Introduction to Identification Methods

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Chapter 8

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8.1. Introduction

The previous chapters of this book clearly show the richness, versatility and usefulness of kinematic or thermal full-field measurements. Such measurement methodologies yield a large amount of data, in practice often in the form of digitized image files. Experimental procedures based on the characterization of materials and structures have naturally evolved so as to accommodate this kind of data and exploit it to the full, in particular, when the measured kinematic or thermal fields are heterogeneous. The overall goal is to optimally exploit this kind of data to identify constitutive parameters and, in particular, to estimate as many parameters as possible using as few experiments as possible. Given both the obvious advantages and potentialities afforded by full-field measurements and the widespread needs they cover, their application to the characterization of the mechanical response of materials and structures has become a very active field of research. In this context, this chapter aims at presenting the problem of identification in general terms and surveying the main computational identification approaches applicable to heterogeneous field data.

8.2. Identification and inversion: a conceptual overview

8.2.1. *Inversion*

To know and understand better a physical system requires the gathering and exploitation of relevant experimental data. In many situations, the quantities that are actually being measured do not directly yield the sought information. Rather, the

latter is hidden in the physical system under examination, the measurement being the consequence of a cause that is the real quantity of interest.

EXAMPLE 8.1.– A body whose material properties are characterized by a (thermal, electrostatic, etc.) heterogeneous conductivity coefficient $k(\boldsymbol{x})$ occupies the spatial region Ω. A flux q^{D} is prescribed on the boundary $\partial\Omega$. The functions u (temperature, electrostatic potential, etc.), q^{D} and k satisfy the equations:

$$\begin{aligned} \mathrm{div}\,(k\boldsymbol{\nabla}u) &= 0 \quad \text{in } \Omega \text{ (local equilibrium equation)} \\ k\boldsymbol{\nabla}u\cdot\boldsymbol{n} &= q^{\mathrm{D}} \quad \text{on } \partial\Omega \text{ (prescribed flux)} \end{aligned} \qquad [8.1]$$

The potential u, assumed to be measurable over $\partial\Omega$, is thus implicitly related to the sought conductivity k.

Exploiting the available experimental data (symbolically denoted by d) thus requires us to formulate a model describing the underlying physics so as to introduce a mathematical (and thus a quantitative) link to the hidden quantities (symbolically denoted by $\boldsymbol{\theta}$) of interest. The symbolic notation

$$\boldsymbol{g}(\boldsymbol{\theta},\boldsymbol{d}) = \boldsymbol{0} \qquad [8.2]$$

for the physical model then expresses the fact that $\boldsymbol{d}$ and $\boldsymbol{\theta}$ are related through the equations describing the relevant physics (such as those used in example 8.1, where $\boldsymbol{\theta} \equiv k$ and $\boldsymbol{d} \equiv u|_{\partial\Omega}$).

Models that accurately describe the mechanical response of solids or materials do not usually enable exact, analytical solutions due to the complexity of the considered configurations. The relevant equations are thus, as a rule, solved numerically. Computational mechanics and engineering has undergone a tremendous development over the last few decades, from the viewpoint of both computational power and algorithmic development.

Generally, the mathematical model is most frequently solved for the physical response of the system assuming the parameters that characterize its geometry, constitutive properties and kinematic constraints and the excitations (prescribed loads, displacements, temperatures, etc.) are known. However, in identification situations, the commonly available measured information pertains to the response of the system to given excitations. In other words, standard computational methods allow us to solve the *forward problem*, that is to find $\boldsymbol{d}$ from [8.2] with given $\boldsymbol{\theta}$. To evaluate unknown system parameters $\boldsymbol{\theta}$ from measurements of the response $\boldsymbol{d}$ entails solving the physical model [8.2] in a reverse fashion (relative to the standard situations), hence the term *inverse problem*.

In example 8.1, the forward problem consists of finding the potential u by solving equations [8.1] for given conductivity $k(\boldsymbol{x})$, geometry Ω and excitation q^{D}. A typical

inverse problem consists of reconstructing the unknown conductivity field $k(\boldsymbol{x})$ from measured values of u on the boundary, the flux q^{D} being again considered as *a priori* known. Note that the measurement $u|_{\partial\Omega}$ depends on the conductivity k in a nonlinear way.

8.2.1.1. *Forward and inverse problems*

The concepts of forward and inverse problems may be conveniently formulated and explained by considering the action of solving the (mechanical, thermal, etc.) governing equations as the prediction of the response $\boldsymbol{d}$ (displacement, stress, temperature, etc.) of the system under consideration to applied excitations $\boldsymbol{X}$ (forces, sources, prescribed displacements, initial stresses, heat fluxes, etc.).

The system (e.g. the mechanical structure being tested, or the region where wave propagation or flow takes place, etc.) usually depends on parameters symbolically denoted by $\boldsymbol{\theta}$: geometry (the region in space occupied by the body), physical characteristics of the constitutive materials, kinematic constraints, etc. The forward problem consists of evaluating the response $\boldsymbol{d}$ given the excitation $\boldsymbol{X}$ and the system parameters $\boldsymbol{\theta}$. The mathematical model describing the relevant physics is usually such that the response $\boldsymbol{d}$ is an implicit function of $(\boldsymbol{X}, \boldsymbol{\theta})$:

$$\text{find } \boldsymbol{d} = \boldsymbol{d}(\boldsymbol{\theta}; \boldsymbol{X}) \text{ such that } \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{\theta}, \boldsymbol{d}) = \boldsymbol{0} \quad (\text{with } \boldsymbol{X}, \boldsymbol{\theta} \text{ given}) \qquad [8.3]$$

In most cases, the forward problem is well-posed in the Hadamard sense, that is its solution (1) exists, (2) is unique and (3) depends continuously on the data. Condition 3 ensures that the response will be only moderately sensitive to small errors caused by, for example, discretization or imperfect data.

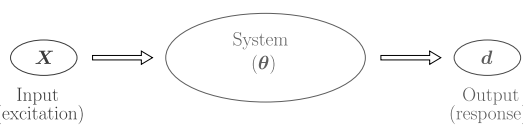

Figure 8.1. *Forward problem*

The inverse problem usually corresponds to situations where the system is at least partially unknown because of incomplete available information on features such as the geometry of the system, constitutive materials and initial conditions. To compensate for, and reconstruct, the missing information on the system parameters $\boldsymbol{\theta}$, supplementary (possibly partial) information about the response $\boldsymbol{d}$ must be sought in addition to the known excitations $\boldsymbol{X}$. The "inverse" qualifier serves as a reminder that the supplementary information is used in a reverse way relative to the usual solution methodologies applied to the physical model: from (partial) information about the

response, we use the model equations backward to find hidden system characteristics that usually cannot be measured directly:

$$\text{find } \boldsymbol{\theta} \in \Theta \text{ such that } \quad \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{\theta}, \boldsymbol{d}^{\text{obs}}) = \boldsymbol{0} \qquad [8.4]$$

where Θ denotes the parameter space in which $\boldsymbol{\theta}$ is sought. The symbolic notation used in [8.3] and [8.4] fails to emphasize the fact that the forward and inverse problems have very different characteristics and mathematical properties. Inverse problems are often ill-posed in that at least one of the Hadamard well-posedness conditions is violated. In particular, any solution $\boldsymbol{\theta}$ is typically highly sensitive to experimental errors.

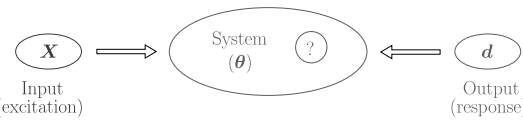

Figure 8.2. *Inverse problem*

8.2.1.2. *Reformulation as an optimization problem*

Practically, it is often neither convenient nor desirable to base the inversion of experimental data on solving exactly an equation of the type [8.4], which is usually multidimensional and nonlinear. The dimension of the data space $\mathcal{D}$ does not, in general, coincide with that of the parameter space Θ (such dimensions are, in practice, finite even though theoretical analyses often consider continuous models with observable data and unknown parameters modeled as functions). In fact, since reducing the adverse effect of uncertainties and making inversion methods more robust are main concerns, we often aim to have much more data than unknowns, making the observation equation [8.4] overdetermined. Therefore, the latter cannot, in general, be solved exactly (unless the data happen to verify certain solvability conditions), for at least two reasons: (1) the model equations $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{\theta}, \boldsymbol{d})$ only approximately describe the actual physical behavior of the system and (2) the experimental data $\boldsymbol{d}^{\text{obs}}$ suffer from measurement uncertainties. To explain the general impossibility of solving exactly [8.4] in another way, we can note that the above considerations (1) and (2) imply that the observed data $\boldsymbol{d}^{\text{obs}}$ cannot, in general, be exactly reproduced by predictions $\boldsymbol{d}(\boldsymbol{\theta}; \boldsymbol{X})$ of the physical model [8.3].

These considerations often (although not always, as will be seen in section 8.4) suggest the reformulation of the data inversion problem as a minimization problem, whose typical form is:

$$\boldsymbol{\theta} = \underset{\boldsymbol{\vartheta} \in \Theta}{\arg\min}\, \mathcal{J}(\boldsymbol{\vartheta}), \quad \mathcal{J}(\boldsymbol{\vartheta}) = \|\boldsymbol{d}(\boldsymbol{\vartheta}; \boldsymbol{X}) - \boldsymbol{d}^{\text{obs}}\| \qquad [8.5]$$

where $\|\cdot\|$ denotes the norm to be specified. This norm is often chosen as the usual quadratic norm (also known as the L^2-norm), possibly weighted; this choice is often

convenient because the L^2-norm, unlike other norms such as the L^1 and L^∞ norms, is differentiable. Problem [8.5] is in that case referred to as a least-squares minimization. It is important at this point to emphasize that the cost function $\mathcal{J}$ features an implicit dependence on $\boldsymbol{\theta}$ through the forward problem, which can be symbolized as:

$$\mathcal{J}(\boldsymbol{\theta}) = J(\boldsymbol{d}) \quad \text{with } \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{\theta}, \boldsymbol{d}) = \boldsymbol{0}. \qquad [8.6]$$

The cost function $\mathcal{J}$ does not have general properties that would *a priori* guarantee (i.e. for any inversion problem) that a local minimizer $\boldsymbol{\theta}$ is unique, or global.

Upon reformulation as a minimization task, seeking a solution $\boldsymbol{\theta}$ consists of minimizing the residual of the observation equation, rather than setting it to zero. This weakening of the concept of solution ensures its existence, and makes more practical sense than the (often unfeasible) task of attempting to exactly match imperfect observation by adjusting the parameters of an approximate model.

8.2.1.3. *Regularization*

As already mentioned, inverse problems are often mathematically ill-posed; in particular, a solution $\boldsymbol{\theta}$ is often highly sensitive to small changes to, or errors in, the experimental data. This property has strongly influenced the design of inversion methodologies since the pioneering works of [TIK 77] and [TWO 77], and is analyzed in many monographs, for example [ENG 96] and [HAN 98]. Such methodologies are based on the formulation and exploitation of *prior information* that is available in addition to experimental data:

– Quantitative prior information about the parameters to be identified (e.g. positiveness and variation range) may be specified via equality or inequality constraints that restrict the search space Θ.

– Qualitative prior information may be prescribed via a *stabilizing functional* $\mathcal{R}(\boldsymbol{\theta}) \geq 0$, expressing a requirement that a certain non-negative function of $\boldsymbol{\theta}$ be as small as possible. Classic examples include $\mathcal{R}(\boldsymbol{\theta}) = \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|$ (desired closeness of $\boldsymbol{\theta}$ to a reference value $\boldsymbol{\theta}_0$) and $\mathcal{R}(\boldsymbol{\theta}) = \|\boldsymbol{\nabla\theta}\|$ (to avoid too-oscillatory solutions $\boldsymbol{\theta}$). This leads to the regularized form of the minimization [8.5]:

$$\boldsymbol{\theta} = \underset{\vartheta \in \Theta}{\arg\min}\, \mathcal{J}(\boldsymbol{\vartheta}), \quad \mathcal{J}(\boldsymbol{\vartheta}) = J(\boldsymbol{d}(\boldsymbol{\vartheta}; \boldsymbol{X})) + \alpha \mathcal{R}(\boldsymbol{\vartheta}) \qquad [8.7]$$

where $J(\cdot)$ again defines the distance between the measurements and their prediction by the forward problem, $\mathcal{R}$ is the stabilizing functional and $0 < \alpha \ll 1$ is the *regularization parameter*.

It is useful for the purpose of adjusting the inversion algorithm to have an available estimation δ of the measurement error. For example, some algorithms select the optimal value of the regularization parameter α on the basis of δ [TIK 95].

8.2.1.3.1. Regularization using total variation

When the dimension of the search space Θ is high, it is, in practice, necessary to use some form of regularization because the inversion is highly sensitive to experimental errors, even with linear forward models (which in that case are ill-conditioned). Such situations arise, for instance, in the reconstruction of heterogeneous material parameters (modulus, wave velocity, conductivity, damage, etc.), whose spatial discretization requires a large number of unknowns. A stabilizing functional of the form $\mathcal{R}(\boldsymbol{\theta}) = \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|$ may be useful, for example when attempting to reconstruct medium properties that deviate moderately from a given reference value $\boldsymbol{\theta}_0$. The choice $\mathcal{R}(\boldsymbol{\theta}) = \|\boldsymbol{\nabla\theta}\|^2$ (with $\|\cdot\|$ denoting the L^2-norm of a square-integrable function) allows us to filter non-physical spatial oscillations that might affect the reconstruction of $\boldsymbol{\theta}$ due to the amplification of experimental uncertainties by the inversion algorithm. However, the squared norm, while very convenient from a computational standpoint because the corresponding stabilizing functional is differentiable, tends to yield oversmoothed solutions $\boldsymbol{\theta}$ that do not reproduce existing contrasts (e.g. when reconstructing piecewise homogeneous media). For this reason, it is often preferable for this kind of inversion problem to use stabilizing functionals of the form

$$\mathcal{R}(\boldsymbol{\theta}) = \left[\|\boldsymbol{\nabla\theta}\|^2 + \eta\right]^{1/2} \quad 0 < \eta \ll 1$$

known as *total variation* functionals [ACA 94], which still filter random spatial oscillations while permitting a limited amount of contrast (the small parameter η serves to define a regularized form of the total variation functional that is differentiable at $\boldsymbol{\nabla\theta} = \mathbf{0}$). This type of regularization is, for example, used in [EPA 08] for the reconstruction of three-dimensional (3D) heterogeneous moduli using seismic data.

8.2.1.4. *Bayesian formulations*

Other approaches adopt a probabilistic viewpoint to model prior information and various uncertainties. They proceed by constructing an *a posteriori* probability density function on $\boldsymbol{\theta}$ by considering the available prior information and the physical model as two independent sources of information [MEN 84, TAR 05, KAI 05]. The starting point for such formulations is the Bayes theorem, written as

$$f_{\Theta|D}(\boldsymbol{\theta}|\boldsymbol{d}^{\text{obs}}) f_D(\boldsymbol{d}) = f_{D|\Theta}(\boldsymbol{d}^{\text{obs}}|\boldsymbol{\theta}) f_\Theta(\boldsymbol{\theta}) \qquad [8.8]$$

where $f_{X|Y}(x|y)$ is the conditional probability density on x knowing y, and the density $f_X(x)$ is defined by the marginalization of $f(x,y)$ (and similarly by switching the roles of x and y). Here, the probability densities $f_\Theta(\boldsymbol{\theta})$, modeling the prior information on $\boldsymbol{\theta}$, and $f_{D|\Theta}(\boldsymbol{d}^{\text{obs}}|\boldsymbol{\theta})$, describing the forward physical model and enabling measurement or modeling uncertainties to be taken into account, are chosen *a priori*. The Bayes theorem [8.8] then yields

$$f_{\Theta|D}(\boldsymbol{\theta}|\boldsymbol{d}^{\text{obs}}) = \frac{f_{D|\Theta}(\boldsymbol{d}^{\text{obs}}|\boldsymbol{\theta}) f_\Theta(\boldsymbol{\theta})}{\int f_{D|\Theta}(\boldsymbol{d}^{\text{obs}}|\boldsymbol{\theta}) f_\Theta(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}} \qquad [8.9]$$

and therefore allows us to evaluate the probability of $\boldsymbol{\theta}$ knowing $\boldsymbol{d}^{\text{obs}}$, once all available (prior and model) information is taken into account. Practical information about $\boldsymbol{\theta}$ and the error sensitivity of its estimation are obtained by analyzing the posterior density $f_{\Theta|D}$. For instance, we may estimate $\boldsymbol{\theta}$ by seeking the value achieving the maximum likelihood, that is $\boldsymbol{\theta} = \arg\max_{\vartheta} f_{\Theta|D}(\boldsymbol{\vartheta}|\boldsymbol{d}^{\text{obs}})$.

The simplest form of this approach corresponds to the case of a forward finite-dimensional linear model $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{\theta}, \boldsymbol{d}) = \boldsymbol{G}(\boldsymbol{X})\boldsymbol{\theta} - \boldsymbol{d}$ (where $\boldsymbol{G}$ is a matrix) and consists of setting $f_{\Theta}(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta}_0, \boldsymbol{C}_\theta)$ and $f_{D|\Theta}(\boldsymbol{d}|\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{d}^{\text{obs}}, \boldsymbol{C}_D)$ (with $\mathcal{N}(\boldsymbol{x}, \boldsymbol{C})$ denoting the multidimensional Gaussian random variable with mean $\boldsymbol{x}$ and covariance matrix $\boldsymbol{C}$). The posterior conditional probability density function $f_{\Theta|D}(\boldsymbol{\theta}|\boldsymbol{d})$ then corresponds to the Gaussian variable $\mathcal{N}(\bar{\boldsymbol{\theta}}, \boldsymbol{C})$ with

$$\begin{aligned} \bar{\boldsymbol{\theta}} &= \underset{\theta}{\arg\min} \left(\boldsymbol{G}\boldsymbol{\theta} - \boldsymbol{d}^{\text{obs}}\right)^{\mathrm{T}} \boldsymbol{C}_D^{-1} \left(\boldsymbol{G}\boldsymbol{\theta} - \boldsymbol{d}^{\text{obs}}\right) + \left(\boldsymbol{\theta} - \boldsymbol{\theta}_0\right)^{\mathrm{T}} \boldsymbol{C}_\theta^{-1} \left(\boldsymbol{\theta} - \boldsymbol{\theta}_0\right), \\ \boldsymbol{C} &= \left[\boldsymbol{G}\boldsymbol{C}_D^{-1}\boldsymbol{G} + \boldsymbol{C}_\theta^{-1}\right]^{-1} \end{aligned} \quad [8.10]$$

The Bayesian approach constitutes a form of regularization in that [8.9] simultaneously takes into account experimental data and prior information. For example, the quadratic cost function featured in [8.10] corresponds to a regularized cost function, where $\boldsymbol{C}_D\boldsymbol{C}_\theta^{-1}$ acts as a small regularization parameter α. Examples where such approaches are applied to the mechanics of materials and structures include [ARN 08], [DAG 07], [GOG 08] and [GUZ 02].

8.2.1.5. *Inversion versus identification*

A distinction is sometimes made between inverse problems and identification problems. In both cases, the aim is to determine certain quantities $\boldsymbol{\theta}$ that enter into the definition of the analyzed structure or material sample by exploiting experimental information concerning its response. The term *inverse problem* primarily refers to situations where the quantity to be reconstructed is mathematically formulated in terms of functions (such as heterogeneous moduli, time-dependent forces or sources, solution-dependent constitutive properties and domain shape or topology), whose discretization is expected to entail a large number of unknowns. Cases where we *a priori* consider a moderate number of unknowns, such as parameters entering into most usual constitutive models, are then referred to as *identification problems*. Both types of problems are solved using similar methods, often entailing the minimization of a cost function. Because of their smaller size, parameter identification problems are, on average, less sensitive to (experimental, modeling, etc.) uncertainties than inverse problems, and do not always require regularization (the cost functions used may thus differ at least in this respect).

8.2.2. *Constitutive parameter identification*

8.2.2.1. *Forward problem*

This chapter is primarily concerned with parameter identification problems for (linear or nonlinear) constitutive models. Consider, for instance, a solid body whose undeformed configuration occupies the domain Ω and whose mechanical state is governed in the small-deformation framework (chosen for expository convenience, but not mandatory) by the equilibrium equations

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} = \mathbf{0} & \text{in } \Omega \quad [8.11a] \\ \boldsymbol{\sigma}.\boldsymbol{n} = \bar{\boldsymbol{T}} & \text{on } S_f \quad [8.11b] \end{cases}$$

(assuming no body forces are present) and the kinematic compatibility equations

$$\begin{cases} \boldsymbol{\varepsilon} = \varepsilon[\boldsymbol{u}] = \dfrac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla}^t \boldsymbol{u}) & \text{in } \Omega \quad [8.12a] \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on } S_u \quad [8.12b] \end{cases}$$

(where $\boldsymbol{u}$ is the displacement, $\boldsymbol{\varepsilon}$ is the linearized stress tensor, $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$). The surfaces S_u (supporting prescribed displacements $\bar{\boldsymbol{u}}$) and S_f (supporting prescribed tractions $\bar{\boldsymbol{T}}$) are such that $S_u \cup S_f = \partial\Omega$ and $S_u \cap S_f = \emptyset$, so as to define well-posed boundary conditions.

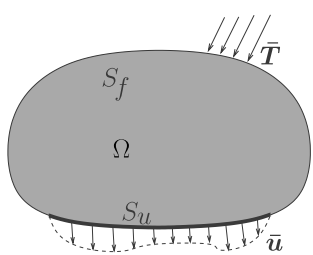

Figure 8.3. *Forward problem: notations*

For elastic linear material properties, the constitutive relation has the well-known form

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{A}}(\boldsymbol{\theta}) : \boldsymbol{\varepsilon} \quad \text{in } \Omega \quad [8.13]$$

where the elasticity tensor $\boldsymbol{\mathcal{A}}$ may be constant (homogeneous material) or spatially variable (heterogeneous material, e.g. due to damage or defects). In the context of this chapter, the elasticity tensor depends on a vector of parameters $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_M\}$, which may be emphasized in [8.13] with the notation $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$ (e.g. $\boldsymbol{\theta} = \{E, \nu\}$ and $M = 2$ for homogeneous isotropic elasticity).

The set of equations [8.11b]–[8.13] may be replaced by the weak formulation

$$\int_\Omega \boldsymbol{\varepsilon}[\tilde{\boldsymbol{u}}] : \boldsymbol{\mathcal{A}}(\boldsymbol{\theta}) : \boldsymbol{\varepsilon}[\boldsymbol{u}] \mathrm{d}V - \int_{S_f} \tilde{\boldsymbol{u}} . \boldsymbol{T} \mathrm{d}S = 0 \qquad \text{for all } \tilde{\boldsymbol{u}} \text{ k.a. with } \mathbf{0} \qquad [8.14]$$

A Galerkin discretization using the finite element interpolation functions as an approximation and trial space leads to the standard finite element matrix equation

$$\boldsymbol{K}(\boldsymbol{\theta})\boldsymbol{U} = \boldsymbol{F} \qquad [8.15]$$

where $\boldsymbol{K}(\boldsymbol{\theta})$ is the stiffness matrix, here restricted to the unconstrained degrees of freedom (DOFs), $\boldsymbol{F}$ is the vector of generalized (nodal) forces resulting from the applied excitations [8.11b] and [8.12b] and the vector $\boldsymbol{U}$ gathers the unknown DOFs. The notation $\boldsymbol{K}(\boldsymbol{\theta})$ emphasizes the obvious but essential fact that the stiffness matrix depends on the constitutive parameters (in this case, elastic moduli).

For more complex constitutive properties (plasticity, damage, etc.) that are history-dependent, we often have to solve a (time-discrete) evolution problem using an incremental and iterative algorithm (typically involving an implicit treatment such as the radial return algorithm [SIM 98, BES 01]). Such treatment is based on a spatially continuous weak formulation of the form

$$\int_\Omega \boldsymbol{\varepsilon}[\tilde{\boldsymbol{u}}] : \boldsymbol{\sigma}[\boldsymbol{u}_n; \boldsymbol{u}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}] \mathrm{d}V - \int_{S_f} \tilde{\boldsymbol{u}} . \boldsymbol{T}_n \mathrm{d}S = 0$$

$$\text{for all } \tilde{\boldsymbol{u}} \text{ k.a. with } \mathbf{0} \qquad [8.16]$$

where $\boldsymbol{T}_n$ denotes the applied loading at time t_n, $\boldsymbol{u}_n = \boldsymbol{u}(\cdot, t_n)$ is the unknown displacement at time t_n and $\boldsymbol{\sigma}[\boldsymbol{u}_n; \boldsymbol{u}_{n-1}, \mathcal{S}_{n-1}]$ denotes the stress at time t_n predicted by the (time-discretized) constitutive model for an assumed value of the displacement $\boldsymbol{u}_n$ and knowing the displacement $\boldsymbol{u}_{n-1}$ and all other mechanical quantities $\mathcal{S}_{n-1}$ (strains, stresses and internal variables) at time t_{n-1}. For finite element discretization in space, the incremental weak formulation [8.16] takes the form

$$\boldsymbol{R}_n(\boldsymbol{U}_n; \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) = \mathbf{0} \qquad [8.17]$$

It is solved by an iterative Newton–Raphson-type method, the iteration $i+1$ of the solution being obtained by solving the linear system

$$\boldsymbol{K}_n^{(i)}(\boldsymbol{\theta})\boldsymbol{U}_n^{(i)} = \boldsymbol{K}_n^{(i)}(\boldsymbol{\theta})\boldsymbol{U}_n^{(i-1)} - \boldsymbol{R}_n(\boldsymbol{U}_n^{(i-1)}; \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) \qquad [8.18]$$

where $\boldsymbol{K}_n^{(i)}(\boldsymbol{\theta}) \equiv \partial \boldsymbol{R}_n / \partial \boldsymbol{U}_n(\boldsymbol{U}_n^{(i-1)}; \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta})$ is the tangent stiffness matrix, also known as the "consistent tangent operator" [SIM 98].

For a given set of parameters $\boldsymbol{\theta}$, formulations such as [8.14] and [8.16] define the *forward problem*, allowing the computation of the response of a structure whose material has known properties. Solving the forward problem uses classic computational structural mechanics approaches, the main approach of which is the finite element method (FEM).

8.2.2.2. *Identification problem*

In cases where parameters $\boldsymbol{\theta}$, associated with a constitutive model or with other characteristics of the sample being analyzed such as internal defects, are unknown, the forward problem [8.14] or [8.16] cannot be solved as it is. Finding the parameters $\boldsymbol{\theta}$ requires supplementary information, obtained from experiments, in addition to the boundary data $\bar{\boldsymbol{u}}, \bar{\boldsymbol{T}}$ entering into the definition of [8.14] or [8.16]. Such supplementary data may, in particular, consist of kinematic field measurements.

It is worth emphasizing, moreover, that the available experimental information regarding applied loads is limited, for many practical identification situations, to resultant loads or couples. Depending on the chosen solution approach, we will then either treat distributed applied loads as supplementary unknowns, define kinematic boundary conditions from the measured kinematic fields or use virtual or adjoint fields so that the formulation of the identification would involve only the (known) resultant load.

8.3. Numerical methods based on optimization

For the sake of generality, consider the task of identifying parameters for a nonlinear, incremental constitutive model using a sequence of kinematic field measurements, obtained at various stages of the loading history, applied to the examined sample. Assume for definiteness that the kinematic response $\boldsymbol{d}_n^{\text{obs}}$ is measured at *all* discrete time instants $t_0 = 0, t_1, \ldots, t_N = T$ introduced for the time-marching algorithm. Identification then consists of determining $\boldsymbol{\theta}$ so as to achieve the best fit between the experimental data $\boldsymbol{d}_n^{\text{obs}}$ and its computed prediction $\boldsymbol{d}_n(\boldsymbol{\theta})$. This approach naturally leads to the problem of minimizing a cost function of the form

$$\mathcal{J}(\boldsymbol{\theta}) = \sum_{n=1}^{N} J(\boldsymbol{U}_n(\boldsymbol{\theta})) \qquad [8.19]$$

where $\boldsymbol{U}_n$ depends on $\boldsymbol{\theta}$ through the equilibrium problem [8.16]. For instance, for the least-squares method, the function J is defined by

$$J(\boldsymbol{V}) = \frac{1}{2} \| \boldsymbol{P}\boldsymbol{V} - \boldsymbol{d}_n^{\text{obs}} \|^2 \qquad [8.20]$$

8.3.1. *Gradient-based methods*

The evaluation of each cost function may entail substantial computational work because it requires a complete analysis of the structure, possibly under dynamic and/or nonlinear conditions. Therefore, we often prefer to rely on gradient-based optimization algorithms that allow us to reduce the total number of evaluations of

$\mathcal{J}(\boldsymbol{\theta})$. For a detailed presentation of the main optimization algorithms, the reader may refer to many monographs [FLE 00, NOC 06].

Gradient-based optimization algorithms assume that both the cost function $\mathcal{J}$ and its gradient $\boldsymbol{\nabla}_\theta \mathcal{J}$ can be numerically evaluated for a given $\boldsymbol{\theta}$. They are usually designed so that each iteration ensures that the cost function is decreased. Classic algorithms of this kind include the steepest descent method (historically the first of its kind, but currently more or less abandoned due to its too-slow convergence), conjugate gradient and quasi-Newton. The latter consists, in fact, of solving the necessary first-order optimality condition $\boldsymbol{\nabla}_\theta \mathcal{J} = \mathbf{0}$ using the Newton–Raphson method, where the true Hessian $\boldsymbol{H} = \boldsymbol{\nabla}^2_\theta \mathcal{J}$, *a priori* required by the Newton–Raphson method, is replaced by the (positive definite) approximation of the inverse Hessian that is updated (using the BFGS or DFP formula [BER 99, FLE 00, NOC 06]) after each iteration.

Other algorithms exploit the nonlinear least-squares cost function structure [8.19–8.20] often used for identification problems. They are also based on solving equation $\boldsymbol{\nabla}_\theta \mathcal{J} = \mathbf{0}$ by the Newton–Raphson method, using an approximation of the Hessian $\boldsymbol{H}$ in order to avoid the often complex task of its numerical evaluation. The Gauss–Newton algorithm uses the approximation $\boldsymbol{H} \approx (\boldsymbol{\nabla}_\theta \boldsymbol{U})^{\mathrm{T}} \boldsymbol{\nabla}_\theta \boldsymbol{U}$, in which second-order derivatives of the residuals $\boldsymbol{PU} - \boldsymbol{d}^{\mathrm{obs}}$ are dropped (this approximation is correct at the local minimum reached upon convergence provided that the converged residuals themselves are small). The matrix $(\boldsymbol{\nabla}_\theta \boldsymbol{U})^{\mathrm{T}} \boldsymbol{\nabla}_\theta \boldsymbol{U}$ may, however, be non-invertible, or ill-conditioned. The Levenberg–Marquardt algorithm addresses this issue by using in the Gauss–Newton algorithm an approximate Hessian of the form $\boldsymbol{H} \approx (\boldsymbol{\nabla}_\theta \boldsymbol{U})^{\mathrm{T}} \boldsymbol{\nabla}_\theta \boldsymbol{U} + \alpha \boldsymbol{I}$, where the parameter $\alpha > 0$ is updated after each iteration (this approximation is thus positive definite by construction). The Gauss–Newton and Marquardt–Levenberg algorithms require repeated evaluations of the complete Jacobian matrix $\boldsymbol{\nabla}_\theta \boldsymbol{U}$, that is of all the partial derivatives of the solution $\boldsymbol{U}$ to the forward problem rather than the partial derivatives of the cost function $\mathcal{J}(\boldsymbol{\theta})$ only.

It is sometimes useful, or even necessary, to reduce the search space Θ by prescribing constraints (e.g. moduli are positive and the Poisson ratio must belong to a certain interval), if only to avoid the occurrence of non-physical forward problems caused by forbidden values of $\boldsymbol{\theta}$ that may otherwise be reached by the iterative algorithm. Many constrained optimization algorithms are available; readers can refer to monographs such as [BER 99], [FLE 00] and [NOC 06]. Some of these are based on iteratively solving the Karush–Kuhn–Tucker necessary optimality conditions (generally using gradients of $\mathcal{J}$ and the functions defining the constraints, and approximate Hessians). Others consist of recasting the constrained optimization in terms of a sequence of unconstrained optimization problems *via* the introduction of penalty, barrier, etc., functions. For example, an augmented Lagrangian method was implemented for the combined shape and material identification of elastic inclusions in [BON 09b].

Various approaches, briefly reviewed next, are available to evaluate the gradient $\boldsymbol{\nabla}_\theta \mathcal{J}$ or the Jacobian matrix $\boldsymbol{\nabla}_\theta \boldsymbol{U}$. A detailed presentation of parameter sensitivity analysis is, for example, available in [KLE 97].

8.3.1.1. *Numerical differentiation*

Numerical differentiation is based on the approximate formula

$$\boldsymbol{\nabla}_\theta \mathcal{J} \cdot \Delta\boldsymbol{\theta} \approx \mathcal{J}(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}) - \mathcal{J}(\boldsymbol{\theta}) \qquad [8.21]$$

for small but finite increments $\Delta\boldsymbol{\theta}$ of $\boldsymbol{\theta}$. If $\boldsymbol{\theta} \in \mathbb{R}^p$, one evaluation of the full gradient $\boldsymbol{\nabla}_\theta \mathcal{J}$ then requires $1+p$ forward solutions, corresponding to the unperturbed configuration $\Delta\boldsymbol{\theta} = \mathbf{0}$ (for the evaluation of $\mathcal{J}(\boldsymbol{\theta})$) and perturbed configurations defined by $\Delta\boldsymbol{\theta} = (\Delta\theta_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \Delta\theta_p)$. Similarly, we may use a numerical differentiation of the forward solution for the purpose of setting up the Jacobian matrix:

$$\boldsymbol{\nabla}_\theta \boldsymbol{U}_n \cdot \Delta\boldsymbol{\theta} \approx \boldsymbol{U}_n(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}) - \boldsymbol{U}_n(\boldsymbol{\theta})$$

This approach is simple to implement and does not require modifications to the forward solution code (it is thus said to be *non-intrusive*). It is, however, computationally expensive because a gradient evaluation entails p complete forward solutions, each new choice for $\boldsymbol{\theta}$ redefining the mechanical system anew and implying a complete analysis. Thus, it is sometimes desirable to replace numerical differentiation by other approaches based on a preliminary analytical differentiation.

8.3.1.2. *Direct differentiation of $\mathcal{J}$*

The direct differentiation approach consists of evaluating the gradient of $\mathcal{J}$ by means of the chain rule:

$$\boldsymbol{\nabla}_\theta \mathcal{J} = \sum_{n=1}^{N} \boldsymbol{\nabla} J(\boldsymbol{U}_n) \cdot \boldsymbol{\nabla}_\theta \boldsymbol{U}_n \qquad [8.22]$$

This expression requires the derivatives $\boldsymbol{\nabla}_\theta \boldsymbol{d}_k$ of the kinematic field that solves the forward incremental problem [8.17]. Differentiating the latter with respect to $\boldsymbol{\theta}$ yields:

$$\boldsymbol{K}_n(\boldsymbol{\theta}) \boldsymbol{\nabla}_\theta \boldsymbol{U}_n = -\boldsymbol{\nabla}_{\mathcal{S}_{n-1}} \boldsymbol{R}_n \boldsymbol{\nabla}_\theta \mathcal{S}_{n-1} - \boldsymbol{\nabla}_\theta \boldsymbol{R}_n \quad (0 = 1, 2, \ldots, N-1) \quad [8.23]$$

The above derivative equations define a linear incremental problem, whose governing matrix is the tangent stiffness $\boldsymbol{K}_n(\boldsymbol{\theta})$ reached on convergence of the Newton–Raphson algorithm [8.18]. A natural time-stepping procedure then consists, for each n $(0 = 1, 2, \ldots, N-1)$, of solving [8.17] (which requires an iterative algorithm) and then the linear problem [8.23]. The fields $\boldsymbol{U}_n$ and $\boldsymbol{\nabla}_\theta \boldsymbol{U}_n$ then enable the evaluation of the corresponding contribution to both $\mathcal{J}$ and $\boldsymbol{\nabla}_\theta \mathcal{J}$. Moreover, this approach allows the computation of Jacobian matrices of the form $\boldsymbol{\nabla}_\theta \boldsymbol{U}_n$, required by the Gauss–Newton or Marquardt–Levenberg methods.

8.3.1.3. *Adjoint state*

Another strategy for evaluating the gradient of $\mathcal{J}$ without recourse to numerical differentiation consists of adopting the viewpoint of minimizing $\mathcal{J}$ subject to the constraint defined by the forward problem [8.17]. Accordingly, we introduce the Lagrangian:

$$\mathcal{L} = \sum_{n=1}^{N} \Big\{ J(\boldsymbol{U}_n) + \tilde{\boldsymbol{U}}_n^{\mathrm{T}} \boldsymbol{R}_n(\boldsymbol{U}_n; \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) + \tilde{\mathcal{S}}_n^{\mathrm{T}} \big[\mathcal{S}_n - \mathcal{P}(\boldsymbol{U}_n, \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) \big] \Big\} \qquad [8.24]$$

where the relation $\mathcal{S}_n - \mathcal{P}(\boldsymbol{U}_n, \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) = 0$ expresses in symbolic notation the process of updating the mechanical quantities at $t = t_n$ once $\boldsymbol{U}_n$ is found, and $\tilde{\boldsymbol{U}}_n$, $\tilde{\mathcal{S}}_n$ are Lagrange multipliers. The first-order variation of $\mathcal{L}$ then reads

$$\delta\mathcal{L} = \boldsymbol{\nabla}_{\mathcal{S}_n}\mathcal{L} \cdot \delta\mathcal{S}_n + \boldsymbol{\nabla}_{U_n}\mathcal{L} \cdot \delta\boldsymbol{U}_n + \boldsymbol{\nabla}_{\tilde{\mathcal{S}}_n}\mathcal{L} \cdot \delta\tilde{\mathcal{S}}_n + \boldsymbol{\nabla}_{\tilde{U}_n}\mathcal{L} \cdot \delta\tilde{\boldsymbol{U}}_n + \boldsymbol{\nabla}_{\theta}\mathcal{L} \cdot \delta\boldsymbol{\theta}$$

The cofactors $\boldsymbol{\nabla}_{\tilde{\mathcal{S}}_n}\mathcal{L}$, $\boldsymbol{\nabla}_{\tilde{U}_n}\mathcal{L}$ vanish whenever $\boldsymbol{U}_n, \mathcal{S}_n$ satisfy the forward problem equations. The cofactors $\boldsymbol{\nabla}_{\mathcal{S}_n}\mathcal{L}$, $\boldsymbol{\nabla}_{U_n}\mathcal{L}$ are given by the formulas

$$\boldsymbol{\nabla}_{\mathcal{S}_n}\mathcal{L} = \tilde{\boldsymbol{U}}_{n+1}^{\mathrm{T}} \boldsymbol{\nabla}_{\mathcal{S}_n}\boldsymbol{R}_{n+1} - \tilde{\mathcal{S}}_{n+1}^{\mathrm{T}} \boldsymbol{\nabla}_{\mathcal{S}_n}\mathcal{P} + \tilde{\mathcal{S}}_n \qquad [8.25a]$$

$$\boldsymbol{\nabla}_{U_n}\mathcal{L} = \boldsymbol{\nabla}J(\boldsymbol{U}_n) + \tilde{\boldsymbol{U}}_n^{\mathrm{T}} \boldsymbol{K}_n + \tilde{\boldsymbol{U}}_{n+1}^{\mathrm{T}} \boldsymbol{K}_{n+1,n} - \tilde{\mathcal{S}}_n^{\mathrm{T}} \mathcal{P}_n - \tilde{\mathcal{S}}_{n+1}^{\mathrm{T}} \mathcal{P}_{n+1,n} \qquad [8.25b]$$

having set $\boldsymbol{K}_{n+1,n} := \boldsymbol{\nabla}_{U_n}\boldsymbol{R}_{n+1}$, $\mathcal{P}_n := \boldsymbol{\nabla}_{U_n}\mathcal{P}(\boldsymbol{U}_n, \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta})$, and $\mathcal{P}_{n,n-1} := \boldsymbol{\nabla}_{U_{n-1}}\mathcal{P}(\boldsymbol{U}_n, \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta})$. They can be made to vanish by a judicious selection of the Lagrange multipliers $\tilde{\boldsymbol{U}}_n, \tilde{\mathcal{S}}_n$. To this end, we can note that for $n = N$, no quantity bearing the subscript $N+1$ should appear in expressions [8.25a] and [8.25b], which leads to the equalities

$$0 = \boldsymbol{\nabla}_{\mathcal{S}_N}\mathcal{L} = \tilde{\mathcal{S}}_N, \qquad 0 = \boldsymbol{\nabla}_{U_N}\mathcal{L} = \boldsymbol{\nabla}J(\boldsymbol{U}_N) + \tilde{\boldsymbol{U}}_N^{\mathrm{T}} \boldsymbol{K}_N - \tilde{\mathcal{S}}_N^{\mathrm{T}} \mathcal{P}_N$$
$$\Longrightarrow \quad \tilde{\mathcal{S}}_N = 0, \;\; \tilde{\boldsymbol{U}}_N = -\boldsymbol{K}_N^{-1} \boldsymbol{\nabla}J(\boldsymbol{U}_N) \qquad [8.26]$$

Moreover, setting $\boldsymbol{\nabla}_{\mathcal{S}_n}\mathcal{L}$, $\boldsymbol{\nabla}_{U_n}\mathcal{L}$ to zero for $n < N$ yields:

$$\tilde{\mathcal{S}}_n^{\mathrm{T}} = \tilde{\mathcal{S}}_{n+1}^{\mathrm{T}} \boldsymbol{\nabla}_{\mathcal{S}_n}\mathcal{P} - \tilde{\boldsymbol{U}}_{n+1}^{\mathrm{T}} \boldsymbol{\nabla}_{\mathcal{S}_n}\boldsymbol{R}_{n+1} \qquad [8.27a]$$

$$\boldsymbol{K}_n \tilde{\boldsymbol{U}}_n = \tilde{\mathcal{S}}_n^{\mathrm{T}} \mathcal{P}_n + \tilde{\mathcal{S}}_{n+1}^{\mathrm{T}} \mathcal{P}_{n+1,n} - \boldsymbol{\nabla}J(\boldsymbol{U}_n) - \tilde{\boldsymbol{U}}_{n+1}^{\mathrm{T}} \boldsymbol{K}_{n+1,n} \qquad [8.27b]$$

A backward *adjoint state* is hence defined by (1) initialization [8.26] and (2) (backward) transition ([8.27a] and [8.27b]). The latter is linear irrespective of the possible nonlinearity of the forward problem. Equation [8.27b] uses the tangent stiffness matrix $\boldsymbol{K}_n$, reached on convergence of the Newton–Raphson step [8.18].

This allows for a very efficient computation of the adjoint state, provided that the converged tangent stiffness matrix has been stored, preferably in a factored form. In view of the backward nature of the adjoint problem, it is necessary to store *all* tangent stiffness matrices $\boldsymbol{K}_n$ and $\boldsymbol{\nabla}_{\mathcal{S}_{n-1}}\boldsymbol{R}_n$ arising in the course of solving the forward incremental problem, as well as the complete forward solution history $\boldsymbol{U}_n, \mathcal{S}_n$, *before* solving the adjoint problem.

This approach finally leads to the result

$$\boldsymbol{\nabla}_\theta \mathcal{J} = \boldsymbol{\nabla}_\theta \mathcal{L} = \sum_{n=1}^{N} \Big\{ \tilde{\boldsymbol{U}}_n^{\mathrm{T}} \boldsymbol{\nabla}_\theta \boldsymbol{R}_n(\boldsymbol{U}_n; \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) - \tilde{\mathcal{S}}_n^{\mathrm{T}} \boldsymbol{\nabla}_\theta \mathcal{P}(\boldsymbol{U}_n, \boldsymbol{U}_{n-1}, \mathcal{S}_{n-1}, \boldsymbol{\theta}) \Big\} \quad [8.28]$$

where $\boldsymbol{U}_n, \mathcal{S}_n$ is the forward solution and $\tilde{\boldsymbol{U}}_n, \tilde{\mathcal{S}}_n$ is the adjoint solution.

For linear constitutive behavior and equilibrium problems, the cost function depends only (assuming no regularization term) on the solution $\boldsymbol{U}(\boldsymbol{\theta})$ of the elastic equilibrium problem [8.15]: $\mathcal{J}(\boldsymbol{\theta}) = J(\boldsymbol{U})$. The gradient of $\mathcal{J}$ and the adjoint solution $\tilde{\boldsymbol{U}}$ are then defined simply by

$$\boldsymbol{\nabla}_\theta \mathcal{J} = \tilde{\boldsymbol{U}}^{\mathrm{T}} \boldsymbol{\nabla}_\theta \boldsymbol{K}(\boldsymbol{\theta}) \boldsymbol{U}, \qquad \tilde{\boldsymbol{U}} = -\boldsymbol{K}^{-1} \boldsymbol{\nabla} J \quad [8.29]$$

It can thus be seen that adjoint state methods exploit a shortcut (the adjoint state) that enables us to completely avoid computing the Jacobian matrix $\boldsymbol{\nabla}_\theta \boldsymbol{U}$ associated with the solution. They are designed to maximize the evaluation efficiency for the gradient of cost functions (such as $\mathcal{J}(\boldsymbol{\theta})$) and are not suited to the evaluation of Jacobian matrices used in Gauss–Newton or Marquardt–Levenberg methods.

8.3.2. *Other methods*

8.3.2.1. *No-derivative methods*

In addition to previously discussed methods that exploit gradient information regarding the solution or the cost function, no-derivative minimization methods are also available. The theory and algorithms for no-derivative minimization methods are less developed than those for the more widely used gradient-based methods. No-derivative methods, which include the Nelder–Mead simplex algorithm [NEL 65, LAG 98] (not to be confused with the simplex method of linear programming), are inefficient for high-dimensional optimization problems.

8.3.2.2. *Evolutionary algorithms*

These algorithms aim at performing a global exploration of the search space Θ, and are designed by analogy with (and using the terminology of) Darwinian evolution;

see [MIC 96], [LER 07] or [BUR 08] for applications to identification. Among their advantages is the ability to identify multiple optimal solutions (and hence remove the dependency on an initial guess which traditional optimization algorithms suffer from), and to solve optimization problems that are ill-suited to classic algorithms due to the presence of combinatorial features. This, of course, comes with a price: such algorithms necessitate very large numbers of forward solutions. This has prompted investigations into the combined use of evolutionary algorithms and model reduction methods such as the *proper orthogonal decomposition* (POD) [BRI 07].

8.3.2.3. *Topological sensitivity*

Classic iterative inversion methods sometimes require significant computational work. In the context of flaw identification, where objects are sought whose geometry (location, size, shape) is unknown and whose support is small relative to the size of the sample being tested, alternative approaches aiming at the construction of a defect indicator function have recently been proposed. In particular, the concept of topological sensitivity, which aims to evaluate the asymptotic behavior of the featured cost function as the characteristic size of a trial defect becomes vanishingly small, leads to a global probing approach that is approximate but computationally fast (with a computational cost of the order of one forward solution) [AMS 05, BON 09a, BEL 09]. This approach is not restricted to specific types of cost functions or data, and is, in particular, easily amenable to the exploitation of full-field measurements.

8.4. Methods specifically designed for full-field measurements: an overview

8.4.1. *Finite element model updating*

Finite element model updating (FEMU) is primarily aimed at the identification of constitutive parameters by using equation [8.15], or similar equations, based on an FEM model of the sample. FEMU is usually based on the minimization of a discrepancy between a measured quantity and its prediction by the model for an assumed value of $\boldsymbol{\theta}$. Such a discrepancy is usually defined for measured displacements, strains or forces.

The cost functions introduced in this context are defined in direct relation to observable quantities. The latter may, advantageously, take the form of kinematic *fields*, but this is not necessary. Indeed, any information that is supplementary relative to boundary conditions ensuring the well-posedness of a forward problem with known material properties may, in this framework, be exploited. In cases where overdetermined data are available (e.g. the simultaneous knowledge of forces and displacements over some part of the sample under examination), part of the data may be considered as contributing to the boundary conditions, the remaining part being considered as overdetermined (the misfit with its simulation being used to define the cost function), with variations arising according to which part of the data is

considered as overdetermined (e.g. cost functions focusing on either kinematic fields or forces). The choice of variant may depend on the precise nature of the available data and be determined by considerations such as algorithmic robustness, computational efficiency and ease of implementation.

A detailed presentation of FEMU is given in Chapter 9. Methodologies reviewed in section 8.3 are particularly relevant to FEMU. Applications of FEMU to constitutive parameter identification are the subject of many investigations, such as [FOR 04], [KAJ 04], [LEC 07], [COO 07], [MAH 02], [MAH 96], [PAG 07] and [SIL 09] (to cite just a few examples). FEMU in the context of structural dynamics and vibrations is also an active research topic; see, for instance, the survey article [FRI 95].

8.4.2. *Constitutive relation error*

Constitutive relation error (CRE) is an energy measure of the discrepancy between a stress field $\boldsymbol{\tau}$ given *a priori* and another stress field evaluated from a given displacement field $\boldsymbol{v}$ using a constitutive model. For example, for a linear elastic constitutive model defined by the (possibly heterogeneous) elasticity tensor $\boldsymbol{\mathcal{A}}$, the CRE between $\boldsymbol{\tau}$ and $\boldsymbol{v}$ is defined by

$$\mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{A}}) = \frac{1}{2}\int_\Omega (\boldsymbol{\tau} - \boldsymbol{\mathcal{A}}{:}\boldsymbol{\varepsilon}[\boldsymbol{v}]){:}\boldsymbol{\mathcal{A}}^{-1}{:}(\boldsymbol{\tau} - \boldsymbol{\mathcal{A}}{:}\boldsymbol{\varepsilon}[\boldsymbol{v}])\ \mathrm{d}V \qquad [8.30]$$

Note that using the compliance tensor $\boldsymbol{\mathcal{A}}^{-1}$ for the purposes of weighting lends units of energy to $\mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{A}})$.

The CRE concept, which was initially introduced for linear elasticity by [LAD 83] in connection with error estimation for the FEM, quickly turned out to be very useful for model updating [REY 90, CHO 96, BAR 04]. More general formulations of CRE, applicable to incremental nonlinear constitutive models, have been proposed in [LAD 99], based on Drucker stability inequality, and in [MOË 99], based on free energy and dissipation potentials. The following remarks explain the usefulness of the CRE concept for identification:

1) The solution $(\boldsymbol{u},\boldsymbol{\sigma})$ to a forward problem for a linear elastic solid is characterized by

$$(\boldsymbol{u},\boldsymbol{\sigma}) = \underset{(v,\tau)\in\mathcal{C}(\bar{u})\times\mathcal{S}}{\arg\min}\ \mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{A}}) \qquad \text{and} \quad \mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{A}}) = 0 \qquad [8.31]$$

where $\mathcal{S}$ and $\mathcal{C}(\bar{\boldsymbol{u}})$ denote the spaces of kinematically and statically admissible fields, respectively, corresponding to well-posed boundary conditions.

2) For a constitutive parameter identification problem for which overdetermined data are used, it is possible to modify the definitions of the admissible field spaces

$\mathcal{S}$ and $\mathcal{C}(\bar{\boldsymbol{u}})$ so as to include *all* the available experimental information. The constitutive parameter identification problem then typically takes the form

$$\boldsymbol{\theta}^\star = \arg\min_{\theta\in\mathbb{A}} J(\boldsymbol{\theta}) \qquad \text{with} \quad J(\boldsymbol{\theta}) = \min_{(v,\tau)\in\mathcal{C}\times\mathcal{S}} \mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{A}}(\boldsymbol{\theta})) \qquad [8.32]$$

where $\mathbb{A}$ is the set of physically admissible parameters $\boldsymbol{\theta}$. Equation [8.32] summarizes the CRE approach, which alternates minimizations with respect to (1) the admissible fields $(\boldsymbol{v},\boldsymbol{\tau})$ (with fixed $\boldsymbol{\theta}$) and (2) the parameters $\boldsymbol{\theta}$ (with fixed admissible fields).

The concept of CRE is, in principle, applicable to any identification problem for which overdetermined data are available, that is which does not specifically require full-field measurements. This versatility has given rise to many applications in the context of model updating. However, the CRE concept is generally applicable to full-field measurements, as explained in Chapter 10.

8.4.3. *Methods based on equilibrium satisfaction*

This class of approach specifically relies on the experimental availability of a kinematic *field* (displacement $\bar{\boldsymbol{u}}$ or strain $\bar{\boldsymbol{\varepsilon}}$), or of a sequence of such fields when considering the identification of parameters for incremental constitutive models. Assuming, for simplicity, quasi-static conditions and no body forces, the local equilibrium equation

$$\operatorname{div}\boldsymbol{\sigma}[\bar{\boldsymbol{\varepsilon}};\boldsymbol{\theta}] = \mathbf{0}$$

yields, at any point of the sample where $\bar{\boldsymbol{\varepsilon}}$ is known, an equality that must be satisfied by *any* constitutive model predicting $\boldsymbol{\sigma}$ for known $\bar{\boldsymbol{\varepsilon}}$ (the necessity of knowing experimentally the *field* $\bar{\boldsymbol{\varepsilon}}$ is, in particular, a consequence of the fact that the equilibrium equation involves spatial derivatives of $\bar{\boldsymbol{\varepsilon}}$ via the divergence operator). We may thus conceivably identify, for example, parameters $\boldsymbol{\theta}$ associated with a model, or some spatial distribution of heterogeneous properties, by enforcing satisfaction of local equilibrium. Measurement or modeling errors generally implying the unfeasibility of exact equilibrium satisfaction, we may instead consider minimizing a global equilibrium residual $J(\boldsymbol{\theta})$, such as

$$J(\boldsymbol{\theta}) = \int_\Omega \|\operatorname{div}\boldsymbol{\sigma}[\bar{\boldsymbol{\varepsilon}};\boldsymbol{\theta}]\|^2 \,\mathrm{d}V$$

In principle, the formulation of an equilibrium residual requires a 3D kinematic field measurement, which may be obtained by means of some recently developed experimental methods. Measurements of kinematic fields on sample surfaces are otherwise commonly done, for instance, using digital image correlation. Such measurements must then be extrapolated to the whole 3D sample by exploiting kinematic modeling assumptions pertaining to, for example, plane-strain or plane-stress settings, thin or elongated structures, etc.

8.4.3.1. *Local form: the equilibrium gap method*

The enforcement of local equilibrium equations underlies the approach initiated in [CLA 04] for the identification of spatial distributions of a scalar damage variable $0 \leq D(\boldsymbol{x}) \leq 1$ [LEM 90] such that

$$\boldsymbol{\mathcal{A}}(\boldsymbol{x}) = (1 - D(\boldsymbol{x}))\boldsymbol{\mathcal{A}}_0 \qquad [8.33]$$

($\boldsymbol{\mathcal{A}}_0$ denoting the elasticity tensor of the undamaged material) from displacement fields measured by means of digital image correlation. In this approach, whose detailed presentation is the subject of Chapter 12, the local equilibrium is exploited through equations generated by the FEM (i.e. the differential equilibrium equations written in weak form using locally supported trial functions) rather than pointwise local equations; the FE-generated equations may be considered local because they act over the element length scale h. The FE mesh is defined so that its nodes coincide with measurement points for the displacement field. The equilibrium equation associated with the mth DOF (assumed to be unloaded) thus has the form

$$\sum_{e|m\in E_e} (1 - D_e)\{\boldsymbol{e}_m\}^{\mathrm{T}}[\boldsymbol{K}_{e0}]\{\boldsymbol{u}_e\} = 0 \qquad [8.34]$$

where $\{\bar{\boldsymbol{u}}_e\}$ denotes the restriction to element E_e of the measured displacement, $[\boldsymbol{K}_{e0}]$ is the element stiffness for the undamaged element and D_e is the (unknown) value of the damage variable in E_e. The set of all possible equations [8.34] is usually overdetermined. It is thus solved for $\{D\}$ in the least-squares sense and subject to the constraints $0 \leq \{D\} \leq 1$. This work has since then been extended to the identification of parameters associated with damage laws [PÉR 09].

Other procedures based on the satisfaction of local field equations by an experimentally known field variable have been proposed, in connection with, for example, the identification of heterogeneous thermal diffusivities [BAM 09] or medical applications of elastography [SIN 05] based on 3D displacement field measurements using magnetic resonance imaging (MRI). They directly exploit finite-difference approximations of the local field equations and do not resort to weak finite element-type formulations. The main difficulty raised by this kind of approach lies in the fact that local field equations require (usually second-order) partial derivatives of the measured field quantity. Numerical differentiation of sampled data unfortunately often causes significant amplification of the original measurement errors. Following a similar approach, the identification of heat sources from infrared thermography data, treated as the right-hand side of the local heat diffusion equation (written in 2D form after integration along the thickness so as to exploit data available on the sample surface), has been investigated in [MOR 07]; there, the measured temperature field is replaced in the field equation by its projection onto a predefined finite-dimensional approximation space.

8.4.3.2. *Weak form: the virtual fields method*

This approach relies on the experimental availability of the strain field $\bar{\varepsilon}$ (possibly via the approximate differentiation of a measured displacement field) and assumes that loading conditions are known (we may, however, manage by using information on global resultant loads by using well-chosen virtual fields). The identification of constitutive parameters $\boldsymbol{\theta}$ then exploits the equilibrium equation in weak (virtual work) form, for example

$$-\int_{\Omega} \boldsymbol{\sigma}[\bar{\varepsilon};\boldsymbol{\theta}] : \boldsymbol{\varepsilon}[\boldsymbol{u}^{\star}] \, \mathrm{d}V + \int_{S_f} \boldsymbol{T}.\boldsymbol{u}^{\star} \, \mathrm{d}S = 0 \qquad [8.35]$$

where $\boldsymbol{u}^{\star}$ denotes the virtual field admissible with zero kinematic data and which is otherwise arbitrary, under quasi-static conditions and assuming zero body forces. Each possible choice of virtual field $\boldsymbol{u}^{\star}$ thus yields a scalar equation that must be verified by the constitutive model $\boldsymbol{\sigma}[\bar{\varepsilon};\boldsymbol{\theta}]$ that predicts the stress value for given strain and constitutive parameters.

The virtual fields method, initiated in [GRÉ 89], consists of exploiting an identity of type [8.35] with judiciously chosen virtual fields according to the specificities of the identification problem at hand and presumes an *a priori* chosen constitutive model whose parameters are to be identified. This approach is detailed more in Chapter 11. Many applications and extensions of the virtual fields method have been investigated, for example by [CHA 06] (anisotropic elasticity with damage) or by [GRÉ 06, AVR 08] (elastoplasticity).

It is worth noting that the FE-based equilibrium gap method coincides with the virtual fields method if each virtual field is chosen as the interpolation function associated with a finite element DOF. Moreover, it is shown in [AVR 07] that the stationarity conditions for cost functionals associated with FEMU, CRE or equilibrium gap can be interpreted in terms of the virtual fields method for specific suitably chosen virtual fields.

8.4.4. *Reciprocity gap*

The reciprocity gap method mainly concerns situations where the field measurements are available at the boundary. Denoting by $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{T}})$ the displacement and density fields on the boundary, we can define a reciprocity gap functional based on the virtual power principle. For example, we can consider measurements performed on a solid Ω whose behavior is defined by the elasticity tensor field $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$, which we are seeking to identify. An auxiliary displacement $\boldsymbol{u}^{\star}$, often called "adjoint", is defined. This displacement is generated, in a solid of identical geometry Ω but for a fictitious reference material characterized by the elastic tensor $\boldsymbol{\mathcal{A}}^{\star}(\boldsymbol{x})$, by a force density $\boldsymbol{T}^{\star}$ applied on $\partial\Omega$. By combining the equations given by the virtual work principle for

$(\hat{\boldsymbol{u}}, \boldsymbol{u}^\star)$ and $(\boldsymbol{u}^\star, \hat{\boldsymbol{u}})$ (or else, equivalently, by writing the Maxwell–Betti reciprocity theorem for states $\hat{\boldsymbol{u}}$ and $\boldsymbol{u}^\star$), we obtain

$$\int_\Omega \boldsymbol{\varepsilon}[\boldsymbol{u}] : [\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^\star] : \boldsymbol{\varepsilon}[\boldsymbol{u}^\star] \, \mathrm{d}V = \int_{\partial\Omega} (\hat{\boldsymbol{T}}.\boldsymbol{u}^\star - \boldsymbol{T}^\star.\hat{\boldsymbol{u}}) \, \mathrm{d}S \equiv R(\boldsymbol{\mathcal{A}}^\star, \boldsymbol{u}; \boldsymbol{u}^\star) \quad [8.36]$$

Identity [8.36] provides an independent scalar equation that must be verified by the unknown elasticity tensor field $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$, or by parameters $\boldsymbol{\theta}$ defining it, for each choice of adjoint loading $\boldsymbol{T}^\star$; this defines the essence of the reciprocity gap method, which is presented in detail in Chapter 13. We can consider the reciprocity gap method as a variant of the virtual fields method for which the kinematic fields are only available at the boundary. In the absence of arguments enabling the kinematic extrapolation of the data to Ω in its entirety, the real displacement field $\boldsymbol{u}$ is *a priori* unknown in Ω and must be reconstructed together with $\boldsymbol{\mathcal{A}}$, at least on the geometrical support of the contrast $\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^\star$. A linearized version of [8.36] for weak contrasts ($\|\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^\star\| \ll \|\boldsymbol{\mathcal{A}}^\star\|$) enables the theoretical analysis of the identifiability of heterogeneous elastic moduli [IKE 90] or of heterogeneous stiffnesses in plate bending [IKE 93], the identifiability condition generally being the knowledge of the Dirichlet–Neumann map, that is, of all the possible $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{T}})$ pairs. Reciprocity gap functionals are also useful for identifying cracks [AND 97, BEN 99] or discrete point sources [ELB 00].

8.5. Conclusion

In this introductory chapter on identification, we proposed an overview of the concepts of inversion and identification, and the numerical solution methods commonly used in this context. Finally, the main methodologies devoted to constitutive parameter identification using full-field kinematic measurements are briefly described; they will be analyzed in more detail in Chapters 9 to 13.

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