



HAL
open science

A remark on Bohr-Sommerfeld Quantization Rules for a self-adjoint 1-D h-Pseudo-differential operator

Michel L. Rouleux, Abdelwaheb Ifa

► **To cite this version:**

Michel L. Rouleux, Abdelwaheb Ifa. A remark on Bohr-Sommerfeld Quantization Rules for a self-adjoint 1-D h-Pseudo-differential operator. 2024. hal-04663314

HAL Id: hal-04663314

<https://cnrs.hal.science/hal-04663314v1>

Preprint submitted on 27 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A remark on Bohr-Sommerfeld Quantization Rules for a self-adjoint 1-D h -Pseudo-differential operator

A.IFA ¹, M.ROULEUX ²

¹ Université de Tunis El-Manar, Département de Mathématiques, 1091 Tunis, Tunisia
 & Université de Kairouan, 3100 Kairouan, Tunisia
 email: abdelwaheb.ifa@fsm.rnu.tn

² Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
 email: rouleux@univ-tln.fr

Abstract

We revisit the well known Bohr-Sommerfeld quantization rule (BS) of order 2 for a self-adjoint 1-D h -Pseudo-differential operator within the algebraic and microlocal framework of Helffer and Sjöstrand; BS holds precisely when the Gram matrix consisting of scalar products of some WKB solutions with respect to the “flux norm” is not invertible. We simplify somewhat our previous proof [1] by working in spatial representation only, as in complex WKB theory for Schrödinger operator.

1 Introduction

Let $p(x, \xi; h)$ be a smooth real classical Hamiltonian on $T^*\mathbb{R}$; we will assume that p belongs to the space of symbols $S^0(m)$ for some order function m with

$$S^N(m) = \{p \in C^\infty(T^*\mathbb{R}) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0, \forall (x, \xi) \in T^*\mathbb{R}; |\partial_{(x,\xi)}^\alpha p(x, \xi; h)| \leq C_\alpha h^N m(x, \xi)\}, \quad (1.1)$$

and has the semi-classical expansion

$$p(x, \xi; h) \sim p_0(x, \xi) + hp_1(x, \xi) + \dots, \quad h \rightarrow 0. \quad (1.2)$$

We call as usual p_0 the principal symbol, and p_1 the sub-principal symbol. We also assume that $p + i$ is elliptic. This allows to take Weyl quantization of p

$$P(x, hD_x; h)u(x; h) = p^w(x, hD_x; h)u(x; h) = (2\pi h)^{-1} \iint e^{\frac{i}{h}(x-y)\eta} p\left(\frac{x+y}{2}, \eta; h\right) u(y) dy d\eta, \quad (1.3)$$

so that $P(x, hD_x; h)$ is essentially self-adjoint on $L^2(\mathbb{R})$. We make the geometrical hypothesis of [2], namely:

Fix some compact interval $I = [E_-, E_+]$, $E_- < E_+$, and assume that there exists a topological ring $\mathcal{A} \subset p_0^{-1}(I)$ such that $\partial\mathcal{A} = \mathcal{A}_- \cup \mathcal{A}_+$ with \mathcal{A}_\pm a connected component of $p_0^{-1}(E_\pm)$. Assume also that p_0 has no critical point in \mathcal{A} , and \mathcal{A}_- is included in the disk bounded by \mathcal{A}_+ (if it is not the case, we can always change p to $-p$).

We define the microlocal well W as the disk bounded by \mathcal{A}_+ . For $E \in I$, let $\gamma_E \subset W$ be a periodic orbit in the energy surface $\{p_0(x, \xi) = E\}$, so that γ_E is an embedded Lagrangian manifold.

The paradigm of such an Hamiltonian, $p(x, \xi; h) = p_0(x, \xi) = \xi^2 + V(x)$, is associated with Schrödinger equation

$$(P - E)u_h = (-h^2 \Delta + V(x) - E)u_h = 0. \quad (1.4)$$

We can also introduce a sub-principal symbol by considering a Sturm-Liouville equation such as $-h^2(p(x)u'(x))' + q(x)u(x) = E$.

Moreover $p_0(x, \xi)$ need not be invariant under $\xi \mapsto -\xi$, we can take for instance $p_0(x, \xi) = \xi^2 + f(x)\xi + V(x)$ where f, V are smooth functions. We assume V that is even and f is odd in x , so that $P(x, hD_x)$ verifies PT symmetry, and the family of Lagrangian submanifolds defined by $p_0(x, \xi) = E$, look like “tilded ellipses” filling \mathcal{A} , provided $f(x)^2 \geq 4(V(x) - E)$.

As for the Pseudo-differential case, we may consider Harper operator $\cosh D_x + \cos x$ on $L^2(\mathbf{R})$ “restricted to a potential well”, i.e. a component of $p_0(x, \xi) = \cos \xi + \cos x = E$ where $E \in [-2, 2] \setminus [-\varepsilon_0, \varepsilon_0]$, see [3].

We call $a = a_E = (x_E, \xi_E) \in \gamma_E$ a *focal point* whenever the Hamilton vector field H_{p_0} turns vertical at a . For $p_0(x, \xi) = \xi^2 + f(x)\xi + V(x)$ the focal points are given by (x_E, ξ_E) where $f(x_E)^2 = 4(V(x_E) - E)$ and $\xi = -f(x_E)/2$. We can assume, without loss of generality, that γ_E contains only 2 focal points, say a_E and a'_E , with $x'_E < x_E$. If $\xi_E = 0$ as in (1.4), x_E is called a *turning point*, and for convenience we keep this terminology in the general case.

Then if $E_+ < E_0 = \liminf_{|x, \xi| \rightarrow \infty} p_0(x, \xi)$, all eigenvalues of P in I are indeed given by *Bohr-Sommerfeld quantization condition* (BS). Bohr-Sommerfeld quantization rules hold for a general Hamiltonian in any dimension up to $\mathcal{O}(h^2)$ [4]. They are symplectic equivariant formulas, also known as EBK quantization rules in higher dimension. For general smooth 1-D Hamiltonians, BS holds with an accuracy $\mathcal{O}(h^N)$ for any N , see [5], [6], [2], [7], [8], [1] and references therein. Exponential accuracy i.e. $\mathcal{O}(e^{-1/Ch})$, is obtained through the complex WKB method in the case of Schrödinger operator $-h^2 \Delta + V(x)$ with an analytic potential, see [9], [10], [11]. It carries to more general operators like Harper’s operator [3], or matrix Hamiltonians (see [12], [13] and references therein). The so-called exact WKB method provides an infinite accuracy in case of Schrödinger operator, see [14], [15], or for the approach based on Ecalle’s resurgence theory, [16], [17], [18] and references therein. In case of more general symbols (e.g. polynomials of order 3 in ξ) exact WKB method turns out to be much more complicated [15].

We recall here BS, when computed at second order for an Hamiltonian of type (1.3):

Theorem 1.1. *With the notations and hypotheses stated above, for $h > 0$ small enough there exists a smooth function $\mathcal{S}_h : I \rightarrow \mathbb{R}$, called the semi-classical action, with asymptotic expansion $\mathcal{S}_h(E) \sim S_0(E) + hS_1(E) + h^2S_2(E) + \dots$ such that $E \in I$ is an eigenvalue of P iff it satisfies the implicit equation (Bohr-Sommerfeld quantization condition) $\mathcal{S}_h(E) = 2\pi n h$, $n \in \mathbb{Z}$. The semi-classical action consists of :*

(i) *the classical action along γ_E*

$$S_0(E) = \oint_{\gamma_E} \xi(x) dx = \int \int_{\{p_0 \leq E\} \cap W} d\xi \wedge dx,$$

(ii) *Maslov correction and the integral of the sub-principal 1-form $p_1 dt$*

$$S_1(E) = \pi - \int_{\gamma_E} p_1(x(t), \xi(t)) dt,$$

(iii) *the second order term*

$$S_2(E) = \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt,$$

where

$$\Delta(x, \xi) = \frac{\partial^2 p_0}{\partial x^2} \frac{\partial^2 p_0}{\partial \xi^2} - \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right)^2.$$

We recall that $S_3(E) = 0$. Our integrals are oriented integrals, t denoting the variable in Hamilton's equations.

Example: In case of (1.4) the semi-classical action takes the form

$$\mathcal{S}_h(E) = \oint_{\gamma_E} \xi(x) dx + \pi h + \frac{h^2}{12} \frac{d}{dE} \int_{\gamma_E} V''(x(t)) dt + \mathcal{O}(h^4),$$

where $(\xi(x))^2 = E - V(x)$.

A quite short proof, based on h -Pseudo Differential calculus, is given in [5], [6], [2]. In [1], we presented instead a derivation of BS, based on the construction of a Hermitian vector bundle of quasi-modes as in ([19], [3]), using different canonical charts, in the terminology of [4]. Namely, if $K_h^N(E)$ denotes the microlocal kernel of $P - E$ of order N , i.e. the space of microlocal solutions of $(P - E)u_h = \mathcal{O}(h^{N+1})$ along the covering of γ_E , the problem amounts to find the set of $E = E(h)$ such that $K_h^N(E)$ contains a global section, i.e. to construct a sequence of quasi-modes $(u_n(h), E_n(h))$ of a given order N (practically $N = 2$).

The main step of [1] consists in computing the homology class of the semi-classical action over γ_E up to order 2 in h , the leading term being $\oint_{\gamma_E} \xi dx$. Our starting point was to write down the microlocal solution $\widehat{u}^a(\xi; h)$ near a focal point $a = (x_E, \xi_E)$ in Fourier representation mod $\mathcal{O}(h^2)$, see [1] formula (3.4). Once we know $\widehat{u}^a(\xi; h)$, we get the corresponding branches $u_{\pm}^a(x; h)$ of $u^a(x; h)$ by stationary phase (inverse Fourier transform), in a punctured neighborhood of a , mod $\mathcal{O}(h^2)$, see ([1] formula (3.27), corrected in [20] formula (*)). We repeat the same procedure starting from the other focal point a'_E and then build up Gram matrix $G^{(a, a')}(E)$ (see [1], formula (2.7) for a definition), whose determinant vanishes precisely when E is an eigenvalue of P mod $\mathcal{O}(h^3)$.

Thus we have made use of 3 canonical charts starting from a (one Fourier and two position representations) and another 3 starting from a' . Our purpose here is to simplify the previous approach, and avoid Fourier representation, by invoking (at least heuristically) some ideas of complex WKB method, which we make rigorous in case of Schrödinger equation (1.4) with analytic coefficients. This reduces to 2+2 the number of canonical charts, the Ansatz allowing to “glue” the branches $\rho = \pm$ of the solutions.

Complex WKB method is based on computing the solutions of (1.4) on various complex domains (Stokes regions) according to their oscillatory or exponentially growing/decaying behavior, as superposition of different branches of a global analytic function. We are interested *in fine* in their boundary values on the real domain. These branches meet at turning points, in the oscillatory (or classically allowed) region they differ by the sign of $\xi(x)$, and also by a phase factor accounting for the change of Maslov index. To fix the ideas, at leading order in h the microlocal solution u of $(P - E)u = 0$ in a punctured neighborhood of a takes the form (up to normalization)

$$u^a(x, h) = e^{i\pi/4} (E - V)^{-1/4} e^{iS(a, x)/h} + e^{-i\pi/4} (E - V)^{-1/4} e^{-iS(a, x)/h} + \mathcal{O}(h), \quad (1.5)$$

with the variation of Maslov index from the lower to the upper branch. See (3.2) for a more intrinsic formula.

The quantization condition requires that the subspace of wave functions satisfying Schrödinger equation with exponential decay at $+\infty$ coincides with those with exponential decay at $-\infty$. It is characterized by the property that the monodromy operator, which takes the wave function around γ_E , has eigenvalue 1. This express the fact that eigenvalues $E = E_n(h)$ are the zeros

of so-called Jost function $J(E)$ (in the terminology of [18]). Jost function is nothing but the determinant of our Gram matrix $G^{(a,a')}(E)$.

In the analytic category, it turns out that Schrödinger operator in (1.4) has a normal form near a simple turning point $V(x) = E$, such that an exact solution of (1.4) (whose asymptotics in h takes the form (1.5)) is mapped onto the solution of semi-classical Airy equation

$$-h^2 v''(z; h) + x v(z; h) = 0, \quad (1.6)$$

near 0 by means of a Micro-differential operator in the sense of M.Sato. See [21] and Sect.3 below.

By analogy we postulate in case of (1.3) (see Ansatz (2.30) below) that the normalized branches $u_{\pm}^a(x; h)$ coincide again in a punctured neighborhood of the focal point a (at least at second order in h) with the normalized WKB solutions, when multiplied by the phase factors $e^{\pm i\pi/4}$. This Ansatz of course is justified *a posteriori* by formula (3.17) in [1].

Thus Fourier representation in proving Theorem 1.1 can (formally) be avoided. This remark makes the argument substantially shorter than in [1].

It would be of course quite challenging to implement these ideas within the context of complex WKB theory for a general symbol (1.2) with analytic coefficients.

The paper is organized as follows : In Sect.2.1 we compute WKB solutions mod $O(h^2)$ in the spatial representation. They are normalized in Sect.2.2 using the microlocal Wronskian. In Sect.2.3 we determine the homology class of the generalized action. In Sect.2.4 we derive Bohr-Sommerfeld quantization rule from Ansatz (2.30). First two parts of this paper rely strongly on [1], but we have recalled the main steps of the proof for the reader's convenience, and outlined the parallel with the present proof. In Sect.3 instead, we consider the particular case of Schrödinger operator with analytic coefficients. We follow the approach of [21] based on M.Sato's Microdifferential Calculus [22]. So we reduce (1.4) to (1.6) near the turning point by the exact WKB method, which gives also the asymptotics (practically up to the 4:th order in h), and eventually compute the (exact) solution of (1.4) from Airy function. Using the full asymptotics of Airy function in the classically allowed region, we recover the asymptotics (3.2) of (1.4) obtained already in [23], and make it more precise. This justifies in particular our Ansatz (2.30) in case of (1.4).

Acknowledgments: We thank André Voros for useful comments on complex WKB methods. The second author also thanks Johannes Sjöstrand and Michael Hitrik for organizing a Workshop on WKB theory, held on-line during the Covid time.

2 Quasi-modes and BS mod $\mathcal{O}(h^2)$ in the spatial representation

2.1 WKB solutions mod $\mathcal{O}(h^2)$ in the spatial representation

In a first step we compute WKB solutions $u_{\rho}^a(x; h) = u_{\pm}^a(x; h)$ starting from the focal point $a = a_E$, and uniformly valid with respect to h for x in any $I \subset \subset]x'_E, x_E[$. So let

$$u_{\rho}^a(x; h) = b_{\rho}(x; h) e^{\frac{i}{h} \Phi_{\rho}(x)}, \quad (2.1)$$

where $b_{\rho}(x; h)$ is a formal series in h , which we shall compute with h^2 accuracy

$$b_{\rho}(x; h) = b_{\rho,0}(x) + h b_{\rho,1}(x) + h^2 b_{\rho,2}(x) + \dots$$

The phase $\varphi_\rho(x)$ is a real smooth function that satisfies the eikonal equation

$$p_0(x, \varphi'_\rho(x)) = E. \quad (2.2)$$

This is the main difference with [1] where $\varphi_\rho(x)$ are the branches of the Legendre transform $x\xi + \psi(\xi)$ near the turning points. For simplicity we shall omit indices $\rho = \pm$ whenever no confusion may occur. Let $Q(x, hD_x; h) = e^{-\frac{i}{h}\varphi(x)}P(x, hD_x)e^{\frac{i}{h}\varphi(x)}$, this is a h -PDO and

$$(Q - E)b(x; h) = (2\pi h)^{-1} \int \int e^{\frac{i}{h}(x-y)\theta} p\left(\frac{x+y}{2}, \theta + F(x, y); h\right) b(y; h) dy d\theta,$$

with $F(x, y) = \int_0^1 \varphi'(x + t(y-x)) dt$. Applying asymptotic stationary phase at order 2, we find

$$\begin{aligned} (Q(x, hD_x; h) - E)b(x; h) &= \left(p(x, \varphi'(x); h) - E\right) b(x; h) + \frac{h}{i} \left(\beta(x; h) \partial_x b(x; h) + \frac{1}{2} \partial_x \beta(x; h) b(x; h)\right) \\ &- h^2 \left(\frac{1}{8} \partial_x r(x; h) b(x; h) + \frac{1}{8} \varphi''(x) \partial_x \theta(x; h) b(x; h) + \frac{1}{2} \partial_x \gamma(x; h) \partial_x b(x; h) + \frac{1}{2} \gamma(x; h) \frac{\partial^2 b(x; h)}{\partial x^2} + \frac{1}{8} \varphi'''(x) \theta(x; h) b(x; h)\right) + \mathcal{O}(h^3). \end{aligned} \quad (2.3)$$

Suppose now that $p(x, \xi; h)$ is real, $p_0(x_E, \xi_E) = E$, $(\frac{\partial p_0}{\partial \xi})(x_E, \xi_E) \neq 0$. We look for formal solutions (i.e in the sense of formal classical symbols) of

$$(P(x, hD_x; h) - E)(b(x; h) e^{\frac{i}{h}\varphi(x)}) = 0 \iff (Q(x, hD_x; h) - E)b(x; h) = 0 \quad (2.4)$$

Once the eikonal equation (2.2) holds, we obtain by annihilating the term in h in (2.3) the first transport equation

$$\beta_0(x) b'_0(x) + \left(i p_1(x, \varphi'(x)) + \frac{1}{2} \beta'_0(x)\right) b_0(x) = 0, \quad (2.5)$$

whose solutions are of the form

$$b_0(x) = C_0 |\beta_0(x)|^{-\frac{1}{2}} \exp\left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy\right), \quad (2.6)$$

C_0 being so far an arbitrary constant. Again we have omitted the index ρ .

Annihilating the term in h^2 in (2.3), we next show that $b_1(x)$ is a solution of the differential equation

$$\begin{aligned} \beta_0(x) b'_1(x) + \left(i p_1(x, \varphi'(x)) + \frac{1}{2} \beta'_0(x)\right) b_1(x) &= -\beta_1(x) b'_0(x) - \left(i p_2(x, \varphi'(x)) + \frac{1}{2} \beta'_1(x)\right) b_0(x) \\ &+ i \left(\frac{1}{8} r'_0(x) b_0(x) + \frac{1}{8} \varphi''(x) \theta'_0(x) b_0(x) + \frac{1}{2} \gamma'_0(x) b'_0(x) + \frac{1}{2} \gamma_0(x) b''_0(x) + \frac{1}{6} \varphi'''(x) \theta_0(x) b_0(x)\right), \end{aligned} \quad (2.7)$$

where we have set

$$\beta_0(x) = \left(\frac{\partial p_0}{\partial \xi}\right)(x, \varphi'(x)); \quad r_0(x) = \left(\frac{\partial^3 p_0}{\partial x \partial \xi^2}\right)(x, \varphi'(x)); \quad \gamma_0(x) = \left(\frac{\partial^2 p_0}{\partial \xi^2}\right)(x, \varphi'(x)); \quad \theta_0(x) = \left(\frac{\partial^3 p_0}{\partial \xi^3}\right)(x, \varphi'(x)).$$

The homogeneous equation associated with (2.7) is the same as (2.5); so we are looking for a particular solution of (2.7), integrating from x_E , of the form

$$b_1(x) = D_1(x) |\beta_0(x)|^{-\frac{1}{2}} \exp\left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy\right). \quad (2.8)$$

Alternatively, we could integrate (2.7) from x'_E instead of x_E . So our main task will consist in computing $D_1(x)$ as a multivalued function, due to the presence of the turning points, in the same way we have determined $D_1(\xi)$ in [1] (Formula (3.5)), using Fourier representation.

We solve (2.7) by the method of variation of constants, and find

$$\frac{1}{C_0} \operatorname{Re}(D_1(x)) = -\frac{1}{2} \left[\partial_\xi \left(\frac{p_1}{\partial_\xi p_0} \right) (y, \varphi'(y)) \right]_{x_E}^x, \quad (2.9)$$

$$\begin{aligned} \frac{1}{C_0} \operatorname{Im}(D_1(x)) &= \int_{x_E}^x \frac{1}{\beta_0} \left(-p_2 + \frac{1}{8} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} + \frac{\varphi''}{12} \frac{\partial^4 p_0}{\partial y \partial \xi^3} - \frac{(\varphi'')^2}{24} \frac{\partial^4 p_0}{\partial \xi^4} \right) dy - \frac{1}{8} \int_{x_E}^x \frac{(\beta_0')^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2} dy \\ &+ \frac{1}{6} \int_{x_E}^x \varphi'' \frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3} dy + \int_{x_E}^x \frac{p_1}{\beta_0^2} \left(\partial_\xi p_1 - \frac{p_1}{2\beta_0} \frac{\partial^2 p_0}{\partial \xi^2} \right) dy + \left[\frac{\varphi''}{6\beta_0} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^x - \left[\frac{\beta_0'}{4\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^x, \end{aligned} \quad (2.10)$$

Function $D_1(x)$ can be normalized by

$$D_1(x_E) = 0$$

These are the expressions found in [1] (formula (3.26), corrected in [20]). The general solution of (2.7) is given by

$$b_1(x) = (C_1 + D_1(x)) |\beta_0(x)|^{-\frac{1}{2}} \exp \left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy \right). \quad (2.11)$$

It follows that

$$b(x; h) = \left(C_0 + h(C_1 + D_1(x)) + \mathcal{O}(h^2) \right) |\beta_0(x)|^{-\frac{1}{2}} \exp \left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy \right)$$

We repeat this construction with the other branch $\rho = -1$, and thus get the 2 branches of WKB solutions

$$u_\pm(x; h) = |\beta_0^\pm(x)|^{-\frac{1}{2}} e^{\frac{i}{h} S_\pm(x_E, x; h)} \left(C_0 + h(C_1 + D_1^\pm(x)) + \mathcal{O}(h^2) \right), \quad (2.12)$$

with

$$\begin{aligned} S_\pm(x_E, x; h) &= \varphi_\pm(x_E) + \int_{x_E}^x \xi_\pm(y) dy - h \int_{x_E}^x \frac{p_1(y, \varphi'_\pm(y))}{\beta_0^\pm(y)} dy, \\ \beta_0^\pm(x) &= (\partial_\xi p_0)(x, \varphi'_\pm(x)). \end{aligned} \quad (2.13)$$

Here we have used that $\varphi_\rho(x) = \varphi_\rho(x_E) + \int_{x_E}^x \xi_\rho(y) dy$, where $p_0(x, \xi_\rho(x)) = E$. Note that we recover the expressions found in [1] (formula (3.27), corrected into (*) in [20]).

2.2 Well normalized quasi-modes mod $\mathcal{O}(h^2)$ in the spatial representation

WKB solutions (2.12) are not yet normalized. We determine the integration constants $C_0, C_1 = C_1(a_E)$ as follows.

In [1] we called *microlocal Wronskian* a key ingredient introduced in [19], [3], which allows to normalize microlocal solutions. Namely choose an orientation on γ_E and denote by $\rho = \pm 1$ its oriented segments near $a \in \gamma_E$. Let $\chi^a \in C_0^\infty(\mathbf{R}^2)$ be a smooth cut-off equal to 1 near a , and ω_ρ^a a small neighborhood of $\operatorname{supp}[P, \chi^a] \cap \gamma_E$ near ρ , where χ^a holds for Weyl quantization $\chi^a(x, hD_x)$ as in (1.3). Let $u^a, \bar{v}^a \in K_h(E)$ (the microlocal kernel of $P - E$) be supported microlocally on γ_E . The microlocal Wronskian of (u^a, \bar{v}^a) is defined as $\mathcal{W}_\rho^a(u^a, \bar{v}^a) = \left(\frac{i}{h} [P, \chi^a]_\rho u^a | \bar{v}^a \right)$. Here $\frac{i}{h} [P, \chi^a]_\rho$ denotes the part of the commutator supported on ω_ρ^a . Without loss of generality, we can take $\chi^a(x, \xi) = \chi_1(x) \chi_2(\xi)$, so that $\chi_2 \equiv 1$ on small neighborhoods ω_\pm^a , of $\operatorname{supp}[P, \chi^a] \cap \{p_0(x, \xi) = E\}$ in $\pm(\xi - \xi_E) > 0$. Thus we need only consider the variations of χ_1 .

Weyl symbol of $\frac{i}{h} [P, \chi^a]$ is given by

$$d(x, \xi; h) = d_0(x, \xi) + h d_1(x, \xi) + \mathcal{O}(h^2),$$

with

$$d_j(x, \xi) = \partial_\xi p_j(x, \xi) \chi_1'(x), \quad \forall j \in \{0, 1\}.$$

Let

$$F_{\pm}^a = \frac{i}{h} [P, \chi^a]_{\pm} u^a.$$

Evaluating by stationary phase, we find

$$\begin{aligned} F_{\pm}^a(x; h) &= C_0 |\beta_0^{\pm}(x)|^{-\frac{1}{2}} \exp \left[\frac{i}{h} \left(\varphi_{\pm}(x) - h \int_{x_E}^x \frac{p_1(y, \varphi'_{\pm}(y))}{\beta_0^{\pm}(y)} dy \right) \right] \\ &\times \left(d_0^{\pm}(x) + h \frac{C_1}{C_0} d_0^{\pm}(x) + \frac{h}{C_0} d_0^{\pm}(x) D_1^{\pm}(x) + h d_1^{\pm}(x) + \frac{h}{2i} (s'_{\pm}(x) + 2s_{\pm}(x) \theta_{\pm}(x)) + \mathcal{O}(h^2) \right). \end{aligned} \quad (2.14)$$

Here we have set

$$\begin{aligned} d_j^{\pm}(x) &= d_j(x, \varphi'_{\pm}(x)) = \beta_j^{\pm}(x) \chi_1'(x), \quad \forall j \in \{0, 1\}, \\ s_{\pm}(x) &= (\partial_{\xi}^2 d_0)(x, \varphi'_{\pm}(x)) = \left(\frac{\partial^2 p_0}{\partial \xi^2} \right)(x, \varphi'_{\pm}(x)) \chi_1'(x) = \gamma_0^{\pm}(x) \chi_1'(x), \\ \theta_{\pm}(x) &= -\frac{1}{\beta_0^{\pm}(x)} \left(\frac{1}{2} (\beta_0^{\pm})'(x) + i p_1(x, \varphi'_{\pm}(x)) \right). \end{aligned}$$

It follows

$$(u^a | F_+^a) = C_0^2 \int_{-\infty}^{x_E} \frac{1}{\beta_0^+} \left(d_0^+ + 2h \frac{C_1}{C_0} d_0^+ + 2h \frac{d_0^+}{C_0} \operatorname{Re}(D_1^+) + h d_1^+ + \frac{ih}{2} (s'_+ + 2s_+ \overline{\theta_+}) \right) dx.$$

Now, using the fact that

$$\begin{aligned} \int_{-\infty}^{x_E} \frac{d_0^+}{\beta_0^+} dx &= \int_{-\infty}^{x_E} \chi_1'(x) dx = 1, \\ \int_{-\infty}^{x_E} \frac{d_1^+}{\beta_0^+} dx &= \int_{-\infty}^{x_E} \frac{\beta_1^+}{\beta_0^+} \chi_1'(x) dx, \\ \int_{-\infty}^{x_E} \frac{d_0^+ \operatorname{Re}(D_1^+)}{\beta_0^+} dx &= \frac{C_0}{2} \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right)(a_E) - \frac{C_0}{2} \int_{-\infty}^{x_E} \frac{\beta_1^+}{\beta_0^+} \chi_1'(x) dx + \frac{C_0}{2} \int_{-\infty}^{x_E} \frac{s_+ p_1}{(\beta_0^+)^2} dx, \\ \int_{-\infty}^{x_E} \frac{s'_+}{\beta_0^+} dx &= \left[\frac{s_+}{\beta_0^+} \right]_{-\infty}^{x_E} + \int_{-\infty}^{x_E} \frac{(\beta_0^+)' }{(\beta_0^+)^2} s_+ dx = \int_{-\infty}^{x_E} \frac{(\beta_0^+)' }{(\beta_0^+)^2} s_+ dx \quad (\text{integration by parts}), \end{aligned}$$

and

$$\int_{-\infty}^{x_E} \frac{s_+ \overline{\theta_+}}{\beta_0^+} dx = -\frac{1}{2} \int_{-\infty}^{x_E} \frac{(\beta_0^+)' }{(\beta_0^+)^2} s_+ dx + i \int_{-\infty}^{x_E} \frac{s_+ p_1}{(\beta_0^+)^2} dx,$$

we get easily

$$(u^a | F_+^a) = C_0^2 + h \left(2C_0 C_1 + C_0^2 \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right)(a_E) \right) + \mathcal{O}(h^2). \quad (2.15)$$

Without loss of generality we may take $C_0, C_1 \in \mathbb{R}$. Similarly

$$(u^a | F_-^a) = -C_0^2 - h \left(2C_0 C_1 + C_0^2 \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right)(a_E) \right) + \mathcal{O}(h^2). \quad (2.16)$$

(the mixed terms such as $(u_{\pm}^a | F_{\mp}^a)$ are $\mathcal{O}(h^{\infty})$ because the phase is non stationary), so that

$$(u^a | F_+^a - F_-^a) = 2C_0^2 + 2h \left(2C_0 C_1 + C_0^2 \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right)(a_E) \right) + \mathcal{O}(h^2). \quad (2.17)$$

The condition that u^a be normalized mod $\mathcal{O}(h^2)$ is then

$$C_0 = 2^{-\frac{1}{2}}; \quad C_1 = C_1(a_E) = -2^{-\frac{3}{2}} \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right)(a_E). \quad (2.18)$$

These are of course the same constants C_0 and $C_1(a_E)$ obtained in [1], formula (3.6), using Fourier representation of WKB solutions.

2.3 The homology class of the generalized action

Here we identify the various terms in (2.12), which are responsible for the holonomy of u^a . Again we proceed in a very similar way to [1] but in the position representation only. First on γ_E (i.e. Λ_E) we have $\varphi(x) = \int \xi(x) dx + \text{Const}$. By Hamilton equations

$$\dot{\xi}(t) = -\partial_x p_0(x(t), \xi(t)), \quad \dot{x}(t) = \partial_\xi p_0(x(t), \xi(t)),$$

so

$$\int \frac{p_1(x, \xi(x))}{\beta_0(x)} dx = \int_{\gamma_E} \frac{p_1(x, \xi)}{\partial_\xi p_0(x, \xi)} dx = \int_0^{T(E)} p_1(x(t), \xi(t)) dt.$$

The form $p_1 dt$ is called the subprincipal 1-form. Next we consider $\sqrt{2}D_1(x)$ as the integral over γ_E of the 1-form, defined near x_E in spatial representation, $\Omega_1(x) = T_1(x) dx$, i.e. $\sqrt{2}D_1(x) = \int_{x_E}^x T_1(y) dy$.

Since γ_E is Lagrangian, Ω_1 is a closed form that we are going to compute modulo exact forms in \mathcal{A} , i.e. modulo the variations $[\cdot]_{x_E}^x$. Using integration by parts, the integrals (2.9), (2.10) of $\text{Re}\Omega_1(x)$, $\text{Im}\Omega_1(x)$ in spatial representation simplify to

$$\sqrt{2}\text{Re}(D_1(x)) = -\frac{1}{2}\partial_\xi\left(\frac{p_1}{\partial_\xi p_0}\right)(x, \xi(x)) - \sqrt{2}C_1(a_E), \quad (2.19)$$

$$\sqrt{2}\text{Im}(D_1(x)) = \int_{x_E}^x T_1(y) dy + \left[\frac{\varphi''}{6\beta_0} \frac{\partial^3 p_0}{\partial \xi^3}\right]_{x_E}^x - \left[\frac{\beta_0'}{4\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2}\right]_{x_E}^x, \quad (2.20)$$

$$T_1(y) = \frac{1}{\beta_0} \left(-p_2 + \frac{1}{8} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} + \frac{\varphi''}{12} \frac{\partial^4 p_0}{\partial y \partial \xi^3} - \frac{(\varphi'')^2}{24} \frac{\partial^4 p_0}{\partial \xi^4} \right) - \frac{1}{8} \frac{(\beta_0')^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2} + \frac{1}{6} \varphi'' \frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3} + \frac{p_1}{\beta_0^2} \left(\partial_\xi p_1 - \frac{p_1}{2\beta_0} \frac{\partial^2 p_0}{\partial \xi^2} \right). \quad (2.21)$$

There follows, as in Lemma 3.2 of [1]:

Lemma 2.1. *Modulo the integral of an exact form in \mathcal{A} , with T_1 as in (2.21) we have:*

$$\begin{aligned} \text{Re}(D_1(x)) &\equiv 0, \\ \sqrt{2}\text{Im}(D_1(x)) &\equiv \int_{x_E}^x T_1(y) dy. \end{aligned} \quad (2.22)$$

If $f(x, \xi), g(x, \xi)$ are smooth functions on \mathcal{A} we set $\Omega(x, \xi) = f(x, \xi) dx + g(x, \xi) d\xi$. By Stokes formula

$$\int_{\gamma_E} \Omega(x, \xi) = \int \int_{\{p_0 \leq E\}} (\partial_x g - \partial_\xi f) dx \wedge d\xi,$$

where, following [2], we have extended p_0 in the disk bounded by \mathcal{A} so that it coincides with a harmonic oscillator in a neighborhood of a point inside, say $p_0(0, 0) = 0$. Making the symplectic change of coordinates $(x, \xi) \mapsto (t, E)$ in $T^*\mathbb{R}$:

$$\int \int_{\{p_0 \leq E\}} (\partial_x g - \partial_\xi f) dx \wedge d\xi = \int_0^E \int_0^{T(E')} (\partial_x g - \partial_\xi f) dt \wedge dE',$$

where $T(E')$ is the period of the flow of Hamilton vector field H_{p_0} at energy E' . Taking derivative with respect to E , we find

$$\frac{d}{dE} \int_{\gamma_E} \Omega(x, \xi) = \int_0^{T(E)} (\partial_x g - \partial_\xi f) dt. \quad (2.23)$$

We compute $\int_{x_E}^x T_1(y) dy$ with T_1 as in (2.21), and start to simplify $J_1 = \int \omega_1$, with ω_1 the last term on the RHS of (2.21). Let

$$g_1(x, \xi) := \frac{p_1^2(x, \xi)}{\partial_\xi p_0(x, \xi)}.$$

By (2.23) we get

$$\begin{aligned}
J_1 &= \frac{1}{2} \int_{\gamma_E} \frac{\partial_\xi g_1(x, \xi)}{\partial_\xi p_0(x, \xi)} dx = \frac{1}{2} \int_0^{T(E)} \partial_\xi g_1(x(t), \xi(t)) dt \\
&= -\frac{1}{2} \frac{d}{dE} \int_{\gamma_E} g_1(x, \xi) dx = -\frac{1}{2} \frac{d}{dE} \int_{\gamma_E} \frac{p_1^2(x, \xi)}{\partial_\xi p_0(x, \xi)} dx \\
&= -\frac{1}{2} \frac{d}{dE} \int_0^{T(E)} p_1^2(x(t), \xi(t)) dt,
\end{aligned} \tag{2.24}$$

which is the contribution of p_1 to the second term S_2 of generalized action in ([2], Theorem 2). Here $T(E)$ is the period on γ_E . This is precisely the expression [1] (Formula (3.15)) using Fourier representation. We also have

$$\int_{x_E}^x \frac{1}{\beta_0(y)} p_2(y, \xi(y)) dy = \int_{\gamma_E} \frac{p_2(x, \xi)}{\partial_\xi p_0(x, \xi)} dx = \int_0^{T(E)} p_2(x(t), \xi(t)) dt. \tag{2.25}$$

To compute T_1 modulo exact forms we are left to simplify in (2.21) the expression

$$\begin{aligned}
J_2 &= \int_{x_E}^x \frac{1}{\beta_0} \left(\frac{1}{8} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} + \frac{\varphi''}{12} \frac{\partial^4 p_0}{\partial y \partial \xi^3} - \frac{(\varphi'')^2}{24} \frac{\partial^4 p_0}{\partial \xi^4} \right) dy - \frac{1}{8} \int_{x_E}^x \frac{(\beta_0')^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2} dy + \\
&\quad \frac{1}{6} \int_{x_E}^x \varphi'' \frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3} dy + \left[\frac{\varphi''}{6\beta_0} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^x - \left[\frac{\beta_0'}{4\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^x.
\end{aligned}$$

Let $g_0(x, \xi) = \frac{\Delta(x, \xi)}{\partial_\xi p_0(x, \xi)}$, where we have set according to [2]

$$\Delta(x, \xi) = \frac{\partial^2 p_0}{\partial x^2} \frac{\partial^2 p_0}{\partial \xi^2} - \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right)^2.$$

Taking second derivative of eikonal equation $p_0(x, \xi(x)) = E$, we get

$$\frac{(\partial_\xi g_0)(x, \xi(x))}{\beta_0(x)} = -\frac{\varphi'''}{\beta_0} \frac{\partial^3 p_0}{\partial \xi^3} - 2\varphi'' \frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3} + \frac{\beta_0''}{\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2} - 2\frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial x \partial \xi^2} + \frac{(\beta_0')^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2}.$$

Integration by parts of the first and third term on the RHS gives altogether

$$\begin{aligned}
\int_{x_E}^x \frac{(\partial_\xi g_0)(y, \xi(y))}{\beta_0(y)} dy &= -3 \int_{x_E}^x \frac{1}{\beta_0} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} dy - 2 \int_{x_E}^x \frac{\varphi''}{\beta_0} \frac{\partial^4 p_0}{\partial y \partial \xi^3} dy + \int_{x_E}^x \frac{(\varphi'')^2}{\beta_0} \frac{\partial^4 p_0}{\partial \xi^4} dy \\
&\quad + 3 \int_{x_E}^x \frac{(\beta_0')^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2} dy - 4 \int_{x_E}^x \varphi'' \frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3} dy - \left[\frac{\varphi''}{\beta_0} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^x \\
&\quad + \left[\frac{\beta_0'}{\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^x + 3 \left[\frac{1}{\beta_0} \frac{\partial^3 p_0}{\partial y \partial \xi^2} \right]_{x_E}^x,
\end{aligned}$$

and modulo the integral of an exact form in \mathcal{A}

$$\begin{aligned}
J_2 &\equiv -\frac{1}{24} \int_{x_E}^x \frac{(\partial_\xi g_0)(y, \xi(y))}{\beta_0(y)} dy = -\frac{1}{24} \int_{\gamma_E} \frac{\partial_\xi g_0(x, \xi)}{\partial_\xi p_0(x, \xi)} dx \\
&= -\frac{1}{24} \int_0^{T(E)} \partial_\xi g_0(x(t), \xi(t)) dt = \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} g_0(x, \xi) dx \\
&= \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \frac{\Delta(x, \xi)}{\partial_\xi p_0(x, \xi)} dx = \frac{1}{24} \frac{d}{dE} \int_0^{T(E)} \Delta(x(t), \xi(t)) dt.
\end{aligned}$$

This is again the J_2 computed in [1] before Proposition 3.3, using Fourier representation. Using these expressions, we recover the well known action integrals (see e.g. [2]):

Proposition 2.1. *Let Γdt be the restriction to γ_E of the 1-form*

$$\omega_0(x, \xi) = \left(\frac{\partial^2 p_0}{\partial x^2} \frac{\partial p_0}{\partial \xi} - \frac{\partial^2 p_0}{\partial x \partial \xi} \frac{\partial p_0}{\partial x} \right) dx + \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \frac{\partial p_0}{\partial \xi} - \frac{\partial^2 p_0}{\partial \xi^2} \frac{\partial p_0}{\partial x} \right) d\xi.$$

We have $\operatorname{Re} \oint_{\gamma_E} \Omega_1 = 0$, whereas

$$\begin{aligned} \operatorname{Im} \oint_{\gamma_E} \Omega_1 &= \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt \\ &= \frac{1}{48} \left(\frac{d}{dE} \right)^2 \int_{\gamma_E} \Gamma dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt. \end{aligned} \quad (2.26)$$

2.4 Bohr-Sommerfeld quantization rule

We have shown that the (normalized) WKB solutions of the eigenvalues equation $(P(x, hD_x; h) - E)u(x; h) = 0$ are given by $u(x; h) = \sum_{\pm} u_{\pm}(x; h) = \sum_{\pm} u_{\pm}^a(x; h)$, where

$$u_{\pm}^a(x; h) = |\beta_0^{\pm}(x)|^{-1/2} e^{\frac{i}{h} S_{\pm}(x_E, x; h)} \left(C_0 + h(C_1(a_E) + D_1^{a_E, \pm}(x)) + \mathcal{O}(h^2) \right), \quad (2.27)$$

with C_0 and $C_1(a_E)$

$$C_0 = 2^{-1/2}; \quad C_1(a_E) = -2^{-3/2} \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right) (a_E). \quad (2.28)$$

the phase function $S_{\pm}(x_E, x; h)$ is given by (2.13), and $D_1^{a_E, \pm}(x)$ as in Lemma 2.1.

Starting from focal point a'_E instead, we can construct in a completely similar way

$$S_{\pm}(x'_E, x; h) = \varphi_{\pm}(x'_E) + \int_{x'_E}^x \xi_{\pm}(y) dy - h \int_{x'_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\beta_0^{\pm}(y)} dy. \quad (2.29)$$

and the corresponding symbols. So we denote again by $u^a(x; h) = \sum_{\pm} u_{\pm}^a(x; h)$ the microlocal solution of $(P(x, hD_x; h) - E)u(x; h) = 0$ valid uniformly with respect to h for x in any $I \subset \subset]x'_E, x_E[$.

The branches labelled by \pm are linearly related by some (constant) phase factors as in the special case of Schrödinger operator (1.4), see (1.5) for the leading order term. Computing the microlocal solutions near a_E and a'_E in Fourier representation as we did in [1] shows that these phase factors are indeed $e^{\pm i\pi/4}$. We avoid instead this computation by stating the following Ansatz:

$$\begin{aligned} \varphi_+(x_E) &= \varphi_-(x_E), \quad \varphi_+(x'_E) = \varphi_-(x'_E), \\ u_{\pm}^a(x; h) &= e^{\pm i\frac{\pi}{4}} |\beta_0^{\pm}(x)|^{-1/2} e^{\frac{i}{h} S_{\pm}(x_E, x; h)} \left(C_0 + hC_1^a + hD_1^{a, \pm}(x) + \mathcal{O}(h^2) \right), \\ u_{\pm}^{a'}(x; h) &= e^{\mp i\frac{\pi}{4}} |\beta_0^{\pm}(x)|^{-1/2} e^{\frac{i}{h} S_{\pm}(x'_E, x; h)} \left(C_0 + hC_1^{a'} + hD_1^{a', \pm}(x) + \mathcal{O}(h^2) \right). \end{aligned} \quad (2.30)$$

We will justify this Ansatz in Sect.3 in the special case of Schrödinger operator with analytic coefficients using the normal form of [21]. The point is that $e^{\pm i\frac{\pi}{4}}$ are not only in factor of the principal symbol of $u_{\pm}^a(x; h)$, $u_{\pm}^{a'}(x; h)$, but also of the lower order terms.

Remember that

$$\begin{aligned} F_{\pm}^a(x; h) &= C_0 e^{\pm i\frac{\pi}{4}} |\beta_0^{\pm}(x)|^{-\frac{1}{2}} e^{\frac{i}{h} S_{\pm}(x_E, x; h)} \\ &\times \left(d_0^{a, \pm}(x) + h \frac{C_1^a}{C_0} d_0^{a, \pm}(x) + \frac{h}{C_0} d_0^{a, \pm}(x) D_1^{a, \pm}(x) + h d_1^{a, \pm}(x) + \frac{h}{2i} (s'_{a, \pm}(x) + 2s_{a, \pm}(x) \theta_{\pm}(x)) + \mathcal{O}(h^2) \right), \end{aligned} \quad (2.31)$$

$$\begin{aligned}
F_{\pm}^{a'}(x; h) &= C_0 e^{\mp i \frac{\pi}{4}} |\beta_0^{\pm}(x)|^{-\frac{1}{2}} e^{\frac{i}{h} S_{\pm}(x'_E, x; h)} \\
&\times \left(d_0^{a', \pm}(x) + h \frac{C_1^{a'}}{C_0} d_0^{a', \pm}(x) + \frac{h}{C_0} d_0^{a', \pm}(x) D_1^{a', \pm}(x) + h d_1^{a', \pm}(x) + \frac{h}{2i} (s'_{a', \pm}(x) + 2s_{a', \pm}(x) \theta_{\pm}(x)) + \mathcal{O}(h^2) \right),
\end{aligned} \tag{2.32}$$

where we have set

$$d_0^{a', \pm}(x) = \beta_0^{\pm}(x) (\chi_1^{a'})'(x); \quad d_1^{a', \pm}(x) = \beta_0^{\pm}(x) (\chi_1^{a'})'(x); \quad \pm \beta_0^{\pm}(x) > 0.$$

A short computation shows that modulo $\mathcal{O}(h^2)$

$$\begin{aligned}
(u^a | F_{\pm}^{a'}) &= i e^{\frac{i}{h} \tilde{A}_{\pm}(x_E, x'_E; h)} \left(-C_0^2 - h C_0 (C_1^a + C_1^{a'}) + h C_0 \int_{x'_E}^{+\infty} (D_1^{a, +} + \overline{D_1^{a', +}}) (\chi_1^{a'})' dx + h C_0^2 \int_{x'_E}^{+\infty} \frac{\beta_1^+}{\beta_0^+} (\chi_1^{a'})' dx + \right. \\
&\left. \frac{ih C_0^2}{2} \int_{x'_E}^{+\infty} \frac{1}{\beta_0^+} (s'_{a', +}(x) + 2s_{a', +}(x) \overline{\theta_+(x)}) dx \right),
\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}
\tilde{A}_{\pm}(x_E, x'_E; h) &= S_{\pm}(x_E, x; h) - S_{\pm}(x'_E, x; h) \\
&= \varphi_{\pm}(x_E) - \varphi_{\pm}(x'_E) + \int_{x'_E}^{x_E} \xi_{\pm}(y) dy - h \int_{x'_E}^{x_E} \frac{p_1(y, \xi_{\pm}(y))}{\beta_0^{\pm}(y)} dy.
\end{aligned} \tag{2.34}$$

We know that

$$\begin{aligned}
D_1^{a, +}(x) &= -\frac{C_0}{2} \left[\partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right) (y, \xi_{\pm}(y)) \right]_{x_E}^x + i C_0 \int_{x'_E}^x T_1^+(y) dy + \frac{i C_0}{6} \left[\frac{\varphi_{\pm}''}{\beta_0^+} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^x - \frac{i C_0}{4} \left[\frac{(\beta_0^+)' }{(\beta_0^+)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^x, \\
D_1^{a', +}(x) &= -\frac{C_0}{2} \left[\partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right) (y, \xi_{\pm}(y)) \right]_{x'_E}^x + i C_0 \int_{x'_E}^x T_1^+(y) dy + \frac{i C_0}{6} \left[\frac{\varphi_{\pm}''}{\beta_0^+} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x'_E}^x - \frac{i C_0}{4} \left[\frac{(\beta_0^+)' }{(\beta_0^+)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x'_E}^x,
\end{aligned}$$

where we have set

$$\begin{aligned}
T_1^+(y) &= \frac{1}{\beta_0^+} \left(-p_2 + \frac{1}{8} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} + \frac{\varphi_{\pm}''}{12} \frac{\partial^4 p_0}{\partial y \partial \xi^3} - \frac{(\varphi_{\pm}'')^2}{24} \frac{\partial^4 p_0}{\partial \xi^4} \right) \\
&\quad - \frac{1}{8} \frac{((\beta_0^+)')^2}{(\beta_0^+)^3} \frac{\partial^2 p_0}{\partial \xi^2} + \frac{1}{6} \varphi_{\pm}'' \frac{(\beta_0^+)' }{(\beta_0^+)^2} \frac{\partial^3 p_0}{\partial \xi^3} + \frac{p_1}{(\beta_0^+)^2} \left(\partial_{\xi} p_1 - \frac{p_1}{2\beta_0^+} \frac{\partial^2 p_0}{\partial \xi^2} \right).
\end{aligned} \tag{2.35}$$

Another straightforward computation shows that

$$\begin{aligned}
\int_{x'_E}^{+\infty} (D_1^{a, +}(x) + \overline{D_1^{a', +}(x)}) (\chi_1^{a'})'(x) dx &= -C_0 \int_{x'_E}^{+\infty} \left(\frac{\beta_1^+}{\beta_0^+} - \frac{p_1}{(\beta_0^+)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right) (\chi_1^{a'})'(x) dx + C_1^a + C_1^{a'} - i C_0 \int_{x'_E}^{x_E} T_1^+(y) dy \\
&\quad - \frac{i C_0}{6} \left[\frac{\varphi_{\pm}''}{\beta_0^+} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^{x'_E} + \frac{i C_0}{4} \left[\frac{(\beta_0^+)' }{(\beta_0^+)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^{x'_E}.
\end{aligned}$$

On the other hand, integrating by parts gives

$$\begin{aligned}
\int_{x'_E}^{+\infty} \frac{s'_{a', +}(x)}{\beta_0^+(x)} dx &= \left[\frac{(\chi_1^{a'})' }{\beta_0^+} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x'_E}^{+\infty} + \int_{x'_E}^{+\infty} \frac{(\beta_0^+)'(x)}{(\beta_0^+(x))^2} \left(\frac{\partial^2 p_0}{\partial \xi^2} \right) (x, \xi_{\pm}(x)) (\chi_1^{a'})'(x) dx = \\
&\int_{x'_E}^{+\infty} \frac{(\beta_0^+)'(x)}{(\beta_0^+(x))^2} \left(\frac{\partial^2 p_0}{\partial \xi^2} \right) (x, \xi_{\pm}(x)) (\chi_1^{a'})'(x) dx.
\end{aligned}$$

We also have

$$\int_{x'_E}^{+\infty} \frac{s_{a', +}(x) \overline{\theta_+(x)}}{\beta_0^+(x)} dx = -\frac{1}{2} \int_{x'_E}^{+\infty} \frac{(\beta_0^+)'(x)}{(\beta_0^+(x))^2} \left(\frac{\partial^2 p_0}{\partial \xi^2} \right) (x, \xi_{\pm}(x)) (\chi_1^{a'})'(x) dx + i \int_{x'_E}^{+\infty} \frac{p_1}{(\beta_0^+)^2} \left(\frac{\partial^2 p_0}{\partial \xi^2} \right) (x, \xi_{\pm}(x)) (\chi_1^{a'})'(x) dx,$$

and it follows that

$$\begin{aligned} (u^a|F_+^{a'}) &\equiv i e^{\frac{i}{h}\tilde{A}_+(x_E, x'_E; h)} \left(-C_0^2 - ihC_0^2 \int_{x_E}^{x'_E} T_1^+(y) dy - \frac{ihC_0^2}{6} \left[\frac{\varphi_+''}{\beta_0^+} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^{x'_E} + \frac{ihC_0^2}{4} \left[\frac{(\beta_0^+)'}{(\beta_0^+)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^{x'_E} \right) \bmod \mathcal{O}(h^2) \\ &\equiv -\frac{i}{2} e^{\frac{i}{h}\tilde{A}_+(x_E, x'_E; h)} \left(1 + ih \int_{x_E}^{x'_E} T_1^+(y) dy + \frac{ih}{6} \left[\frac{\varphi_+''}{\beta_0^+} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^{x'_E} - \frac{ih}{4} \left[\frac{(\beta_0^+)'}{(\beta_0^+)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^{x'_E} \right) \bmod \mathcal{O}(h^2), \end{aligned}$$

so

$$(u^a|F_+^{a'}) \equiv -\frac{i}{2} e^{\frac{i}{h}A_+(x_E, x'_E; h)} \bmod \mathcal{O}(h^2), \quad (2.36)$$

and similarly

$$(u^a|F_-^{a'}) \equiv -\frac{i}{2} e^{\frac{i}{h}A_-(x_E, x'_E; h)} \bmod \mathcal{O}(h^2), \quad (2.37)$$

where we have set

$$\begin{aligned} A_\pm(x_E, x'_E; h) &= \tilde{A}_\pm(x_E, x'_E; h) + h^2 \int_{x_E}^{x'_E} T_1^\pm(y) dy + \frac{h^2}{6} \left[\frac{\varphi_\pm''}{\beta_0^\pm} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^{x'_E} - \frac{h^2}{4} \left[\frac{(\beta_0^\pm)'}{(\beta_0^\pm)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^{x'_E} \\ &= \varphi_\pm(x_E) - \varphi_\pm(x'_E) + \int_{x_E}^{x'_E} \xi_\pm(y) dy - h \int_{x_E}^{x'_E} \frac{p_1(y, \xi_\pm(y))}{\beta_0^\pm(y)} dy + h^2 \int_{x_E}^{x'_E} T_1^\pm(y) dy + \\ &\frac{h^2}{6} \left[\frac{\varphi_\pm''}{\beta_0^\pm} \frac{\partial^3 p_0}{\partial \xi^3} \right]_{x_E}^{x'_E} - \frac{h^2}{4} \left[\frac{(\beta_0^\pm)'}{(\beta_0^\pm)^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^{x'_E}. \end{aligned} \quad (2.38)$$

Hence we have

$$(u^a|F_+^{a'} - F_-^{a'}) \equiv \frac{i}{2} (e^{\frac{i}{h}A_-(x_E, x'_E; h)} - e^{\frac{i}{h}A_+(x_E, x'_E; h)}) \bmod \mathcal{O}(h^2). \quad (2.39)$$

A similar computation shows that

$$(u^{a'}|F_+^a - F_-^a) \equiv \frac{i}{2} (e^{-\frac{i}{h}A_-(x_E, x'_E; h)} - e^{-\frac{i}{h}A_+(x_E, x'_E; h)}) \bmod \mathcal{O}(h^2). \quad (2.40)$$

(taking again into account that the mixed terms $(u_\pm^a|F_\mp^{a'})$ and $(u_\pm^{a'}|F_\mp^a)$ are $\mathcal{O}(h^\infty)$ because the phase is non stationary).

We conclude as in [1]. Namely, microlocal solutions u^a and $u^{a'}$ extend as smooth solutions on the whole interval $]x'_E, x_E[$; we denote them by u_1 and u_2 . Since there are no other focal points between a and a' , they are expressed by the same formulae (which makes the analysis particularly simple) and satisfy $\bmod \mathcal{O}(h^2)$:

$$(u_1|F_+^a - F_-^a) \equiv 1, \quad (2.41)$$

$$(u_2|F_+^{a'} - F_-^{a'}) \equiv -1, \quad (2.42)$$

$$(u_1|F_+^{a'} - F_-^{a'}) \equiv \frac{i}{2} (e^{\frac{i}{h}A_-(x_E, x'_E; h)} - e^{\frac{i}{h}A_+(x_E, x'_E; h)}), \quad (2.43)$$

$$(u_2|F_+^a - F_-^a) \equiv \frac{i}{2} (e^{-\frac{i}{h}A_-(x_E, x'_E; h)} - e^{-\frac{i}{h}A_+(x_E, x'_E; h)}). \quad (2.44)$$

Now we define Gram matrix

$$G^{(a, a')}(E) := \begin{pmatrix} (u_1|F_+^a - F_-^a) & (u_2|F_+^a - F_-^a) \\ (u_1|F_+^{a'} - F_-^{a'}) & (u_2|F_+^{a'} - F_-^{a'}) \end{pmatrix}, \quad (2.45)$$

whose determinant $-\cos^2((A_-(x_E, x'_E; h) - A_+(x_E, x'_E; h))/2h)$ vanishes precisely on eigenvalues of P in I , which allows to obtain modulo $\mathcal{O}(h^3)$

$$A_-(x_E, x'_E; h) - A_+(x_E, x'_E; h) = \pi h + 2\pi n h, \quad n \in \mathbb{Z}.$$

If $\varphi_+(x_E) = \varphi_-(x_E)$ and $\varphi_+(x'_E) = \varphi_-(x'_E)$, so modulo exact forms, we have

$$\begin{aligned} A_-(x_E, x'_E; h) - A_+(x_E, x'_E; h) &= \int_{x'_E}^{x_E} (\xi_+(y) - \xi_-(y)) dy - h \int_{x'_E}^{x_E} \left(\frac{p_1(y, \xi_+(y))}{\beta_0^+(y)} - \frac{p_1(y, \xi_-(y))}{\beta_0^-(y)} \right) dy + \\ &h^2 \int_{x'_E}^{x_E} (T_1^+(y) - T_1^-(y)) dy + \mathcal{O}(h^3). \end{aligned}$$

We have

$$\begin{aligned} \int_{x'_E}^{x_E} (\xi_+(y) - \xi_-(y)) dy &= \oint_{\gamma_E} \xi(y) dy, \\ \int_{x'_E}^{x_E} \left(\frac{p_1(y, \xi_+(y))}{\beta_0^+(y)} - \frac{p_1(y, \xi_-(y))}{\beta_0^-(y)} \right) dy &= \int_{\gamma_E} p_1 dt, \\ \int_{x'_E}^{x_E} (T_1^+(y) - T_1^-(y)) dy &= \oint_{\gamma_E} T_1(y) dy = \text{Im} \oint_{\gamma_E} \Omega_1(y) \\ &= \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt. \end{aligned}$$

This takes the proof of Theorem 1.1 to an end.

3 Checking the Ansatz in the case of Schrödinger operator with analytic coefficients

Here we use the normal form, taking a semi-classical Schrödinger operator $P(x, hD_x) = -h^2\Delta + V(x)$ at energy E , near a simple turning point of $V(x)$, say x_E , to Airy operator $-h^2\Delta + y$ near 0, see (1.6). We assume V to be analytic near x_E , such that $V(x) - E \sim x - x_E$, and $x > x_E$ is the classically forbidden region (CFR). Reduction of P to its normal form Q has been achieved in the framework of exact complex WKB analysis, starting from a somewhat heuristic level in [23] and then formalized in [21] using Sato's Microdifferential Calculus, which we follow here closely. We are particularly interested in computing the precise asymptotics of the solutions of (1.4) up to order 4 in h . They are linear combinations of formal WKB solutions $Y_{\text{WKB}}(x; h)$ in $x > x_E$ or $x \in [x'_E, x_E]$, the classically allowed region (CAR). The complex WKB method consists in constructing the branches of $Y_{\text{WKB}}(x; h)$ in the complex plane across Stokes lines, that verify (1.4) up to exponential accuracy. The main contribution of [23] (where the CAR is taken instead to be $x > x'_E$) was to correct some formulas encountered in the previous Physics literature, by taking into account the precise jump of the phase that takes place when crossing the Stokes line $\arg(x - x_E) \sim 0$. The resulting connexion formula reads at leading order

$$u(x; h) \sim \left(\frac{dS}{dx} \right)^{-1/2} [(\tilde{b} - i\tilde{a})e^{i\pi/4} e^{iS/h} + (\tilde{b} + i\tilde{a})e^{-i\pi/4} e^{-iS/h}], \quad \arg(x_E - x) \sim 0 \quad (3.1)$$

$$u(x; h) \sim \left(-\frac{d\tilde{S}}{dx} \right)^{-1/2} [2\tilde{b}e^{\tilde{S}/h} + (\tilde{a} \pm i\tilde{b})e^{-\tilde{S}/h}], \quad \arg(x - x_E) \sim 0, \quad \pm \text{Im} x > 0 \quad (3.2)$$

Here $v = \tilde{a}Ai + \tilde{b}Bi$ is the general solution of $(-h^2\Delta + y)v = 0$ (\tilde{a}, \tilde{b} being constants), $S(x; h) = S^{(0)}(x) + \mathcal{O}(h^2)$, $S^{(0)}(x) = \int^x \sqrt{E - V(t)} dt$ the action in CAR and $\tilde{S}(x; h) = \int^x \sqrt{V(t) - E} dt + \mathcal{O}(h^2)$, its analytic continuation in CFR. The physical solution (purely decaying in the CFR) is obtained with $\tilde{b} = 0$, which we will assume here, though our method applies also to the case $\tilde{b} \neq 0$. It is stressed in [23] that the coefficients of the various components of (3.1) are independent of the order in h to

which S and \widetilde{S} have been calculated; stated differently, it means that (3.1) are not only asymptotics expansions in h , but rather analytic functions of the variables (x, h) .

It is also well-known ([23], Formulas (17)-(19)) (but we will use this fact), that $S(x; h) \sim \sum_{n=0}^{\infty} h^{2n} S^{(n)}(x)$, can be computed in terms of another asymptotic series $\phi(x; h) \sim \sum_{n=0}^{\infty} h^{2n} \phi^{(n)}(x)$, with $S^{(0)}(x) = (\frac{3}{2} \phi^{(0)}(x))^{2/3}$, and the $\phi^{(n)}$ are determined recursively from a Riccati equation. In particular, the expansion of $S(x; h)$ contains only even powers of h , which we recover independently here, with the method of ([21]).

In turn, $S(x; h)$ and $\widetilde{S}(x; h)$ can be interpreted as Borel resummations of symbols in h . Such resummations were given a mathematical justification in the framework of Sato's Microdifferential Calculus or Ecalle's Resurgence Theory (see [21], [17] and references therein). Our aim is to retrieve connexion formula (3.1) from the Microdifferential Calculus. This will in particular justify our Ansatz in this case.

3.1 Reduction to Airy equation

For the reader's convenience we adopt in this Sect. the notations of [21], and allow sometimes for a potential depending also on $h = \eta^{-1}$ as a pre-Borel summable power series of $h = \eta^{-1}$, namely $Q(\tilde{x}, \eta) = Q_0(\tilde{x}) + \eta^{-1} Q_1(\tilde{x}) + \dots$. So we need to reduce the ODE $(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}, \eta)) \tilde{\varphi}(\tilde{x}, \eta) = 0$ to Airy ODE $(\frac{d^2}{dx^2} - \eta^2 x) \varphi(x, \eta) = 0$.

As usual, we identify an analytic function $\check{\phi}(x, y)$ with its "symbol", i.e. Borel sum $\phi(x, \eta) = \int_{\gamma} e^{-y\eta} \check{\phi}(x, y) dy$, where γ is an integration contour in $\text{Re}(y\eta) > 0$.

This formally amounts to quantize η by ∂_y .

So we need to reduce the microdifferential operator $\tilde{A}(\tilde{x}, \partial_{\tilde{x}}, \partial_y) = \frac{\partial^2}{\partial \tilde{x}^2} - Q(\tilde{x}, \frac{\partial}{\partial y}) \frac{\partial^2}{\partial y^2}$ to the microdifferential operator $B(x, \partial_x, \partial_y) = \frac{\partial^2}{\partial x^2} - x \frac{\partial^2}{\partial y^2}$.

Recall the main result of [21]:

Theorem 3.1. *Assume $Q_0(\tilde{x})$ has a simple zero at $\tilde{x} = 0$. Then, in a neighborhood of $\tilde{x} = 0$, and with a holomorphic change of coordinate $x(\tilde{x}) = x$ such that*

$$x(\tilde{x})(x'(\tilde{x}))^2 = Q_0(\tilde{x}), \quad (3.3)$$

with $x(0) = 0$, we can find invertible microdifferential operators S and T with normal (ordered) product

$$\begin{aligned} S &=: (g'(x))^{5/2} \left(1 + \frac{\partial r(x, \eta)}{\partial x}\right)^{3/2} \exp(r(x, \eta)\xi) : \\ T &=: (g'(x))^{1/2} \left(1 + \frac{\partial r(x, \eta)}{\partial x}\right)^{-1/2} \exp(r(x, \eta)\xi) : \end{aligned} \quad (3.4)$$

such that

$$\tilde{A}(\tilde{x}, \partial_{\tilde{x}}, \partial_y)|_{\tilde{x}=g(x)} T = S \tilde{B}(x, \partial_x, \partial_y). \quad (3.5)$$

Here $\tilde{x} = g(x)$ denotes the inverse function of $x = x(\tilde{x})$ near 0, and $r(x, \eta) \sim r_1(x)\eta^{-1} + r_2(x)\eta^{-2} + \dots$ is a symbol of order -1 .

Note that $\exp(r(x, \eta)\xi)$ is simply the symbol of the "shift" operator

$$: \exp(r(x, \eta)\xi) : \varphi(x)|_{x=x(\tilde{x})} = \varphi(x(\tilde{x}) + r(x, \eta)), \quad (3.6)$$

while $(g'(x))^{1/2} (1 + \frac{\partial r(x, \eta)}{\partial x})^{-1/2}$, not containing ξ , is (the symbol of) a multiplication operator.

Theorem 3.1 is a consequence of ([21], Proposition 2.2), that we use to compute the asymptotics of $r(x, \eta)$.

Let $\tilde{r}(\tilde{x}, \eta) = x + r(x, \eta)$. We recall from ([21], Eq.(2.6)) that $\tilde{r}(\tilde{x}, \eta)$ verifies the ‘‘master’’ equation

$$\left(\frac{\partial \tilde{r}}{\partial \tilde{x}}\right)^2 \tilde{r}(x, \eta) - \frac{1}{2} \eta^{-2} \{\tilde{r}, \tilde{x}\} = Q_0(\tilde{x}) + Q_1(\tilde{x}) \eta^{-1} + Q_2(\tilde{x}) \eta^{-2} + \dots \quad (3.7)$$

Here $\{\tilde{r}, \tilde{x}\}$ denotes the Schwarzian derivative

$$\{\tilde{r}, \tilde{x}\} = \frac{\frac{\partial^3 \tilde{r}}{\partial \tilde{x}^3}}{\frac{\partial \tilde{r}}{\partial \tilde{x}}} - \frac{3}{2} \left(\frac{\frac{\partial^2 \tilde{r}}{\partial \tilde{x}^2}}{\frac{\partial \tilde{r}}{\partial \tilde{x}}} \right)^2.$$

Assuming that \tilde{r} has the following asymptotics

$$\tilde{r}(\tilde{x}, \eta) = x_0(\tilde{x}) + x_1(\tilde{x}) \eta^{-1} + x_2(\tilde{x}) \eta^{-2} + \dots, \quad (3.8)$$

we find that the $x_j(\tilde{x})$ solve a hierarchy of ‘‘transport equations’’. Denoting the differentiation with respect to \tilde{x} by a prime, the first one ([21] Eq. (2.7.0)) is of the form (3.3), which trivially holds for $x_0 = x$. So we have

$$x(\tilde{x}) = x_0(\tilde{x}) = \left(\frac{3}{2} \int_0^{\tilde{x}} \sqrt{Q_0(\tilde{y})} d\tilde{y}\right)^{2/3}. \quad (3.9)$$

The second one ([21] Eq. (2.7.1)) yields the first order ODE

$$x'_0(\tilde{x}) (2x_0(\tilde{x}) \frac{d}{d\tilde{x}} + x'_0(\tilde{x})) x_1(\tilde{x}) = Q_1(\tilde{x}),$$

which we solve as

$$x_1(\tilde{x}) = \frac{1}{2} (x_0(\tilde{x}))^{-1/2} \int^{\tilde{x}} \sqrt{Q_0(\tilde{y})} Q_1(\tilde{y}) d\tilde{y}, \quad (3.10)$$

and this vanishes when $Q_1 = 0$. This is consistent with the properties of WKB solution computed through Ricatti Equation.

The next ‘‘transport equation’’ ([21] Eq. (2.7.2)) can be written as

$$x'_0(x'_0 x_2 + 2x_0 x'_2) + x'_1(x'_1 x_0 + 2x'_0 x_1) - \frac{1}{2} \{x_0, \tilde{x}\} = Q_2(\tilde{x}),$$

and we find

$$x_2(\tilde{x}) = \frac{1}{2} (x_0(\tilde{x}))^{-1/2} \int^{\tilde{x}} \sqrt{Q_0(\tilde{y})} \left(\frac{1}{2} \{x_0, \tilde{y}\} + Q_2(\tilde{y}) - x'_1(x'_1 x_0 + 2x'_0 x_1) \right) d\tilde{y}. \quad (3.11)$$

In case $Q_2 = 0$ let us compute Taylor expansion of $x_2(\tilde{x})$ at $x = 0$. We find:

$$x_2(\tilde{x}) = \frac{3}{7} v_3 - \frac{9}{35} v_2^2 + \mathcal{O}(\tilde{x}), \quad (3.12)$$

where the coefficients v_j are defined by:

$$Q_0(\tilde{x}) = \tilde{x} + \sum_{n=2}^{+\infty} v_n \tilde{x}^n.$$

The next ‘‘transport equation’’ ([21] Eq. (2.7.3)) determines $x_3(\tilde{x})$. It is of the form

$$x'_0(x'_0 x_3 + 2x_0 x'_3) + E_1(\tilde{x}) = Q_3(\tilde{x}),$$

and we find

$$x_3(\tilde{x}) = \frac{1}{2} (x_0(\tilde{x}))^{-1/2} \int^{\tilde{x}} \sqrt{Q_0(\tilde{y})} (Q_3(\tilde{y}) - E_1(\tilde{y})) d\tilde{y}, \quad (3.13)$$

where

$$E_1 = 2x'_1(x'_0x_2 + x_0x'_2) + x_1(2x'_0x'_2 + (x'_1)^2) - \frac{1}{2}(x'_0)^{-2}(x'_0x'''_1 - x'_1x'''_0) + \frac{3}{2}x''_0(x'_0)^{-3}(x'_0x''_1 - x'_1x''_0).$$

So again in case $Q_3(\tilde{x}) = 0$ we have $x_3(\tilde{x}) = 0$.

The next “transport equation” ([21] Eq. (2.7.4)) determines $x_4(\tilde{x})$, and takes the form

$$x'_0(x'_0x_4 + 2x_0x'_4) + E_2(\tilde{x}) = Q_4(\tilde{x}),$$

where

$$E_2 = 2x'_0x'_1x_3 + x_2(2x'_0x'_2 + (x'_1)^2) + 2x_1(x'_0x'_3 + x'_1x'_2) + x_0(2x'_1x'_3 + (x'_2)^2) - \frac{1}{2}(x'_0)^{-3}(x'_0(x'_0x'''_2 - x'_2x'''_0) - x'_1(x'_0x'''_1 - x'_1x'''_0)),$$

and we find

$$x_4(\tilde{x}) = \frac{1}{2}(x_0(\tilde{x}))^{-1/2} \int^{\tilde{x}} \sqrt{Q_0(\tilde{y})} (Q_4(\tilde{y}) - E_2(\tilde{y})) d\tilde{y}. \quad (3.14)$$

3.2 WKB solution of order 4

We use Theorem 3.1 to compute $T\varphi$. Using (3.3) and (3.4) we see that

$$T\varphi(x, \eta) = (g'(x))^{1/2} \left(1 + \frac{\partial r(x, \eta)}{\partial x}\right)^{-1/2} \varphi(x(\tilde{x}) + r(x, \eta)). \quad (3.15)$$

Now $\varphi(x, \eta) = \text{Ai}(x\eta^{2/3})$ solves $(\frac{\partial^2}{\partial x^2} - x\frac{\partial^2}{\partial y^2})\varphi(x, \eta) = 0$ (at the level of symbols). Thus by Theorem 3.1 the solution of $\tilde{A}u = 0$, evaluated at $\tilde{x} = g(x)$ is of the form

$$(g'(x))^{1/2} \left(1 + \frac{\partial r(x, \eta)}{\partial x}\right)^{-1/2} \text{Ai}\left(\eta^{2/3}(x(\tilde{x}) + r(x, \eta))\right). \quad (3.16)$$

Substituting this expression in the asymptotics of Airy function in $\eta^{2/3}(x(\tilde{x}) + r(x, \eta)) = z' = \eta^{2/3}z < 0$ gives, with $h = 1/\eta$, in a punctured neighborhood of $x = 0$

$$\begin{aligned} \text{Ai}(z') &\sim z'^{-1/4} \sin\left(\frac{2}{3h}z^{3/2} + \frac{\pi}{4}\right) \left[1 - \frac{385}{4608}h^2z^{-3} + \frac{111546435}{382205952}h^4z^{-6} + \mathcal{O}(h^6)\right] - \\ & z'^{-1/4} \cos\left(\frac{2}{3h}z^{3/2} + \frac{\pi}{4}\right) \left[\frac{5}{48}hz^{-3/2} - \frac{765765}{5971968}h^3z^{-9/2} + \mathcal{O}(h^5)\right] \end{aligned}$$

up to the common factor $\pi^{-1/2}$.

We expand sin and cos and factor out the phase factors $e^{\pm i\pi/4}$, so that (3.4) gives

$$T\varphi(x, \eta) = \frac{1}{2}(g'(x))^{1/2} (x(\tilde{x}) + r(x, \eta))^{-1/4} \left(1 + \frac{\partial r(x, \eta)}{\partial x}\right)^{-1/2} \left[(-R_2 - iR_1)e^{i\pi/4} e^{\frac{2i}{3h}z^{3/2}} + (-R_2 + iR_1)e^{-i\pi/4} e^{-\frac{2i}{3h}z^{3/2}}\right], \quad (3.17)$$

where

$$R_1(z; h) = 1 - \frac{385}{406}h^2z^{-3} + \mathcal{O}(h^4),$$

and

$$R_2(z; h) = \frac{5}{48}hz^{-3/2} + \mathcal{O}(h^3).$$

This shows also that Maslov correction $e^{\pm i\pi/4}$ is common to all terms of the asymptotics. We can check also that at least, at leading order

$$(g'(x))^{1/2} (x(\tilde{x}) + r(x, \eta))^{-1/4} = \left(\frac{dS}{d\tilde{x}}\right)^{-1/2}$$

So we can rewrite (3.17) as (3.1) when $\tilde{b} = 0$, making it more precise by adding the h^2 correction to the prefactors of $e^{\pm iS(x)/h}$. Considering the WKB solutions of (1.4) near the focal point a'_E , it is easily seen that $D_1^{a',+}(x) = \overline{D_1^{a',-}(x)}$, and $S_+(x'_E, x; h) = -S_-(x'_E, x; h)$, so that (2.30) agrees with (3.17). So we have checked our Ansatz (2.30) in case of Schrödinger operator, with an additional accuracy of h^3 .

Remark 1: Asymptotics (3.9) breaks down of course for z (or $x(\tilde{x})$) near 0, i.e. at the caustics. Nevertheless the argument of Airy function is not evaluated at $x(\tilde{x})$, but at $x(\tilde{x}) + r(x, \eta)$. Looking at the asymptotics (3.3) we see that if $x_2(0) < 0$, then $x(\tilde{x}) + r(x, \eta) = x(\tilde{x}) + h^2 x_2(\tilde{x}) + \dots$ is $\sim -h^2$ already for $x(\tilde{x}) = 0$. So asymptotics (3.9) is indeed “regular” at $x(\tilde{x}) = 0$, and exact WKB method “smears out” the geometric singularity given by ordinary asymptotics in that case. By (3.6) this holds when $v_3 = 0$. On the contrary, if $x_2(0) > 0$, asymptotics (3.9) is already “singular” for $x(\tilde{x}) \sim h^2$. These effects of course are irrelevant from the point of view of geometric asymptotics.

Remark 2: Taking into account Bi in the solution of Airy equation, we could as well make (3.2) more precise, by computing the asymptotics of $T\psi(x, \eta)$, with $\psi(x, \eta) = \text{Bi}(x\eta^{2/3})$. Thus we can reveal the full connexion formula, when continuing these exact WKB solutions in the complex domain across Stokes lines. This is of course of interest in tunneling problems.

References

- [1] A. Ifa H. Louati and M. Rouleux. Bohr-Sommerfeld Quantization Rules Revisited: the Method of Positive Commutators. J. Math. Sci. Univ. Tokyo, 25(2):1–37, 2018.
- [2] Y. Colin de Verdière. Bohr-sommerfeld Rules to All orders. Ann. H.Poincaré, 6:925–936, 2005.
- [3] B. Helffer and J. Sjöstrand. Semi-classical analysis for Harper’s equation III. Mémoire No 39, Soc. Math. de France, 117(4), 1988.
- [4] V.P. Maslov and M.V. Fedoriuk. Semi-classical approximation in Quantum Mechanics. D. Reidel Publishing Company, 1981.
- [5] P.N. Argyres. The Bohr-Sommerfeld Quantization Rule and the Weyl correspondence. Physics, (2):131–199, 1965.
- [6] R. Littlejohn. Lie Algebraic Approach to Higher-Order Terms. 6 2013.
- [7] A. Voros. Ann. Inst. Henri Poincaré, Sect. A26, 343, 1977.
- [8] B. Helffer and D. Robert. Puits de potentiel généralisés et asymptotique semi-classique. Annales Inst. H.Poincaré, 41(3):291–331, 1984.
- [9] J. L. Dunham. The WKB Method of Solving the Wave Equation. Phys. Rev, 41(713), 1932.
- [10] Carl M. Bender and Steven A. Orszag. Advanced Mathematical Methods for scientists and engineers. Springer, 1979.
- [11] M.V. Fedoriouk. Méthodes asymptotiques pour les Équations Différentielles Ordinaires Linéaire. Asymptotic Analysis. Springer, Mouscou, Mir edition, 1987.

- [12] M. Rouleux. Tunneling effects for \hbar -Pseudodifferential Operators, Feshbach resonances and the Born-oppenheimer approximation. Adv. Part. Diff. Eq Wiley-VCH, 1999.
- [13] A. Ifa and M. Rouleux. The one dimensional semi-classical Bogoliubov-de Gennes Hamiltonian Preprint hal-02072721. 2019 (submitted).
- [14] A. Voros. Résurgence quantique. Ann. Inst. Fourier, 43(2):1509-1534, 1993.
- [15] T. Aoki T. Kawai and Y. Takei. The Bender-Wu Analysis and the Voros Theory. 1991.
- [16] B. Candelpergher C. Nosmas and F. Pham. Resurgence, quantized canonical transformations, and multi-instanton expansions. Papers Dedicated to Professor Mikio Sato on the Occasion of his Sixtieth Birthday. 2(1):699–726, 1988.
- [17] E. Delabaere and F. Pham. Resurgence methods in semi-classical asymptotics. Ann. Inst. H.Poincaré, 71(1):1-94, 1999.
- [18] E. Delabaere H. Dillinger and F. Pham. Exact semi-classical expansions for 1-D quantum oscillators. J.Math.Phys, 38(12):6126–6184, 1997.
- [19] J. Sjöstrand. Density of states oscillations for magnetic Schrödinger operators, in: Bennewitz (ed.). Diff. Eq. Math. Phys, pages 295–345, 1990.
- [20] A. Ifa H. Louati and M. Rouleux. Bohr-Sommerfeld Quantization Rules Revisited: the Method of Positive Commutators (Erratum). J. Math. Sci. Univ. Tokyo, 27(1):81–85, 2020.
- [21] T. Aoki and J. Yoshida. Microlocal reduction of ordinary differential operators with a large parameter. Publ RIMS, Kyoto Univ, 29:959–975, 1993.
- [22] M. Sato T. Kawai and M. Kashiwara. Microfunctions and pseudo-differential equations. Lecture Notes in Math., Springer, 287:265–529, 1973.
- [23] H. Silverstone. JWKB Connection-Formula Problem revisited via Borel Summation. Physical Rev. Lett., 55 (23):2523–2526, 1986.
- [24] J. Ecalle. Les fonctions réurgentes, 1-3. Publ. Math. Orsay, Univ. Paris-Sud, 1981, 1985.
- [25] L. Hörmander. The Analysis of Linear Partial Differential Operators I, III. Springer-Verlag, 1983.
- [26] A. Ifa and M. Rouleux. The one dimensional semi-classical Bogoliubov-de Gennes Hamiltonian with PT symmetry: generalized Bohr-Sommerfeld quantization rules. Journal of Physics: Conference Series-IOPscience, 1194:1–11, 2019.
- [27] F.W.J. Olver. Introduction to Asymptotics and Special Functions. Academic Press, New York, 1974.
- [28] A. Ifa N. M’hadbi and M. Rouleux. On generalized Bohr-Sommerfeld quantization rules for operators with PT symmetry. Mathematical Notes, 99(5):673–683, 2016.
- [29] D.R. Yafaev. The semiclassical limit of eigenfunctions of the Schrödinger equation and the Bohr-sommerfeld quantization condition, revisited. Algebra i Analiz, 22(6):270–291, 2010.