_0(\delta_j, \mu_j + z)} \epsilon_0(\delta_j, \mu_j + z, \psi).\end{aligned}$$

This gives the second functional equation and ends the proof of assertion (1).

Let $\Phi \in \mathcal{S}(V^+)$. Define $\Psi(Y) = \mathcal{F}(\Phi)(-Y)$, then $\Phi = \mathcal{F}(\Psi)$. From (4.13), we get

$$\Xi^-(\mathcal{F}(\Phi), 1-z, \xi, w) = (\delta_0 \dots \delta_k)(-1)\Xi^-(\Psi, 1-z, \xi, w). \tag{4.15}$$

Using the two functional equations and (4.15) we obtain

$$\begin{aligned}\Xi^-(\mathcal{F}(\Phi), 1-z, \xi, w) &= \varepsilon^+(\pi_{\delta,\mu}, z, \psi)\Xi^+(\Phi, z, w) = \varepsilon^+(\pi_{\delta,\mu}, z, \psi)\Xi^+(\mathcal{F}(\Psi), z, w) \\ &= \varepsilon^+(\pi_{\delta,\mu}, z, \psi)\varepsilon^-(\pi_{\delta,\mu}, 1-z, \psi)\Xi^-(\Psi, 1-z, \xi, w) \\ &= (\delta_0 \dots \delta_k)(-1)\varepsilon^+(\pi_{\delta,\mu}, z, \psi)\varepsilon^-(\pi_{\delta,\mu}, 1-z, \psi)\Xi^-(\mathcal{F}(\Phi), 1-z, \xi, w),\end{aligned}$$

which implies that

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi)\varepsilon^-(\pi_{\delta,\mu}, 1-z, \psi) = (\delta_0 \dots \delta_k)(-1). \tag{4.16}$$

We know from Proposition 4.5.3, that there exist finite families $(\Phi_i^\pm)_{i\in I} \in \mathcal{S}(V^\pm)$ and $(w_i^\pm)_{i\in I} \in I_{\delta,\mu}$ such that $\sum_{i\in I} \Xi^\pm(\Phi_i^\pm, z, \xi, w_i^\pm) = 1$ and such that $\Xi^\pm(\mathcal{F}(\Phi_i^\pm), z, \xi, w_i^\pm)$ are polynomials in q^{-z} and q^z. The two preceding functional equations imply then that $\varepsilon^\pm(\pi_{\delta,\mu}, z, \psi)$ are polynomials in q^{-z} and q^z. Finally equation (4.16) implies that there are in fact monomials in q^{-z}. This ends the proof of the Theorem. □

Remark 4.5.6. The relation

$$\varepsilon^+(\pi_{\delta,\mu},z,\psi)\varepsilon^-(\pi_{\delta,\mu},1-z,\psi)=(\delta_0\dots\delta_k)(-1)$$

generalizes the one for $GL(n,F)$ (see [7] p.33 and [5] (23.4.2) p. 142)

Corollary 4.5.7. .
(1) There exists a positive integer $d(\delta,\mu)$ and a polynomial $Q^+_{\delta_\mu}(T)=\prod_{r=1}^{d(\delta,\mu)}(1-a_r(\delta_\mu)T)\in\mathbb{C}[T]$ such that

$$L^+(\pi_{\delta,\mu},z)=\frac{1}{Q^+_{\delta_\mu}(q^{-z})}\prod_{j=0}^{k}L_0(\delta_j^{-1},z-\mu_j).$$

Then if we set $Q^-_{\delta_\mu}(T)=\prod_{r=1}^{d(\delta,\mu)}(1-a_r(\delta_\mu)^{-1}qT)$ we have also

$$L^-(\pi_{\delta,\mu},z)=\frac{1}{Q^-_{\delta_\mu}(q^{-z})}\prod_{j=0}^{k}L_0(\delta_j,z+\mu_j).$$

(2) Let $\delta\in\widehat{\mathcal{O}_F^*}^{k+1}$. There exist roots of the unit u_r and rational numbers $p_r,p_{r,j}\in\mathbb{Q}$ ($r=1,\dots,d(\delta,\mu)$ and $j=0,\dots,k$), such that

$$a_r(\delta_\mu)=u_rq^{-p_r-\sum_{j=0}^k p_{r,j}\mu_j}.$$

Proof. From the definition of Euler factor, there exist polynomials $P^\pm_{\delta_\mu}(T)\in\mathbb{C}[T]$ such that $P^\pm_{\delta_\mu}(0)=1$ and $L^\pm(\pi_{\delta,\mu},z)=\dfrac{1}{P^\pm_{\delta_\mu}(q^{-z})}$. Let us denote by

$$P^+_{\delta,\mu}(T)=\prod_{r=1}^{N}(1-a_rT),\quad\text{and}\quad P^-_{\delta,\mu}(T)=\prod_{t=1}^{N'}(1-b_tT),\qquad(a_r,b_t\in\mathbb{C})$$

the decomposition into prime factors of these two polynomials.

As the factors $\varepsilon_0(\delta_j^{-1},z-\mu_j,\psi)$ and $\varepsilon^+(\pi_{\delta,\mu},z,\psi)$ are monomials in q^{-z} (Theorem 4.5.5 (2)), we obtain that there exists $C\in\mathbb{C}$ and $n\in\mathbb{Z}$ such that

$$Cq^{-nz}=\frac{L^+(\pi_{\delta,\mu},z)}{L^-(\pi_{\delta,\mu},1-z)}\prod_{j=0}^{k}\frac{L_0(\delta_j,1-(z-\mu_j))}{L_0(\delta_j^{-1},z-\mu_j)}.$$

Here $L_0(\chi,s)=P_0(\chi,q^{-s})^{-1}$ where $P_0(\chi,T)\in\mathbb{C}[T]$ and $P_0(\chi,T)=1$ if χ is ramified and $P_0(\chi,T)=1-\chi(\pi)T$ if χ is non ramified (see (4.8)).

Then, we obtain easily

$$CT^nP^+_{\delta_\mu}(T)\prod_{j=0}^{k}P_0(\delta_j,q^{-1-\mu_j}T^{-1})=P^-_{\delta_\mu}(q^{-1}T^{-1})\prod_{j=0}^{k}P_0(\delta_j^{-1},q^{\mu_j}T)\tag{4.17}$$

And if all the δ_j's are ramified, the preceding relation becomes

$$CT^n\prod_{r=1}^{N}(1-a_rT)=\prod_{t=1}^{N'}(1-b_t(qT)^{-1})=T^{-N'}\prod_{t=1}^{N'}(-b_tq^{-1})\prod_{t=1}^{N'}(1-b_t^{-1}qT).$$

This implies that $N=N'$ and that $\{a_1,\dots,a_N\}=\{qb_1^{-1},\dots,qb_N^{-1}\}$, and hence assertion (1) is proved in this case by taking $Q^\pm_{\delta_\mu}(T)=P^\pm_{\delta_\mu}(T)$.

Suppose now that at least one δ_j is non ramified. Up to a change of the indexation of the δ_j's, one can suppose that there exists $k_0 \geq 0$ such that δ_j is non ramified for $0 \leq j \leq k_0$ and δ_j is ramified for $j > k_0$.
Equation (4.17) becomes then

$$\begin{aligned} P^-_{\delta_\mu}(q^{-1}T^{-1}) \prod_{j=0}^{k_0}(1-\delta_j^{-1}(\pi)q^{\mu_j}T) &= CT^n P^+_{\delta_\mu}(T) \prod_{j=0}^{k_0}(1-\delta_j(\pi)q^{-1-\mu_j}T^{-1}) \\ &= CT^{n-(k_0+1)} \left(\prod_{j=0}^{k_0}(-\delta_j(\pi)q^{-1-\mu_j}\right) P^+_{\delta_\mu}(T) \prod_{j=0}^{k_0}(1-\delta_j(\pi)^{-1}q^{1+\mu_j}T). \end{aligned} \tag{4.18}$$

This shows that the polynomial $\prod_{j=0}^{k_0}(1-\delta_j^{-1}(\pi)q^{\mu_j}T)$ divides $P^+_{\delta_\mu}(T)$. We set

$$Q^+_{\delta_\mu}(T) = \frac{P^+_{\delta_\mu}(T)}{\prod_{j=0}^{k_0}(1-\delta_j^{-1}(\pi)q^{\mu_j}T)}.$$

Relation (4.18) can now be written as

$$P^-_{\delta_\mu}(q^{-1}T^{-1}) = CT^n Q^+_{\delta_\mu}(T) \prod_{j=0}^{k_0}(1-\delta_j(\pi)q^{-1-\mu_j}T^{-1}).$$

Therefore $\prod_{j=0}^{k_0}(1-\delta_j(\pi)q^{-\mu_j}T)$ divides $P^-_{\delta_\mu}(T)$ and we set

$$Q^-_{\delta_\mu}(T) = \frac{P^-_{\delta_\mu}(T)}{\prod_{j=0}^{k_0}(1-\delta_j(\pi)q^{-\mu_j}T)}.$$

Then

$$CT^n Q^+_{\delta_\mu}(T) = Q^-_{\delta_\mu}(q^{-1}T^{-1}).$$

As before, we obtain then the asserted form for the functions $L^{\pm}(\pi_{\delta,\mu}, z)$.

Let us now show assertion (2).
From Theorem 3.2.3, we know that for $\omega \in \widehat{\mathscr{O}_F^*}^{k+1}$, there exists a polynomial

$$R^+(\omega, s) = \prod_{r=1}^{d_0}(1-q^{-N_r-\sum_{j=0}^k N_{r,j}s_j}), \quad N_r \in \mathbb{N}, N_{r,j} \in \mathbb{N} \tag{4.19}$$

such that, for all $\Phi \in \mathcal{S}(V^+)$, the function $R^+(\omega, s)\mathcal{K}^+(\Phi, \omega, s)$ is a polynomial is the variables q^{s_j} and q^{-s_j}.
Remember that $\omega(\delta) = (\omega_0, \ldots, \omega_k) = (\delta_0^{-1}, \delta_0\delta_1^{-1}, \ldots, \delta_{k-1}\delta_k^{-1})$ and that $s(\mu) = (s_0, \ldots, s_k)$ is defined by the relations $s_0 + \ldots + s_j = \frac{d}{4}(k-2j) - \mu_j$ (see Definition 4.3.3). Theorem 4.3.8 implies that for $\Phi \in \mathcal{S}(V^+)$ $w \in I_{\delta,\mu}$ and $\xi \in (I^*_{\delta,\mu})^H$, the function $R^+(\omega(\delta), s(\mu) + z - m)Z^+(\Phi, z, \xi, w)$ is a polynomial in q^{-z} and q^z.
Set $\mathcal{P}(T) = \prod_{r=1}^{d_0}(1 - c_r(\delta_\mu)T^{N_{r,0}})$ with $c_r(\delta_\mu) = q^{-N_r + \frac{N_{r,0}}{2}(m+1) - \sum_{j=0}^k N_{r,j}s_j}$. Then $\mathcal{P}(q^{-z}) = R^+(\omega(\delta), s(\mu) + z - \frac{1}{2}(m+1))$.
As $z + \frac{1}{2}(m-1) - m = z - \frac{1}{2}(m+1)$, it is easily seen that the function $L^+(z) = \dfrac{1}{\mathcal{P}(q^{-z})}$ is an Euler factor in $\mathcal{E}^+$ (see Definition 4.5.1 (2)).

Then from Proposition 4.5.3, 2), the polynomial $P^+_{\delta,\mu}$ divides $\mathcal{P}$ in $\mathbb{C}[T]$. Hence, up to permutations of the families $(N_r, N_{r,0}, \ldots, N_{r,k})$, there exists $r_0^+ \leq d_0$, and for all $r \in \{1, \ldots, r_0^+\}$, a finite family

U_r^+ of $N_{r,0}$-th root of unity such that

$$P^+_{\delta,\mu}(T) = \prod_{r=1}^{r_0^+} \prod_{u\in U_r^+} (1 - u\, b_r(\delta_\mu)T), \quad \text{with} \quad b_r(\delta_\mu) = c_r(\delta_\mu)^{1/N_{r,0}} = q^{-\frac{N_r}{N_{r,0}} + \frac{m+1}{2} - \sum_{j=0}^{k} \frac{N_{r,j}}{N_{r,0}} s_j}.$$

As $s_0 = \dfrac{kd}{4} - \mu_0$ and $s_j = \mu_{j-1} - \mu_j - \dfrac{d}{2}$, we obtain assertion (2). □

We will now describe how the ε-factors $\varepsilon^\pm(\pi_{\delta,\mu}, z, \psi)$ depend on the additive character ψ.

Let $a \in F^*$ Remember that m_a is the element in G which acts by multiplication by a on V^+, by multiplication by a^{-1} on V^- and trivially on $\mathfrak{g}$ (Definition 1.1.5). Therefore m_a is central in G. Let $\mu = \mu_0\lambda_0 + \ldots \mu_k\lambda_k$ and define $\varpi(a) = (\delta_0 \ldots \delta_k)(a)|a|^{\mu_0+\ldots+\mu_k}$. Then $\pi_{\delta,\mu}(m_a) = \varpi(a)\mathrm{Id}_{I_{\delta,\mu}}$.

Proposition 4.5.8. . *Let $a \in F^*$. Denote by ψ^a the character of F given by $\psi^a(t) = \psi(at)$. Let $(d^aX, d^aY) = (c_a dX, d_a dY)$, with $c_a, d_a > 0$, a pair of measures on (V^+, V^-) which are dual for the Fourier transform $\mathcal{F}^a$ defined by ψ^a. Then $c_a d_a = |a|^{\dim V^+}$ and*

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi^a) = \frac{\varpi(a)^{-1}|a|^{(k+1)(\frac{m-1}{2}+z)}}{c_a}\varepsilon^+(\pi_{\delta,\mu}, z, \psi^a),$$
$$\varepsilon^-(\pi_{\delta,\mu}, z, \psi^a) = c_a\varpi(a)|a|^{-(k+1)(\frac{m+1}{2}-z)}\varepsilon^-(\pi_{\delta,\mu}, z, \psi).$$

If $d_a = c_a = |a|^{\frac{\dim V^+}{2}} = |a|^{m\frac{(k+1)}{2}}$, the preceding formulas become

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi^a) = \varpi(a)^{-1}|a|^{(k+1)(z-\frac{1}{2})}\varepsilon^+(\pi_{\delta,\mu}, z, \psi),$$
$$\varepsilon^-(\pi_{\delta,\mu}, z, \psi^a) = \varpi(a)|a|^{(k+1)(z-\frac{1}{2})}\varepsilon^-(\pi_{\delta,\mu}, z, \psi).$$

Proof.
From the definitions we have:

$$\mathcal{F}^a(\Phi)(Y) = c_a \int_{V^+} \Phi(X)\psi(a\, b(X,Y))dX = c_a\mathcal{F}(\Phi)(aY). \tag{4.20}$$

It is easily seen that the measures $c_a dX$ and $d_a dY$ are dual for $\mathcal{F}^a$ if and only if

$$c_a d_a = |a|^{\dim V^+}.$$

From (4.11) and (4.14), the functions $\varepsilon^\pm(\pi_{\delta,\mu}, z, \psi)$ are uniquely determined by $d(\delta, \mu, z)$ which appears in the functional equation satisfied by $Z^\pm(\Phi, z, \xi, w)$. The definition of these functions depends on the choice of the dual measures (dX, dY) on $V^+ \times V^-$. Let $Z^{a,\pm}(\Phi, z, \xi, w)$ be the new zeta functions relative to $(c_a dX, d_a dY)$. Then $Z^{1,\pm}(\Phi, z, \xi, w) = Z^\pm(\Phi, z, \xi, w)$.

Let $\Phi \in \mathcal{S}(V^+)$ and $w \in I_{\delta,\mu}$. Let also $z \in \mathbb{C}$ such that $\mathrm{Re}(\mu_k - z) > 0$. Then $Z^-(\mathcal{F}(\Phi), \frac{1}{2}(m+1) - z, \xi, w)$ is given by an integral (Theorem 4.3.8 (2)(a)). Using (4.20) and the G-invariance of d^*Y under the central element m_a, we obtain:

$$Z^{a,-}(\mathcal{F}^a(\Phi), \frac{m+1}{2} - z, \xi, w) = Z^{a,-}(c_a\mathcal{F}(\Phi)(a\,\cdot)), \frac{m+1}{2} - z, \xi, w)$$
$$= |a|^{\dim V^+} \int_{V^-} \mathcal{F}(\Phi)(m_a^{-1}.Y)|\nabla_0(Y)|^{\frac{m+1}{2}-z}\langle\pi^*_{\delta,\mu}(Y)\xi, w\rangle d^*Y$$
$$= |a|^{\dim V^+} \int_{V^-} \mathcal{F}(\Phi)(Y)\frac{1}{|a|^{(k+1)\left(\frac{m+1}{2}-z\right)}}|\nabla_0(Y)|^{\frac{m+1}{2}-z}\langle\pi^*_{\delta,\mu}(m_a.Y)\xi, w\rangle d^*Y$$

$$= |a|^{\dim V^+} \int_{V^-} \mathcal{F}(\Phi)(Y) \frac{1}{|a|^{(k+1)\left(\frac{m+1}{2}-z\right)}} |\nabla_0(Y)|^{\frac{m+1}{2}-z} \langle \pi^*_{\delta,\mu}(Y)\xi, \pi_{\delta,\mu}(m_a^{-1})w\rangle d^*Y$$

$$= \varpi(a)^{-1}|a|^{\dim V^+} \int_{V^-} \mathcal{F}(\Phi)(Y) \frac{1}{|a|^{(k+1)\left(\frac{m+1}{2}-z\right)}} |\nabla_0(Y)|^{\frac{m+1}{2}-z} \langle \pi^*_{\delta,\mu}(Y)\xi, w\rangle d^*Y$$

As $m = \dfrac{\dim V^+}{k+1}$ we have

$$Z^{a,-}(\mathcal{F}^a(\Phi), \frac{m+1}{2} - z, \xi, w) = \varpi(a)^{-1}|a|^{(k+1)(\frac{m-1}{2}+z)} Z^-(\mathcal{F}(\Phi), \frac{m+1}{2} - z, \xi, w).$$

Therefore $Z^{a,-}(\mathcal{F}^a(\Phi), \frac{m+1}{2} - z, \xi, w) = d^a(\delta, \mu, z) Z^{a,+}(\Phi, z + \frac{m-1}{2}, \xi, w)$ where

$$d^a(\delta, \mu, z) = d(\delta, \mu, z) \frac{\varpi(a)^{-1}|a|^{(k+1)(\frac{m-1}{2}+z)}}{c_a},$$

which implies that

$$\varepsilon^+(\pi_{\delta,\mu}, z, \psi^a) = \frac{\varpi(a)^{-1}|a|^{(k+1)(\frac{m-1}{2}+z)}}{c_a} \varepsilon^+(\pi_{\delta,\mu}, z, \psi).$$

Similarly, taking again $z \in \mathbb{C}$ such that $\mathrm{Re}(\mu_k - z) > 0$, one has

$$\mathcal{F}^a(\Psi)(X) = d_a \int_{V^+} \Psi(Y)\psi(a\ b(X,Y))dY = d_a \mathcal{F}(\Psi)(aX)$$

and

$$Z^{a,+}(\mathcal{F}^a(\Psi), \frac{m+1}{2} - z, \xi, w) = |a|^{\dim V^+} \int_{V^+} \mathcal{F}(\Psi)(aX) |\Delta_0(X)|^{\frac{m+1}{2}-z} \langle \pi^*_{\delta,\mu}(X)\xi, w\rangle d^*X$$

$$= |a|^{\dim V^+} \int_{V^+} \mathcal{F}(\Psi)(X) |a|^{-(k+1)(\frac{m+1}{2}-z)} |\Delta_0(X)|^{\frac{m+1}{2}-z} \langle \pi^*_{\delta,\mu}(m_a^{-1}.X)\xi, w\rangle d^*X$$

$$= |a|^{\dim V^+} \varpi(a) |a|^{-(k+1)(\frac{m+1}{2}-z)} Z^+(\mathcal{F}(\Psi), \frac{m+1}{2} - z, \xi, w)$$

If we divide both sides of the preceding identity by $L(\pi_{\delta,\mu}, 1-z)$ we obtain

$$\begin{aligned}
\Xi^{a,+}(\mathcal{F}^a(\Psi), 1-z, \xi, w) &= |a|^{\dim V^+} \varpi(a) |a|^{-(k+1)(\frac{m+1}{2}-z)} \Xi^+(\mathcal{F}(\Psi), 1-z, \xi, w) \\
&= |a|^{\dim V^+} \varpi(a) |a|^{-(k+1)(\frac{m+1}{2}-z)} \varepsilon^-(\pi_{\delta,\mu}, z, \psi) \Xi^-(\Psi, z, \xi, w) \\
&= \frac{1}{d_a} |a|^{\dim V^+} \varpi(a) |a|^{-(k+1)(\frac{m+1}{2}-z)} \varepsilon^-(\pi_{\delta,\mu}, z, \psi) \Xi^{a,-}(\Psi, z, \xi, w).
\end{aligned}$$

And hence

$$\varepsilon^-(\pi_{\delta,\mu}, z, \psi^a) = c_a \varpi(a) |a|^{-(k+1)(\frac{m+1}{2}-z)} \varepsilon^-(\pi_{\delta,\mu}, z, \psi).$$

□

REFERENCES

[1] P. Blanc, P. Delorme, *Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H* , Ann. Inst. Fourier (Grenoble), 2008, no 1, 213–261.

– cited on p. 2, 4, 51, 52

[2] N. Bopp, H. Rubenthaler, *Local Zeta functions attached to the minimal spherical series for a class of symmetric spaces*, Memoirs of the AMS (2005), Volume 174, n° 821.

– cited on p. 12, 13, 18, 23, 27, 28, 32, 34

[3] N. Bourbaki | *Groupes et Algèbres de Lie, Chapitres 4,5 et 6*, (1981) Masson, Paris. – cited on p. 38, 40

[4] N. Bourbaki | *Groupes et Algèbres de Lie, Chapitres 7 et 8*, (1975) Hermann, Paris. – cited on p. 7

[5] C.J. Bushnell and G. Henniart,*The Local Langlands Conjecture for $GL(2)$*, Springer, Grundlehren der mathematischen Wissenschaften, volume 335.

– cited on p. 68, 71

[6] J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. math.,1-23 (1984).

– cited on p. 42, 43

[7] R. Godement, H. Jacquet, *Zeta Functions of Simple Algebras*, (1981) Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972. ix+188 pp.

– cited on p. 30, 65, 71

[8] P. Harinck and H. Rubenthaler, *Local zeta functions for a class of p-adic symmetric space (I): Structure and orbits.* hal: 02507407 (109 p.)

– cited on p. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 21, 27, 33, 34, 35, 36, 37, 39, 48, 49, 50, 56, 66

[9] J. Igusa, *Some results on p-adic complex powers*, Amer. J. Math., 106 (1984), 1013-1032, – cited on p. 42, 43

[10] N. Lagier, *Terme constant de fonctions sur un espace symétrique réductif p-adique*, Journal of Funct. Analysis 254 (2008), p 1088-1145

– cited on p. 52

[11] T.Y. Lam, *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, 67. American Mathematical Society, Providence, RI, 2005. xxii+550 pp. – cited on p. 7

[12] I. Muller, *Une classe d'espaces préhomogènes de type parabolique faiblement sphériques*, preprint, arXiv:0810.3148

– cited on p. 12, 18, 23, 28, 36, 44

[13] S. Rallis, G. Schiffmann, *Distributions invariantes par le groupe orthogonal*, In Analyse Harmonique sur les groupes de Lie, Lecture Notes in Math. Vol. 497, p. 494-642. – cited on p. 27, 30, 31, 34

[14] D. Renard, *Représentations des groupes réductifs p-adiques* , Cours spécialisé, volume 17, SMF.

– cited on p. 24, 49

[15] F. Sato, *On Functional Equations of Zeta Distributions*, Advances Studies in Pure Math., 15, 1989, p465-508

– cited on p. 2, 3, 38, 42, 43

[16] J. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, Algebraic Number Theory, Cassels and Fröhlich Editors (Proc. Instructional Conf., Brighton, 1965), 305-347. Academic Press, London, 1967 – cited on p. 43

[17] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math.111 (1964) p. 143-211. – cited on p. 27, 30

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