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State constrained two player differential games with decoupled dynamics

Piernicola Bettiol* and Jérémy Rouot†

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Abstract

We consider a two player zero-sum differential game with state constraints, in which the dynamics is decoupled: each player has to stay in a closed (nonempty) set. We prove that, under suitable assumptions, the lower and the upper values are locally Lipschitz continuous and we establish that they are solutions, in the viscosity sense, of the Hamilton-Jacobi-Isaacs equation, which involves an appropriate Hamiltonian, called *inner Hamiltonian*. We finally provide a comparison theorem. It follows that the differential game under consideration admits a value (which coincides with the lower and the upper values). A key step in our analysis is a new nonanticipative Filippov-type theorem, which is valid for general closed sets.

Keywords: Differential Games, State Constraints, Hamilton-Jacobi Equation, Viscosity Solutions.

1 Introduction

We shall consider state-constrained two player differential games in which the dynamics is decoupled in the following sense: a first system is exclusively controlled by one player using measurable functions u

$$\begin{cases} \dot{y}(t) = f_1(t, y(t), u(t)), & \text{for a.e. } t \in [t_0, T] \\ u(t) \in U & \text{for a.e. } t \in [t_0, T] \\ y(t_0) = y_0 \in A_1 \\ y(t) \in A_1 & \text{for all } t \in [t_0, T], \end{cases} \quad (1)$$

whereas, a second player intervenes with measurable functions v modifying the dynamics of a second control system

$$\begin{cases} \dot{z}(t) = f_2(t, z(t), v(t)), & \text{for a.e. } t \in [t_0, T] \\ v(t) \in V & \text{for a.e. } t \in [t_0, T] \\ z(t_0) = z_0 \in A_2 \\ z(t) \in A_2 & \text{for all } t \in [t_0, T]. \end{cases} \quad (2)$$

Here, $T > 0$ is a fixed final time, $f_1(., ., .) : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \rightarrow \mathbb{R}^{n_1}$ and $f_2(., ., .) : \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^{n_2}$ are given functions, $U \subset \mathbb{R}^{m_1}$ and $V \subset \mathbb{R}^{m_2}$ are given sets.

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25 Associated with any initial data $(t_0, x_0) = (t_0, y_0, z_0) \in [0, T] \times A_1 \times A_2$, A_1 (resp. A_2) being
 26 a nonempty closed subset of \mathbb{R}^{n_1} (resp. \mathbb{R}^{n_2}), and with any couple of controls $(u(\cdot), v(\cdot))$ we
 27 shall consider the following cost functional:

$$J(t_0, x_0; u(\cdot), v(\cdot)) := \int_{t_0}^T [L_1(t, x(t), u(t)) + L_2(t, x(t), v(t))] dt + g(x(T)), \quad (3)$$

28 in which $x(t) = x[t_0, x_0; u(\cdot), v(\cdot)](t)$ ($= (y[t_0, y_0; u(\cdot)](t), z[t_0, z_0; v(\cdot)](t))$) denotes the solution
 29 of systems (1) and (2) associated with the controls (u, v) . Set $n := n_1 + n_2$. The functions
 30 $L_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m_1} \rightarrow \mathbb{R}$ and $L_2 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}$ are called *Lagrangians* (or running cost)
 31 and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *final cost*. We shall consider a differential game in which the first player
 32 wants to minimize the functional $J(\cdot)$, while the second player's goal is to maximize $J(\cdot)$.

33 For each starting point $(y_0, z_0) \in A_1 \times A_2$ and subinterval $[t_0, T_0] \subset [0, T]$, we define the set
 34 of *admissible* controls for the two players as follows:

$$\begin{aligned} \mathcal{U}([t_0, T_0], y_0) &:= \{u(\cdot) : [t_0, T_0] \rightarrow U \text{ measurable} \mid y[t_0, y_0; u(\cdot)](t) \in A_1 \forall t \in [t_0, T_0]\}; \\ \mathcal{V}([t_0, T_0], z_0) &:= \{v(\cdot) : [t_0, T_0] \rightarrow V \text{ measurable} \mid z[t_0, z_0; v(\cdot)](t) \in A_2 \forall t \in [t_0, T_0]\}. \end{aligned}$$

35 When $T_0 = T$, which is often the case under consideration, we shall use the simplified notation:

$$\begin{aligned} \mathcal{U}(t_0, y_0) &:= \{u(\cdot) : [t_0, T] \rightarrow U \text{ measurable} \mid y[t_0, y_0; u(\cdot)](t) \in A_1 \forall t \in [t_0, T]\}; \\ \mathcal{V}(t_0, z_0) &:= \{v(\cdot) : [t_0, T] \rightarrow V \text{ measurable} \mid z[t_0, z_0; v(\cdot)](t) \in A_2 \forall t \in [t_0, T]\}. \end{aligned}$$

Our standing assumptions allows us to be in a situation such that, for all $x_0 = (y_0, z_0) \in A_1 \times A_2$
 and $t_0 \in [0, T]$, we have

$$\mathcal{U}(t_0, y_0) \neq \emptyset \text{ and } \mathcal{V}(t_0, z_0) \neq \emptyset.$$

36 As is customary in differential games theory, we consider the upper value function V^\sharp and the
 37 lower value function V^\flat . In the definition of V^\sharp and V^\flat we shall make use of *nonanticipative*
 38 *strategies*, in the Varaya-Roxin-Elliot-Kalton sense. To recall this notion, we take, for instance,
 39 initial data $(t_1, y_0) \in [0, T] \times A_1$ and $(t_2, z_0) \in [0, T] \times A_2$. We say that a mapping $\alpha : \mathcal{V}(t_2, z_0) \rightarrow$
 40 $\mathcal{U}(t_1, y_0)$ is a nonanticipative strategy for the first player if, for any $\tau \in [0, T - t_2]$, for all controls
 41 $v_1(\cdot)$ and $v_2(\cdot)$ belonging to $\mathcal{V}(t_2, z_0)$, which coincide a.e. on $[t_2, t_2 + \tau]$, $\alpha(v_1(\cdot))$ and $\alpha(v_2(\cdot))$
 42 coincide a.e. on $[t_1, (t_1 + \tau) \wedge T]$. Analogously we can define the nonanticipative strategies β
 43 for the second player. For $t_0 \in [0, T]$ and $x_0 = (y_0, z_0) \in A_1 \times A_2$, we write $S_U(t_0, x_0)$ and $S_V(t_0, x_0)$
 44 the sets of the nonanticipative strategies for the first and second player respectively.

45 Now, the lower value V^\flat is defined by:

$$V^\flat(t_0, x_0) := \inf_{\alpha \in S_U(t_0, x_0)} \sup_{v(\cdot) \in \mathcal{V}(t_0, y_0)} J(t_0, x_0; \alpha(v(\cdot)), v(\cdot)), \quad (4)$$

46 where the functional $J(\cdot)$ is represented in (3). The upper value function is defined as follows:

$$V^\sharp(t_0, x_0) := \sup_{\beta \in S_V(t_0, x_0)} \inf_{u(\cdot) \in \mathcal{U}(t_0, z_0)} J(t_0, x_0; u(\cdot), \beta(u(\cdot))). \quad (5)$$

47 In absence of state constraints, differential games have been widely investigated using dif-
 48 ferent approaches, cf. [22, 3, 5, 28, 30, 16, 18]. The case in which the dynamics of the players
 49 are decoupled and that each player has to make sure that his state variable does not violate his
 50 own state-constraint is a classical situation for a number of applications (cf. Isaacs' book [25]
 51 for classical examples of games with decoupled dynamics).

52 State constrained problems are in general more difficult to treat: the main issue is due to
 53 the fact that both players have to use admissible controls and strategies, and the set of controls

54 allowed to each player depends on the position of the initial state variable. To solve this problem
55 one has to be able to provide Filippov-type results (called also distance estimates results) in a
56 nonanticipative way (see the discussion in [11, 12]). State constrained differential games with
57 coupled dynamical constraints have been investigated in [27] and [12]. Koike in [27], under
58 implicit uniform controllability assumptions and considering inner Hamiltonians, shows that the
59 upper and lower value functions are viscosity solutions of the associated Hamilton-Jacobi-Isaacs
60 equation and provides a comparison result. In [12] the implicit controllability assumptions of
61 [27] are replaced by directly verifiable constraint qualifications (inward pointing conditions); it
62 is shown that it is possible to derive a nonanticipative Filippov-type result (for measurable in
63 time dynamic constraint and state constraints with \mathcal{C}^1 boundaries) and, as a consequence, that
64 the upper and lower values are locally Lipschitz continuous; these are also the unique viscosity
65 solution of the related Hamilton-Jacobi-Isaacs equation; moreover conditions under which the
66 game admits a value are provided.

67 Decoupled state constrained differential games have been considered in [6, 17] for pursuit-
68 evasion problems, in [11] for Bolza problems, and in [1] for exit cost problems. Imposing some
69 uniform controllability assumptions Bardi, Koike and Soravia [6] show that the game admits a
70 (continuous) value, and imposing additional constraint qualifications they provide a comparison
71 theorem. Using different (viability) techniques Cardaliaguet, Quincampoix and Saint-Pierre [17],
72 under some suitable assumptions (which do not involve controllability conditions) demonstrate
73 that the game has a value in the class of lower semicontinuous functions. The existence result
74 [17] was subsequently extended to Bolza problems in [11], in which a nonantivipative Filippov-
75 type theorem is also proved (for state constraints with a \mathcal{C}^2 boundary); this is used also to show
76 that the value is locally Lipschitz continuous (when the final cost is locally Lipschitz continuous).
77 Bagagiolo, Maggistro and Zoppello [1] investigate exit cost differential game on domains with
78 \mathcal{C}^2 boundary and provide an existence and uniqueness result; the continuity of the values follow
79 from a nonanticipative Filippov-type result, which is proved for linear dynamic constraints and
80 state constraints that are represented by half-spaces (the boundary are hyperplanes). For an
81 application of the results obtained in [1] to a discontinuous hybrid model we refer the reader to
82 [2].

83 Numerical schemes for Differential games (in presence or in absence of state constraints)
84 have been developed by Falcone, cf. [23] and the references therein (see also his papers in
85 collaboration with Bardi, Bottacin, Soravia and Cristiani [3, 4, 21]). For an alternative approach
86 also we suggest the work done by Cardaliaguet, Quincampoix and Saint-Pierre, cf. [16]. For
87 different methods that transform a reference state constrained differential game problem to a
88 state constraint free problem see [24].

89 In our paper, we first establish that, under a set of assumptions which allow the dynamic
90 constraints and the Lagrangians to be possibly discontinuous w.r.t the time variable (more
91 precisely f_1 , f_2 , L_1 and L_2 are supposed to be merely of bounded variation in t), V^b and V^\sharp are
92 locally Lipschitz continuous. The main difficulty, here, is represented by the presence of the state
93 constraint, and it is, then, necessary to provide a new nonanticipative Filippov-type theorem,
94 which holds for general closed sets and is a crucial result to construct admissible controls and
95 strategies. Then, we show that V^b and V^\sharp are solutions in the viscosity sense of the Hamilton-
96 Jacobi-Isaacs equation associated with the reference problem. We highlight that we focus on
97 fundamental properties of the lower and upper value functions, in particular their Lipschitz
98 regularity and characterization as viscosity solutions, even when the problem data exhibit only
99 a bounded variation behaviour with respect to the time variable. This investigation introduces a
100 novel perspective in the literature on differential games, even for state constraint free problems,
101 extending key insights from recents findings obtained in the context of optimal control [14],
102 [9], [8] and calculus of variations [10], where problems with data of bounded variation with
103 respect to time have been considered. Following [27] (cf. [26] for optimal control problems) and

104 [6, 17, 11] the Hamilton-Jacobi-Isaacs equation shall involve an *inner Hamiltonian*: this is the
105 (standard inf-sup) Hamiltonian which is modified on the boundary of the state constraint set,
106 taking into account only inward pointing (w.r.t. the state constraint set) vectors which belong
107 to the convexified velocity sets of each player. The last step is represented by a comparison
108 theorem, which we prove imposing additional continuity properties on the dynamic functions f_1
109 and f_2 , and on the Lagrangians L_1 and L_2 . As a result we obtain that the differential game has
110 a value. The comparison theorem provided here represents an extension to differential games
111 with decoupled dynamics of the comparison result proved in [31] for optimal control problems:
112 this is based on the stability properties of the interior of the Clarke tangent cone (cf. [29]).

113 2 Standing assumptions

114 We shall assume that the data involved in systems (1) and (2) and the cost (3) above satisfy
115 the following hypotheses:

116 (H1): $f_1(\cdot, y, \cdot)$ is $\mathcal{L} \times \mathcal{B}^{m_1}$ measurable, $f_2(\cdot, z, \cdot)$ is $\mathcal{L} \times \mathcal{B}^{m_2}$ measurable and $L(\cdot, x = (y, z), \cdot, \cdot)$
117 is $\mathcal{L} \times \mathcal{B}^{m_1} \times \mathcal{B}^{m_2}$ measurable for each y and z (here \mathcal{L} denotes the Lebesgue subsets of \mathbb{R}
118 and \mathcal{B}^m the Borel subsets of \mathbb{R}^m); $U \subset \mathbb{R}^{m_1}$ and $V \subset \mathbb{R}^{m_2}$ are closed sets;

(H2): (i) there exists $c_f \in L^1(0, T)$ such that

$$|f_1(t, y, u)| \leq c_f(t)(1 + |y|), \quad |f_2(t, z, v)| \leq c_f(t)(1 + |z|)$$

119 for all $(y, z, u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times U \times V$ and for a.e. $t \in [0, T]$,

(ii) for every $R_0 > 0$, there exists $c_0 > 0$ such that

$$|f_1(t, y, u)| \vee |f_2(t, z, v)| \leq c_0 \quad \text{for all } (t, x = (y, z), u, v) \in [0, T] \times R_0\mathbb{B} \times U \times V,$$

(H3): for every $R_0 > 0$, there exist a modulus of continuity $\omega_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $k_f \in L^1(0, T)$
such that

$$|f_1(t, y', u) - f_1(t, y, u)| \leq \omega_f(|y - y'|), \quad |f_2(t, z', v) - f_2(t, z, v)| \leq \omega_f(|z - z'|)$$

for all $y, y' \in \mathbb{R}^{n_1}$, $z, z' \in \mathbb{R}^{n_2}$ with $|y|, |y'|, |z|, |z'| \leq R_0$, $u \in U$, $v \in V$, and $t \in [S, T]$,
and

$$|f_1(t, y', u) - f_1(t, y, u)| \leq k_f(t)|y - y'|, \quad |f_2(t, z, v) - f_2(t, z', v)| \leq k_f(t)|z - z'|$$

120 for all $y, y' \in \mathbb{R}^{n_1}$, $z, z' \in \mathbb{R}^{n_2}$ with $|y|, |y'|, |z|, |z'| \leq R_0$, $u \in U$, $v \in V$, and a.e. $t \in$
121 $[S, T]$,

(H4): for every $R_0 > 0$, there exist a modulus of continuity $\omega_L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $k_L \in L^1(0, T)$
such that (here $x = (y, z)$)

$$|L(t, x', u, v) - L(t, x, u, v)| \leq \omega_L(|x - x'|) \quad \text{for all } x, x' \in R_0\mathbb{B}, \quad u \in U, v \in V, \quad \text{and } t \in [S, T],$$

$$|L(t, x', u, v) - L(t, x, u, v)| \leq k_L(t)|x - x'| \quad \text{for all } x, x' \in R_0\mathbb{B}, \quad u \in U, v \in V, \quad \text{and a.e. } t \in [S, T],$$

and

$$|L(t, x, u, v)| \leq c_0 \quad \text{for all } x \in R_0\mathbb{B}, \quad u \in U, v \in V \quad \text{and } t \in [S, T].$$

(BV): for every $R_0 > 0$, $f_1(\cdot, y, u)$, $f_2(\cdot, z, v)$ and $L(\cdot, x, u, v)$ have bounded variation uniformly over $x = (y, z) \in R_0\mathbb{B}$, $u \in U$ and $v \in V$ in the following sense: there exists a non-decreasing function of bounded variation $\eta : [0, T] \rightarrow [0, \infty)$ such that

$$|f_1(s, y, u) - f_1(t, y, u)| \vee |f_2(s, z, v) - f_2(t, z, v)| \vee |L(s, x, u, v) - L(t, x, u, v)| \leq \eta(t) - \eta(s),$$

122 for every $[s, t] \subset [0, T]$, $u \in U$, $v \in V$, and $x = (y, z) \in R_0\mathbb{B}$.

123 (H5): for every $R_0 > 0$, there exists $k_g \geq 0$ such that $|g(x) - g(x')| \leq k_g|x - x'|$ for all $x, x' \in R_0\mathbb{B}$

(H6): the Lagrangian L has the following structure

$$L(t, x, u, v) = L_1(t, x, u) + L_2(t, x, v)$$

124 for all $x = (y, z) \in \mathbb{R}^n$, $u \in U$, $v \in V$, and $t \in [S, T]$.

(IPC): (*Convexified Inward Pointing Condition*) for each $t \in [S, T]$, $s \in (S, T]$, $y \in \partial A_1$, and $z \in \partial A_2$,

$$\text{co } f_1(t^+, y, U) \cap \text{int } T_{A_1}(y) \neq \emptyset, \quad \text{co } f_1(s^-, y, U) \cap \text{int } T_{A_1}(y) \neq \emptyset$$

and

$$\text{co } f_2(t^+, z, V) \cap \text{int } T_{A_2}(z) \neq \emptyset, \quad \text{co } f_2(s^-, z, V) \cap \text{int } T_{A_2}(z) \neq \emptyset.$$

125 Here, $T_A(x)$ denotes the Clarke's tangent cone to the set A at x defined by

$$T_A(x) := \left\{ \eta : \text{for any sequences } x_i \xrightarrow{A} x \text{ and } t_i \downarrow 0, \text{ there exists } \{w_i\} \subset A \text{ such that } t_i^{-1}(w_i - x_i) \rightarrow \eta \right\}.$$

126 \mathbb{B} denotes the closed unit ball of the Euclidean space; $\text{co}D$ is the convex hull of the set D . For
127 $a, b \in \mathbb{R}$, we write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The limits in (IPC) are intended
128 in the sense of Kuratowski (see for instance [32] for details on this notion):

$$f_1(t^+, y, U) = \lim_{t' \downarrow t} f_1(t', y, U), \quad f_1(s^-, y, U) = \lim_{s' \uparrow s} f_1(s', y, U),$$

129 and similarly for f_2 .

130 3 The Hamilton-Jacobi-Isaacs equation

We first introduce the Hamiltonian functions of interest, starting from the (un-max-minimized) Hamiltonian:

$$\mathcal{H}(t, x = (y, z), p = (p_y, p_z), u, v) := -f_1(t, y, u) \cdot p_y - f_2(t, z, v) \cdot p_z - L_1(t, x, u) - L_2(t, x, v).$$

We observe that, from the particular game structure (which is decoupled with respect the controls), these coincide, i.e. the Isaacs condition holds. We write H the obtained Hamiltonian function:

$$H(t, x, p) := \inf_{v \in V} \sup_{u \in U} \mathcal{H}(t, x, p, u, v) = \sup_{u \in U} \inf_{v \in V} \mathcal{H}(t, x, p, u, v).$$

We set also

$$Q_1(t, (y, z)) := (\text{co } f_1, L_1)(t^+, (y, z), U), \quad Q_2(t, (y, z)) := (\text{co } f_2, L_2)(t^+, (y, z), V)$$

and

$$\begin{aligned} G_1(t, (y, z)) &:= \{(e_1, \ell_1) \in Q_1(t, (y, z)) : e_1 \in \text{int}T_{A_1}(y)\}, \\ G_2(t, (y, z)) &:= \{(e_2, \ell_2) \in Q_2(t, (y, z)) : e_2 \in \text{int}T_{A_2}(z)\}. \end{aligned}$$

131 This allows us to introduce the *inner Hamiltonian*

$$H_{in}(t, (y, z), p = (p_y, p_z)) := \inf_{(e_2, \ell_2) \in G_2(t, (y, z))} \sup_{(e_1, \ell_1) \in G_1(t, (y, z))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2],$$

132 which is defined on $[0, T] \times (A_1 \times A_2) \times (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Observe that the Isaacs condition is satisfied:

$$\begin{aligned} H_{in}(t, (y, z), p = (p_y, p_z)) &= \inf_{(e_2, \ell_2) \in G_2(t, (y, z))} \sup_{(e_1, \ell_1) \in G_1(t, (y, z))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2] \\ &= \sup_{(e_1, \ell_1) \in G_1(t, (y, z))} \inf_{(e_2, \ell_2) \in G_2(t, (y, z))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2]. \end{aligned} \quad (6)$$

133 We aim to characterize the lower and upper value functions as generalized solutions in the
134 viscosity sense to the following Hamilton-Jacobi-Isaacs equation:

$$\begin{cases} -\partial_t W(t, x) + H_{in}(t, x, \partial_x W(t, x)) = 0 & \text{on } [0, T] \times (A_1 \times A_2) \\ W(T, x) = g(x) & \text{on } A_1 \times A_2. \end{cases} \quad (7)$$

The inner Hamiltonian function H_{in} involved in equation (7) can be discontinuous. One way to overcome this difficulty when we consider the notion of viscosity solution is to make use of the upper and lower semicontinuous envelopes of the Hamiltonian H_{in} (see for instance [20], [6], [7]). We recall that the upper and lower semicontinuous envelopes of a function $\Phi(t, x, p)$, written respectively Φ^* and Φ_* , are defined by

$$\Phi^*(t, x, p) := \limsup_{(t', x', p') \rightarrow (t, x, p)} \Phi(t', x', p')$$

and

$$\Phi_*(t, x, p) := \liminf_{(t', x', p') \rightarrow (t, x, p)} \Phi(t', x', p').$$

135 (The limits here are taken on points where Φ is defined.)

Definition 3.1 (Viscosity super/sub solutions of (7)) *A continuous function $w : [0, T] \times (A_1 \times A_2) \rightarrow \mathbb{R}$ is called viscosity supersolution of (7) on $[0, T] \times (A_1 \times A_2)$ if $w(T, x) = g(x)$ for all $x \in (A_1 \times A_2)$ and it satisfies the following property: for any test function $\varphi \in \mathcal{C}^1$ such that $w - \varphi$ has a local minimum at $(t_0, x_0) \in [0, T] \times (A_1 \times A_2)$ (relative to $[0, T] \times (A_1 \times A_2)$) then*

$$-\partial_t \varphi(t_0, x_0) + (H_{in})^*(t_0, x_0, \partial_x \varphi(t_0, x_0)) \geq 0.$$

A continuous function $w : [0, T] \times (A_1 \times A_2) \rightarrow \mathbb{R}$ is called viscosity subsolution of (7) on $[0, T] \times (A_1 \times A_2)$ if $w(T, x) = g(x)$ for all $x \in (A_1 \times A_2)$ and it satisfies the following property: for any test function $\varphi \in \mathcal{C}^1$ such that $w - \varphi$ has a local maximum at $(t_0, x_0) \in [0, T] \times (A_1 \times A_2)$ (relative to $[0, T] \times (A_1 \times A_2)$) then

$$-\partial_t \varphi(t_0, x_0) + (H_{in})_*(t_0, x_0, \partial_x \varphi(t_0, x_0)) \leq 0.$$

136 **Definition 3.2 (Viscosity Solution of (7))** *Consider the Hamilton-Jacobi-Isaacs equation (7).
137 Then we say that a continuous function is a viscosity solution of (7) if it is a supersolution on
138 $[0, T] \times A_1 \times A_2$ and subsolution on $[0, T] \times A_1 \times A_2$ of (7).*

139 A central role in the analysis of the value functions is the fact that we can guarantee the
140 possibility to construct admissible controls and strategies in a nonanticipative way. This is the
141 objective of next section.

4 State constrained control systems: nonanticipative constructions of admissible controls

Consider the state-constrained control system, described as follows:

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, T] \quad (8)$$

$$u(t) \in U \quad \text{a.e. } t \in [0, T] \quad (9)$$

$$x(t) \in A \quad \text{for all } t \in [0, T], \quad (10)$$

in which $f(., ., .) : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}^N$ is a given function, $A \subset \mathbb{R}^N$ is a given closed set, and $U \subset \mathbb{R}^m$ is a given closed set.

We shall refer to a couple $(x(\cdot), u(\cdot))$, comprising a measurable function $u(\cdot) : [0, T] \rightarrow \mathbb{R}^m$ and an absolutely continuous function $x(\cdot) : [0, T] \rightarrow \mathbb{R}^N$ which satisfy $\dot{x}(t) = f(t, x(t), u(t))$ and $u(t) \in U$ a.e., as a process (on $[0, T]$). The function $x(\cdot)$ is called a state trajectory and the function $u(\cdot)$ is called a control function. If $x(\cdot)$ satisfies the state constraint (10), the process is ‘admissible’.

We shall assume that the control system data satisfy the hypotheses: (H1)’-(H3)’, (BV)’ and (IPC)’ which are the equivalent formulations of the hypotheses (H1)-(H3), (BV) and (IPC) but for the control system (8)-(10) and f_1, A_1, m_1 and n_1 (or f_2, A_2, m_2 and n_2) are replaced by f, A, m and N . Observe that the inward pointing condition now takes the form

(IPC)’: (*Convexified Inward Pointing Condition*) for each $t \in [S, T]$, $s \in (S, T]$, $x \in \partial A$,

$$\text{co } f(t^+, x, U) \cap \text{int } T_A(x) \neq \emptyset, \quad \text{co } f(s^-, x, U) \cap \text{int } T_A(x) \neq \emptyset.$$

Employing the L^∞ -metric on the set of trajectories, we derive linear estimates w.r.t. the left-end points of a reference process and its approximating process. This result, often referred as nonanticipative Filippov’s theorem or ‘distance estimates’, guarantees the possibility to construct approximating admissible controls (and trajectories) in a nonanticipative way, and, therefore, build up suitable nonanticipative strategies (which is crucial when dealing with the differential games considered in this paper).

Theorem 4.1 *Take a closed nonempty set $A \subset \mathbb{R}^N$. Fix $r_0 > 0$ and define $R(t) := (r_0 + 1) \exp\left(\int_0^t c_f(s) ds\right) - 1$. Assume hypotheses (H1)’-(H3)’, (BV)’, (IPC)’ for $R_0 := R(T)$. Then there exist constants $k_0 > 0$, $K_0 > 0$, $\delta_0 > 0$, $\rho_0 > 0$, (whose magnitude depends only on the parameter r_0 and the data of assumptions (H1)’-(H3)’, (BV)’, (IPC)'), with the following property: given any $(t_1, \xi_1) \in [0, T] \times (A \cap R(t_1)\mathbb{B})$ and admissible process $(x_1(\cdot), u_1(\cdot))$ on $[t_1, T]$ such that $x_1(t_1) = \xi_1$, for any $(t_2, \xi_2) \in [0, T] \times (A \cap R(t_2)\mathbb{B})$, there exists an admissible process $(x_2(\cdot), u_2(\cdot))$ on $[t_2, T]$ with $x_2(t_2) = \xi_2$ such that the construction of $u_2(\cdot)$ is nonanticipative,*

$$x_2(t) \in \text{int } A \quad \text{for all } t \in (t_2, T], \quad \text{and} \quad (11)$$

$$\|x_1(\cdot + t_1 - t_2) - x_2(\cdot)\|_{L^\infty(t_2, T_2)} \leq K_0 (|\xi_1 - \xi_2| + |t_1 - t_2|), \quad (12)$$

where $T_2 := (T + t_2 - t_1) \wedge T$.

Moreover if $\rho := (1 + \eta(T)) \exp\left(\int_0^T k_f(t) dt\right) (|\xi_1 - \xi_2| + |t_1 - t_2|) \leq \rho_0$ then there exists a finite sequence $\{\tau_1, \dots, \tau_M\}$, (with $M \leq T/\delta_0 + 1$) such that $t_2 \leq \tau_1$, $\tau_1 + \delta_0 \leq \tau_2, \dots$, $\tau_j + \delta_0 \leq \tau_{j+1}, \dots, \tau_M < T_2$, and the control u_2 on $[t_2, T_2]$ that has the following structure:

$$u_2(t) := \begin{cases} \bar{u}_j(t) & \text{if } t \in [\tau_j, (\tau_j + k_0\rho) \wedge T_2] \\ u_1(t - k_0\rho + t_1 - t_2) & \text{if } t \in (\tau_j + k_0\rho, (\tau_j + \delta_0) \wedge T_2] \text{ for } j = 1, \dots, M \\ u_1(t + t_1 - t_2) & \text{otherwise,} \end{cases}$$

174 for some control functions $\bar{u}_j : [\tau_j, (\tau_j + k_0\rho) \wedge T_2] \rightarrow U$, $j = 1, \dots, M$.

175 **Remark 4.2** A scrutiny of the proof of Theorem 4.1 tells us further useful information about
 176 the existence of admissible controls for processes emerging from some given initial data $(t_0, \xi_0) \in$
 177 $[0, T) \times A$ (even if we are not necessarily interested in comparing it with respect to admissible
 178 processes having different initial data, which is the purpose of Theorem 4.1 statement). Indeed,
 179 we can always fix a control function u_0 on $[0, T]$. Then for any given initial data $(t_0, \xi_0) \in$
 180 $[0, T) \times A$, we can consider a positive number ρ , which now has a different expression from
 181 (14) below and can be written in terms of the state constraint violation of the trajectory $x_0(\cdot)$
 182 associated with the control u_0 on $[t_0, T]$ (cf. [15] for the details in the context of differential
 183 inclusions, the adaptation of which to the control systems is straightforward). The analysis
 184 along the line of the proof of Theorem 4.1 (cf. [15] for the differential inclusions) then provides
 185 an admissible pair (\bar{x}, \bar{u}) such that $\bar{x}(t_0) = \xi_0$, $\bar{x}(t) \in \text{int}A$ for all $t \in (t_0, T]$ and, if ξ_0 is ‘close’
 186 to the boundary ∂A and a reference vector $v \in \text{cof}(t_0^+, \xi_0, U)$ is given, then the control \bar{u} can be
 187 constructed in such a manner that, for each $\varepsilon > 0$ we can find $\sigma > 0$ such that

$$\left| \int_{t_0}^{t_0+\sigma} [f(s, \bar{x}(s), \bar{u}(s)) - v] ds \right| \leq \varepsilon. \quad (13)$$

188 When we consider our reference differential game problem and assumptions (H1)-(H3), (BV)
 189 and (IPC) are satisfied, this translates into the fact that for every initial data $(t_0, x_0 = (y_0, z_0)) \in$
 190 $[0, T) \times A_1 \times A_2$, the sets of admissible controls $\mathcal{U}(t_0, y_0)$ and $\mathcal{V}(t_0, z_0)$ are nonempty and, in
 191 particular, we can always find controls $\bar{u} \in \mathcal{U}(t_0, y_0)$ and $\bar{v} \in \mathcal{V}(t_0, z_0)$ satisfying the properties
 192 described above.

193 Proof of Theorem 4.1

194 We fix a control function u_0 on $[0, T]$. Fix also any $r_0 > 0$. Assume that the function f
 195 and set A in the theorem statement satisfy (H1)', (H2)', (H3)', (IPC)' and (BV)' with constant
 196 $c_0 > 0$, and functions $c_f, k_f \in L^1(0, T)$, for $R_0 := R(T)$. Take any $t_1 \in [0, T]$. The constants R_0
 197 and c_0 bound, respectively, magnitudes and velocities of all processes (x, u) on the subinterval
 198 $[t_1, T] \subset [0, T]$ starting from $R(t_1)\mathbb{B}$. Let $\bar{\eta} > 0$ and $\eta(\cdot)$ be the constant and modulus of variation
 199 appearing in (BV)'. Take any $\xi_i \in A \cap R(t_i)\mathbb{B}$, $i = 1, 2$, with $\xi_1 \neq \xi_2$ (for, otherwise, there is
 200 nothing to prove).

201 Let (x_1, u_1) be any given admissible process on $[t_1, T]$. We shall first construct, in a nonan-
 202 ticipative way, an admissible process (x_2, u_2) on $[t_2, T_2]$ such that (11) and (12) are satisfied
 203 (Steps 1–5). In Step 6, we provide its extension to $[t_2, T]$ (when $T_2 < T$).

204
 205 *Step 1. (A reduction argument).*

206 The reduction techniques and the preliminary analysis used in [15] can be easily adapted to our
 207 control system (8) (considering the multivalued map $(t, x) \rightsquigarrow F(t, x) := f(t, x, U)$), allowing us
 208 to restrict our attention, without loss of generality, to the case when

209
 210 (i): $\xi_2 \in (\partial A + \frac{\bar{\eta}}{2}\mathbb{B}) \cap A \cap R(t_2)\mathbb{B}$ and

$$\rho := (1 + \eta(T)) \exp\left(\int_0^T k_f(t) dt\right) (|\xi_1 - \xi_2| + |t_1 - t_2|) \leq \rho_0, \text{ for some } \rho_0 > 0, \quad (14)$$

211 (ii): we consider a subinterval $[t_2, \tilde{T}_2] \subset [0, T]$ (in place of $[t_2, T_2]$) such that $\tilde{T}_2 - t_2 \leq \delta_0$ for
 212 some $\delta_0 > 0$,

213
 214 (iii): $\eta(\tilde{T}_2) - \eta(t_2) \leq \gamma$.

215

216 Here, $\rho_0 > 0$, $\delta_0 > 0$ and $\gamma > 0$ are constants which depend only on the data given in the
 217 assumptions, see (16) and (17)-(18) below.

218

219 Using well-known stability properties of the interior of Clarke tangent cone [29], owing to [15,
 220 Lemma 5], we can eventually modify the reference function f at most on a finite number of times
 221 $\{\sigma_i\} \subset [0, T]$ and obtain a new function \tilde{f} which satisfies the following property: there exist
 222 $\epsilon \in (0, 1)$ and $\bar{\eta} > 0$ (we can arrange that $\bar{\eta}$ is the same constant appearing in (BV)', otherwise
 223 we can always reduce its size) such that given any $(t, x) \in [0, T] \times ((\partial A + \bar{\eta}\mathbb{B}) \cap R_0\mathbb{B} \cap A)$, we
 224 can find a vector $v \in \text{co } \tilde{f}(t, x, U)$ such that

$$x' + [0, \epsilon](v + \epsilon\mathbb{B}) \subset A, \quad \text{for all } x' \in (x + \epsilon\mathbb{B}) \cap A. \quad (15)$$

225 We also know that a process (x, u) for the dynamics governed by \tilde{f} is also a process for f
 226 and vice-versa. Therefore, it is not restrictive to continue our analysis assuming that, for any
 227 $(t, x) \in [t, T] \times ((\partial A + \bar{\eta}\mathbb{B}) \cap R_0\mathbb{B} \cap A)$ we can find $v \in \text{co } f(t, x, U)$ such that (15) is true for all
 228 $x' \in (x + \epsilon\mathbb{B}) \cap A$.

229

230

Let $\omega : [0, T] \rightarrow [0, \infty)$ be the function

$$\omega(\delta) := \sup \left(\int_I k_f(s) ds \right),$$

231 where the supremum is taken over sub-intervals $I \subset [0, T]$ of length not greater than δ . Observe
 232 that $\omega(\cdot)$ is well-defined on $[0, T]$, for $k_f \in L^1(0, T)$, and that $\omega(\delta) \rightarrow 0$, as $\delta \downarrow 0$.

233

234 Fix $k_0 \geq 1$ such that $k_0 > \epsilon^{-1}$ and take constants $\delta_0 > 0$, $\rho_0 > 0$ and $\gamma > 0$ in such a manner
 235 that

$$\delta_0 \leq \epsilon, \quad \rho_0 + c_0\delta_0 < \epsilon, \quad k_0\rho_0 < \delta_0, \quad \rho_0 \leq \bar{\eta}, \quad 4\delta_0c_0 \leq \bar{\eta}, \quad (16)$$

236 and

$$4e^{\omega(\delta_0)}(\gamma + c_0\omega(\delta_0)) < \epsilon, \quad (17)$$

$$\left[\delta_0/2 + c_0k_0\omega(\delta_0) + k_0e^{\omega(\delta_0)}(\gamma(1 + \omega(\delta_0) + c_0\omega(\delta_0)(3 + \omega(\delta_0))) \right] \rho + \gamma\delta_0 < (k_0\epsilon - 1). \quad (18)$$

237 Set $K := (1 + \eta(T)) \exp \left(\int_0^T k_f(t) dt \right)$ so that $\rho = K(|\xi_1 - \xi_2| + |t_1 - t_2|)$.

238

239 *Step 2. (Admissible control construction – first part of the time interval).*

240 Since we can restrict attention to a situation in which (i) is valid, we can find a vector $v \in$
 241 $\text{co } f(t_2, \xi_2, U)$ satisfying property (15) for $(t, x) = (t_2, \xi_2)$. Now, consider the arc $y : [t_2, \tilde{T}_2] \rightarrow \mathbb{R}^n$
 242 such that $y(t_2) = \xi_2$ and $\dot{y}(t) = v$. It immediately follows that, for all $t \in [t_2, (t_2 + k_0\rho) \wedge T]$,

$$y(t) = \xi_2 + (t - t_2)v. \quad (19)$$

243 Recalling that c_0 constitutes an upper bound for the magnitude for both v and $\|\dot{x}_1\|_{L^\infty}$, we
 244 deduce that

$$\|x_1(\cdot + t_1 - t_2) - y(\cdot)\|_{L^\infty(t_2, (t_2 + (T - t_1) \wedge k_0\rho) \wedge \tilde{T}_2)} \leq 2c_0k_0\rho. \quad (20)$$

245 In addition, from (BV)' we also obtain that, for all $s \in [t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$,

$$\begin{aligned} d_{\text{co } f(s, y(s), U)}(\dot{y}(s)) &\leq (\eta(s) - \eta(t_2)) + k_f(s)|y(s) - y(t_2)| \\ &\leq (\eta(s) - \eta(t_2)) + k_f(s)c_0(s - t_2). \end{aligned} \quad (21)$$

246

247

248 Invoking Filippov's Existence Theorem (cf. [32, Thm. 2.4.3]), in which we take as reference
 249 multivalued function $\tilde{F}(t, x) := \text{co } F(t, x) = \text{co } f(t, x, U)$ and bearing in mind condition (21),
 250 we can find an \tilde{F} -trajectory \tilde{x} on $[t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$ with $\tilde{x}(t_2) = y(t_2) = \xi_2$ and such that, for
 251 any $t \in [t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$

$$\|\tilde{x} - y\|_{L^\infty(t_2, t)} \leq e^{\omega(\delta_0)}(\gamma + c_0 \omega(\delta_0))(t - t_2). \quad (22)$$

252 It follows that for all $t \in [t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$,

$$\begin{aligned} \tilde{x}(t) &\in y(t) + e^{\omega(\delta_0)}(\gamma + c_0 \omega(\delta_0))(t - t_2)\mathbb{B}, \text{ from (22),} \\ &\subset \xi_2 + (t - t_2)(v + e^{\omega(\delta_0)}(\gamma + c_0 \omega(\delta_0))\mathbb{B}), \text{ from (19),} \\ &\subset \xi_2 + (t - t_2)(v + \epsilon/4\mathbb{B}), \text{ from (17)} \\ &\subset \text{int}A, \text{ from (14) - (16).} \end{aligned} \quad (23)$$

253 We take a decreasing sequence $\{\sigma_k\}_{k \geq 1}$ in $(t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$ such that $\sigma_1 := (t_2 + k_0\rho) \wedge \tilde{T}_2$
 254 and $\sigma_k \downarrow t_2$ as $k \rightarrow +\infty$. Observe that, since $\tilde{x}(t) \in \text{int}A$ for all $t \in (t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$, there
 255 exists a decreasing sequence $\epsilon_k \downarrow 0$, with $\epsilon_k \in (0, (\epsilon(\sigma_k - t_2)) \wedge (\delta_0\rho))$, for all $k \geq 1$, such that

$$\tilde{x}(\sigma) + \epsilon_k \mathbb{B} \subset \text{int}A \text{ for all } \tau \in [\sigma_k, \sigma_1].$$

256 Employing the techniques used in the proof of [13, Lemma 5.2, Step 3] we can find an
 257 F -trajectory $x_2(\cdot)$ on $[t_1, (t_2 + k_0\rho) \wedge \tilde{T}_2]$ such that $x_2(t_2) = y(t_2) = \xi_2$ and, for each $k \geq 2$,

$$x_2(t) \in \tilde{x}(t) + \frac{\epsilon_k}{2} \mathbb{B} \subset \text{int}A, \text{ for all } t \in (\sigma_k, \sigma_{k-1}]. \quad (24)$$

258 In particular we have:

$$x_2(\sigma_1) \in \tilde{x}(\sigma_1) + \frac{\epsilon_1}{2} \mathbb{B} \subset y(\sigma_1) + \left[\frac{\delta_0}{2} + e^{\omega(\delta_0)}(\gamma + c_0 \omega(\delta_0))k_0 \right] \rho \mathbb{B}. \quad (25)$$

259 Using the Filippov's Selection Theorem (cf. [32, Thm. 2.3.13]) we can find a control $\bar{u}_2 : [t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2] \rightarrow U$ such that (x_2, \bar{u}_2) is a process on $[t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$.

261 Observe that, in fact, from the analysis of this step we can deduce a much stronger property:
 262 for each initial data (t, ξ) , where $t \in [0, T]$ and

$$\xi \in \left(\partial A + \frac{\bar{\eta}}{2} \mathbb{B} \right) \cap A \cap R(t)\mathbb{B}, \quad (26)$$

263 we can construct a control $\bar{u} : [t, (t + k_0\rho) \wedge T] \rightarrow U$ such that the associated process (\bar{x}, \bar{u})
 264 emerging from $\bar{x}(t) = \xi$ satisfies the condition $\bar{x}(s) \in \text{int}A$ for all $s \in (t, (t + k_0\rho) \wedge T]$ and

$$\bar{x}((t + k_0\rho) \wedge T) \in \bar{y}((t + k_0\rho) \wedge T) + \left[\frac{\delta_0}{2} + e^{\omega(\delta_0)}(\gamma + c_0 \omega(\delta_0))k_0 \right] \rho \mathbb{B},$$

265 where $\bar{y}(s) := \xi + (s - t)v$, $s \in [t, (t + k_0\rho) \wedge T]$ and $v \in \text{co } f(t, \xi, U) \cap \text{int}T_A(\xi)$ is taken according
 266 to Step 1.

267 Therefore for each initial data (t, ξ) such that (26) is satisfied, we consider the associated (fixed)
 268 control obtained in this step, which is admissible on $[t, (t + k_0\rho) \wedge T]$.

269

270 *Step 3. (Admissible control construction - second part of the time interval and distance esti-*
 271 *mates).*

272 Observe that, given any control u on $[t_1, T]$ such that the process $(x(\cdot), u(\cdot))$ with starting point
 273 $x(t_1) = \xi_1$ is admissible on $[t_1, T]$, if we consider the process $(\hat{x}(\cdot), u(\cdot + t_1 - t_2))$ on $[t_2, T + t_2 - t_1]$
 274 such that $\hat{x}(t_2) = \xi_2$ (which in general is not necessarily admissible), then from Gronwall in-
 275 equality we have

$$\begin{aligned} \max_{t \in [t_2, T_2]} d_A(\hat{x}(t)) &\leq \|x(\cdot + t_1 - t_2) - \hat{x}(\cdot)\|_{L^\infty(t_2, T_2)} \\ &\leq \exp\left(\int_0^T k_f(t) dt\right) (1 + \eta(T)) (|\xi_1 - \xi_2| + |t_1 - t_2|) (= \rho). \end{aligned} \quad (27)$$

Take now an admissible process $(x_1(\cdot), u_1(\cdot))$ on $[t_1, T]$ such that $x_1(t_1) = \xi_1$. Define a new control

$$u_2(t) := \begin{cases} \bar{u}_2(t) & \text{if } t \in [t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2] \\ u_1(t + t_1 - t_2 - k_0\rho) & \text{if } t \in (t_2 + k_0\rho, \tilde{T}_2]. \end{cases}$$

276 Write (x_2, u_2) the process associated with the control u_2 with starting point $x_2(t_2) = \xi_2$. It
 277 follows that for any $t \in [t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$

$$\begin{aligned} |x_1(t + t_1 - t_2) - x_2(t)| &\leq \int_{t_2}^t |f(s + t_1 - t_2, x_1(s + t_1 - t_2), u_1(s + t_1 - t_2)) \\ &\quad - f(s, x_2(s), u_2(s))| ds + |\xi_1 - \xi_2| \\ &\leq 2c_0 k_0 \rho + |\xi_1 - \xi_2| \\ &\leq (2c_0 k_0 K + 1) (|\xi_1 - \xi_2| + |t_1 - t_2|). \end{aligned} \quad (28)$$

278 We have to consider now the case in which $t_2 + k_0\rho < \tilde{T}_2$. Write $(\hat{x}_2(\cdot), u_1(\cdot + t_1 - t_2))$ the process
 279 associated with the control $u_1(\cdot + t_1 - t_2)$ with starting point $\hat{x}_2(t_2) = \xi_2$. From (27) applied to
 280 $(\hat{x}_2, u_1(\cdot + t_1 - t_2))$ we deduce that $\max_{t \in [t_2, T_2]} d_A(\hat{x}_2(t)) \leq \rho$. For all $t \in [t_2 + k_0\rho, \tilde{T}_2]$ we have

$$\begin{aligned} |\hat{x}_2(t) - x_2(t)| &= \left| \int_{t_2}^t f(s, \hat{x}_2(s), u_1(s + t_1 - t_2)) ds - \int_{t_2}^t f(s, x_2(s), u_2(s)) ds \right| \\ &= \left| \int_{t_2}^t f(s, \hat{x}_2(s), u_1(s + t_1 - t_2)) ds - \int_{t_2}^{t_2 + k_0\rho} f(s, x_2(s), \bar{u}_2(s)) ds \right. \\ &\quad \left. - \int_{t_2 + k_0\rho}^t f(s, x_2(s), u_1(s + t_1 - t_2 - k_0\rho)) ds \right| \\ &\leq \int_{t - k_0\rho}^t |f(s, x_2(s), u_1(s + t_1 - t_2))| ds + \int_{t_2}^{t_2 + k_0\rho} |f(s, x_2(s), \bar{u}_2(s))| ds \\ &\quad + \int_{t_2}^t |f(s, \hat{x}_2(s), u_1(s + t_1 - t_2)) - f(s, x_2(s), u_1(s + t_1 - t_2))| ds \\ &\quad + \int_{t_2}^{t - k_0\rho} |f(s, x_2(s), u_1(s + t_1 - t_2)) - f(s + k_0\rho, x_2(s), u_1(s + t_1 - t_2))| ds \\ &\quad + \int_{t_2}^{t - k_0\rho} |f(s + k_0\rho, x_2(s), u_1(s + t_1 - t_2)) - f(s + k_0\rho, x_2(s + k_0\rho), u_1(s + t_1 - t_2))| ds \\ &\leq 2c_0 k_0 \rho + \int_{t_2}^t k_f(s) |\hat{x}_2(s) - x_2(s)| ds + \int_{t_2}^{t - k_0\rho} (\eta(s + k_0\rho) - \eta(s)) ds \\ &\quad + \int_{t_2}^{t - k_0\rho} k_f(s) |x_2(s + k_0\rho) - x_2(s)| ds \\ &\leq (2c_0 + \gamma + \omega(\delta_0)c_0) k_0 \rho + \int_{t_2}^t k_f(s) |\hat{x}_2(s) - x_2(s)| ds. \end{aligned} \quad (29)$$

281 Then, from Gronwall inequality (in the integral form), we deduce that, for all $t \in [t_2, \tilde{T}_2]$,

$$|\hat{x}_2(t) - x_2(t)| \leq e^{\omega(\delta_0)}(2c_0 + \gamma + \omega(\delta_0)c_0)k_0\rho. \quad (30)$$

282 Take any $t \in (t_2 + k_0\rho, \tilde{T}_2]$, from the estimates above we obtain

$$\begin{aligned} |x_1(t+t_1-t_2) - x_2(t)| &\leq |x_1(t+t_1-t_2) - \hat{x}_2(t)| + |\hat{x}_2(t) - x_2(t)| \\ &\leq e^{\omega(\delta_0)}(1 + \eta(T))(|\xi_1 - \xi_2| + |t_1 - t_2|) + e^{\omega(\delta_0)}(2c_0 + \gamma + \omega(\delta_0)c_0)k_0\rho \\ &\leq e^{\omega(\delta_0)} [1 + \eta(T) + (2c_0 + \gamma + \omega(\delta_0)c_0)k_0K] (|\xi_1 - \xi_2| + |t_2 - t_1|), \end{aligned} \quad (31)$$

283 from which we deduce the required estimate:

$$\|x_1(\cdot + t_1 - t_2) - x_2(\cdot)\|_{L^\infty(t_2, \tilde{T}_2)} \leq K_0(|\xi_1 - \xi_2| + |t_1 - t_2|), \quad (32)$$

where

$$K_0 := e^{\omega(\delta_0)} [1 + \eta(T) + (2c_0 + \gamma + \omega(\delta_0)c_0)k_0K].$$

284 *Step 4. (The process is admissible on the second part of the time interval).*

285 From (24) we know that $x_2(t) \in \text{int}A$ for all $t \in (t_2, (t_2 + k_0\rho) \wedge \tilde{T}_2]$. So to complete the proof
286 we proceed assuming that $t_2 + k_0\rho < \tilde{T}_2$. Define the arc $\hat{y} : [t_2, \tilde{T}_2] \rightarrow \mathbb{R}^n$ as follows

$$\hat{y}(t) := \begin{cases} y(t) = \xi_2 + v(t - t_2) & \text{if } t \in [t_2, t_2 + k_0\rho) \\ \xi_2 + k_0\rho v + \int_{t_2+k_0\rho}^t \hat{x}_2(s - k_0\rho) ds & \text{if } t \in [t_2 + k_0\rho, \tilde{T}_2] \end{cases}. \quad (33)$$

287 Observe that, when $t \in [t_2 + k_0\rho, \tilde{T}_2]$, we have $\hat{y}(t) = k_0\rho v + \hat{x}_2(t - k_0\rho)$, and writing $z(t)$ a
288 projection on A of the arc $t \mapsto \hat{x}_2(t - k_0\rho)$, it satisfies $|\hat{x}_2(t - k_0\rho) - z(t)| = d_A(\hat{x}_2(t - k_0\rho)) \leq$
289 $\|\hat{x}_2 - x_1(\cdot + t_1 - t_2)\|_{L^\infty(t_2+k_0\rho, \tilde{T}_2)} \leq \rho$ and we deduce that

$$\hat{y}(t) \in z(t) + k_0\rho v + \rho\mathbb{B} \text{ for all } t \in [t_2 + k_0\rho, \tilde{T}_2]. \quad (34)$$

290 Notice also that for all $t \in [t_2 + k_0\rho, \tilde{T}_2]$, making use of (25) and (30), recall that here $\tau_1 = t_2 + k_0\rho$,
291 we obtain

$$\begin{aligned} |x_2(t) - \hat{y}(t)| &\leq |x_2(t_2 + k_0\rho) - y(t_2 + k_0\rho)| \\ &\quad + \left| \int_{t_2+k_0\rho}^t \left[f(s, x_2(s), u_2(s)) - f(s - k_0\rho, \hat{x}_2(s - k_0\rho), u_1(s - k_0\rho + t_1 - t_2)) \right] ds \right| \\ &\leq \left[\delta_0/2 + e^{\omega(\delta_0)}(\gamma + c_0\omega(\delta_0))k_0 \right] \rho + \int_{t_2+k_0\rho}^t (\eta(s) - \eta(s - k_0\rho)) ds \\ &\quad + \int_{t_2+k_0\rho}^t k_f(s) |x_2(s) - x_2(s - k_0\rho) + x_2(s - k_0\rho) - \hat{x}_2(s - k_0\rho)| ds \\ &\leq \left[\delta_0/2 + e^{\omega(\delta_0)}(\gamma + c_0\omega(\delta_0))k_0 + c_0\omega(\delta_0)k_0 + \omega(\delta_0)e^{\omega(\delta_0)}(2c_0 + \gamma + \omega(\delta_0)c_0)k_0 \right] \rho + \gamma\delta_0 \end{aligned} \quad (35)$$

Since $|\hat{x}_2(t - k_0\rho) - \hat{x}_2(t_2)| \leq c_0(\tilde{T}_2 - t_2)$ for all $t \in (t_2 + k_0\rho, \tilde{T}_2]$, appealing once again to (16), we also have

$$|z(t) - \hat{x}_2(t_2)| = |z(t) - \hat{x}_2(t - k_0\rho) + \hat{x}_2(t - k_0\rho) - \xi_2| \leq \rho + c_0\delta_0 \leq \rho_0 + c_0\delta_0 < \varepsilon.$$

Thus bearing in mind (15) and (16), we see that

$$z(t) + k_0\rho(v + \varepsilon\mathbb{B}) \subset A,$$

and, owing to (34),

$$\hat{y}(t) + (k_0\epsilon - 1)\rho\mathbb{B} \subset A.$$

292 Taking into account (18) and (35), we deduce that $x_2(t) \in \text{int } A$ for all $t \in [t_2 + k_0\rho, \tilde{T}_2]$ in this
 293 case as well, confirming all the assertions of the theorem.

294

295 *Step 5. (Iteration, nonanticipativity).*

296 With the help of the reduction argument of Step 1 we constructed (in Steps 2 and 3) an admissible
 297 process on the interval $[t_2, \tilde{T}_2]$ of length at most $\delta_0 > 0$, and the magnitude of δ_0 , depends only
 298 on the data of the problem and the choice of the radius $r_0 > 0$ (which regulates the size of the
 299 region in which the processes are supposed to emerge). We recall that the reduction argument
 300 of Step 1 (we refer the reader to [15] and [13] for full details) allows to reduce attention to
 301 subintervals of length smaller than δ_0 , since, if $T_2 - t_2 > \delta_0$, we partition $[t_2, T_2]$ as a family of
 302 M_0 contiguous intervals, each of length at most δ_0 , where M_0 is the smaller integer such that
 303 $M_0^{-1}(T_2 - t_2) \leq \delta_0$, $\{[\sigma_0^i, \sigma_1^i]\}_{i=1}^{M_0}$, where $\sigma_0^1 = t_2$, $\sigma_1^{M_0} = T_2$, $\sigma_1^i = \sigma_0^i + \delta_0$ for all $i = 1, \dots, M_0 - 1$
 304 and $\sigma_1^{M_0} - \sigma_0^{M_0} \leq \delta_0$. If the starting point ξ_2 belong to $(\partial A + \frac{\eta}{2}\mathbb{B}) \cap A \cap R(t_2)\mathbb{B}$, then we construct
 305 an admissible process (x_2, u_2) on $[\sigma_0^1 = t_2, \sigma_1^1]$ according to Steps 1–4. On the other hand, if
 306 $\xi_2 \in (A \cap R(t_2)\mathbb{B}) \setminus (\partial A + \frac{\eta}{2}\mathbb{B})$, then we just consider the admissible control $u_2(\cdot) := u_1(\cdot + t_1 - t_2)$
 307 on $[\sigma_0^1, \sigma_1^1]$. In a subsequent stage we simply extend the obtained process (x_2, u_2) for $[\sigma_0^1 = t_2, \sigma_1^2]$
 308 taking into account the position of the new initial condition $x_2(\sigma_0^2) := x_2(\sigma_1^1)$ and according to
 309 the criterion employed above: the control depends on whether $x_2(\sigma_0^2) \in (\partial A + \frac{\eta}{2}\mathbb{B}) \cap A$ or not
 310 (observe that it necessarily belongs to $R(\sigma_0^2)\mathbb{B}$). This tells us that to build up an admissible
 311 process (x_2, u_2) on (the full time interval) $[t_2, T_2]$ it is necessary to apply the construction
 312 displayed in the previous steps only a finite number of times $M \leq M_0$.

313 We write $\tau_j \in [t_2, T_2]$, for $j = 1, \dots, M$, the initial time of each interval of length at most δ_0
 314 on which we employ the above construction of Steps 1–4. Observe that we have $t_2 \leq \tau_1 < \dots <$
 315 $\tau_M < T_2$, $\tau_{j+1} - \tau_j \geq \delta_0$ for all $j = 1, \dots, M - 1$. Whenever $t \notin \cup_{j=1}^M [\tau_j, (\tau_j + \delta_0) \wedge T_2]$ we have
 316 $u_2(t) := u_1(t + t_1 - t_2)$. Therefore we shall end up with an admissible control u_2 on $[t_2, T_2]$ that
 317 has the following structure:

$$u_2(t) := \begin{cases} \bar{u}_j(t) & \text{if } t \in [\tau_j, (\tau_j + k_0\rho) \wedge T_2] \\ u_1(t - k_0\rho + t_1 - t_2) & \text{if } t \in (\tau_j + k_0\rho, (\tau_j + \delta_0) \wedge T_2] \text{ for } j = 1, \dots, M \\ u_1(t + t_1 - t_2) & \text{otherwise,} \end{cases}$$

318 for some control functions $\bar{u}_j : [\tau_j, (\tau_j + k_0\rho) \wedge T_2] \rightarrow U$, $j = 1, \dots, M$. Observe that the control
 319 u_2 is constructed starting from u_1 (shifted of the quantity $t_1 - t_2$) and it is modified on the
 320 intervals $[\tau_j, (\tau_j + \delta_0) \wedge T_2]$ according to Steps 1–4.

321 Now, we show that this construction is nonanticipative. Take two admissible processes
 322 (x_1, u_1) and (x'_1, u'_1) on $[t_1, T]$ such that $x_1(t_1) = x'_1(t_1) = \xi_1$. Take any $\sigma \in [0, T - t_1]$. It is not
 323 restrictive to consider only the following two situations:

324 (i) $t_2 + \sigma \in [t_2, \tau_1]$ and $t_2 < \tau_1$,

325 (ii) $t_2 + \sigma \in [\tau_1, (\tau_1 + \delta_0) \wedge T_2]$,

326 since the analysis for all the other cases can be easily traced back to these ones.

327 Case (i) : $t_2 + \sigma \in [t_2, \tau_1]$ ($t_2 < \tau_1$). In this case if $u_1 = u'_1$ a.e. on $[t_1, t_1 + \sigma]$, then clearly
 328 for a.e. $t \in [t_2, (t_2 + \sigma) \wedge T_2]$ we have $u_2(t) = u_1(t + t_1 - t_2) = u'_1(t + t_1 - t_2) = u'_2(t)$.

329 Case (ii) : $t_2 + \sigma \in [\tau_1, (\tau_1 + \delta_0) \wedge T_2]$. Suppose that $u_1 = u'_1$ a.e. on $[t_1, t_1 + \sigma]$. If
 330 $t_2 + \sigma \in [\tau_1, \tau_1 + k_0\rho]$ then we have

$$\begin{cases} u_2(t) = u'_2(t) \text{ a.e. on } [t_2, \tau_1] \\ u_2(t) = \bar{u}_1(t) = u'_2(t) \text{ a.e. } t \in [\tau_1, \tau_1 + k_0\rho] \end{cases} \quad (36)$$

331 On the other hand, when $t_2 + \sigma \in [\tau_1 + k_0\rho, (\tau_1 + \delta_0) \wedge T_2]$, in addition to (36) we have
 332 $u_2(t) = u_1(t - k_0\rho + t_1 - t_2) = u'_1(t - k_0\rho + t_1 - t_2) = u'_2(t)$ a.e. on $[\tau_1 + k_0\rho, (t_2 + \sigma) \wedge T_2]$. In
 333 both cases, we obtain that the mapping that provides the control u_2 is nonanticipative.

334
 335 *Step 6. (Construction completion).*

336 If $T_2 = T$ (which means that $t_2 - t_1 \geq 0$) then the construction of the admissible process (x_2, u_2)
 337 is complete. Otherwise if $T_2 < T$ (i.e. $T_2 := T + t_2 - t_1$ and $t_2 < t_1$), we have to extend the
 338 process (x_2, u_2) on $[t_2, T_2]$ obtained above to the interval $[t_2, T]$. Observe that in this extension
 339 procedure we restrict attention only to condition (11) since estimate (12) involves the restriction
 340 of the trajectory $x_2(\cdot)$ to the time interval $[t_2, T_2]$.

341 We consider the process $(w_2, u_0|_{[T_2, T]})$ on $[T_2, T]$ such that $w_2(T_2) = x_2(T_2)$, where u_0 is the
 342 control (on $[0, T]$) that we fixed at the beginning of the proof. Since $w_2(T_2) \in \text{int}A$ and $T - T_2 =$
 343 $t_1 - t_2 < \delta_0$, if $x_2(T_2) \in (\text{int}A) \setminus (\partial A + \frac{\eta}{2}\mathbb{B})$, then from (16) we deduce that $w_2(t) \in \text{int}A$ for all
 344 $t \in [T_2, T]$. In this case we extend (x_2, u_2) to $[t_2, T]$ obtaining $x_2(t) = w_2(t)$ and $u_2(t) = u_0(t)$
 345 on $[T_2, T]$.

346 On the other hand, if $x_2(T_2) \in (\text{int}A) \cap (\partial A + \frac{\eta}{2}\mathbb{B})$, then there exists a vector $v_2 \in \text{cof}(T_2, x_2(T_2), U)$
 347 which satisfies condition (15). Then employing exactly the argument displayed in *Step 2* on the
 348 whole interval $[T_2, T]$ (in place of $[T_2, (T_2 + k_0\rho) \wedge T]$), we can find a process (\bar{x}_2, \bar{u}_2) on $[T_2, T]$
 349 such that $\bar{x}_2(T_2) = x_2(T_2)$ and $\bar{x}_2(t) \in \text{int}A$ for all $t \in [T_2, T]$. Therefore, in this situation, the
 350 extension of (x_2, u_2) will be obtained setting $(x_2, u_2) = (\bar{x}_2, \bar{u}_2)$ on $[T_2, T]$.

351 The nonanticipative property of our construction in the last interval $[T_2, T]$ (when $T - T_2 =$
 352 $t_1 - t_2 > 0$) follows from the fact that we have two possible situations: either $x_2(T_2) \in \text{int}A \setminus$
 353 $(\partial A + \frac{\eta}{2}\mathbb{B})$ and we complete u_2 with the (fixed) control u_0 , or $x_2(T_2) \in \partial A + \frac{\eta}{2}\mathbb{B}$ and then we
 354 extend u_2 using the control \bar{u} provided by Step 2.

355 Finally, observe that if

$$(1 + \eta(T)) \exp\left(\int_0^T k_f(t) dt\right) (|t_1 - t_2| + |\xi_1 - \xi_2|) > \rho_0 \quad (37)$$

356 then to construct the admissible process (x_2, u_2) we merely make use of the argument of *Step*
 357 *6* (repeating it at most for a finite number of times). The analysis of *Step 6* provides a state
 358 trajectory $x_2(\cdot)$ on $[t_2, T]$ satisfying $x_2(t_2) = \xi_2$ and condition (11), but it gives no information
 359 on the L^∞ distance between the two trajectories $x_1(\cdot)$ and $x_2(\cdot)$. However when the distance
 360 between the initial data is big enough and (37) is in force we immediately deduce that

$$\|x_1(\cdot + t_1 - t_2) - x_2(\cdot)\|_{L^\infty(t_2, T_2)} \leq \frac{4}{\rho_0} c_0 TK (1 + \eta(T)) (|t_1 - t_2| + |\xi_1 - \xi_2|).$$

361 Then, possibly adjusting the constant K_0 we deduce in this case also the validity of (12).

362 □

363 5 Regularity properties of the lower and upper value functions

364 **Proposition 5.1 (Lipschitz continuity)** *Let $A_1 \subset \mathbb{R}^{n_1}$ and $A_2 \subset \mathbb{R}^{n_2}$ be nonempty closed*
 365 *sets. Suppose that assumptions (H1)-(H6), (BV) and (IPC) are satisfied. Then the lower value*
 366 *function V^b and the upper value function V^\sharp are locally Lipschitz continuous on $[0, T] \times A_1 \times A_2$.*

367 **Proof.** Fix a real number $r_0 > 0$. Take any $(t_1, x_1 = (\xi_1, \zeta_1)), (t_2, x_2 = (\xi_2, \zeta_2)) \in$
 368 $[0, T] \times (A \cap r_0\mathbb{B})$. Define $T_1 := (T + t_1 - t_2) \wedge T$, $T_2 := (T + t_2 - t_1) \wedge T$ and take $R_0 :=$
 369 $\exp\left(\int_0^T c_f(t) dt\right) (r_0 + 1)$.

370 Let $\varepsilon > 0$ be a given number. From the definition of the upper value V^\sharp , we can find a
 371 nonanticipative strategy $\beta_2 \in S_V(t_2, x_2)$ such that

$$V^\sharp(t_2, x_2) \leq \inf_{u \in \mathcal{U}(t_2, \xi_2)} J(t_2, x_2; u(\cdot), \beta_2(u)(\cdot)) + \varepsilon. \quad (38)$$

372 We consider the nonanticipative map $\Psi : \mathcal{U}([t_1, T], \xi_1) \rightarrow \mathcal{U}([t_2, T], \xi_2)$ provided by Theorem
 373 4.1 applied to the control system $\dot{y} = f_1(t, y, u)$, which with any admissible process (y_1, u_1) on
 374 $[t_1, T]$ such that $y_1(t_1) = \xi_1$ associates a constant K_0 (depending only on r_0) and an admissible
 375 process $(y_2, u_2 = \Psi(u_1))$ such that $y_2(t_2) = \xi_2$, $y_2(t) \in \text{int}(A)$ for all $t \in (t_2, T]$ and

$$\|y_1(\cdot + t_1 - t_2) - y_2(\cdot)\|_{L^\infty(t_2, T_2)} \leq K_0(|\xi_1 - \xi_2| + |t_1 - t_2|). \quad (39)$$

376 We also know that we can restrict our attention to the case in which

$$\begin{aligned} \rho_\xi &:= (1 + \eta(T)) \exp\left(\int_0^T k_f(t) dt\right) (|\xi_1 - \xi_2| + |t_1 - t_2|) \leq \rho_0; \\ \rho_\zeta &:= (1 + \eta(T)) \exp\left(\int_0^T k_f(t) dt\right) (|\zeta_1 - \zeta_2| + |t_1 - t_2|) \leq \rho_0 \end{aligned} \quad (40)$$

377 for some $\rho_0 \geq 0$ and let δ_0 such that $\rho_\xi, \rho_\zeta \leq \delta_0/k_0$. We have also the information that the
 378 control u_2 has the following structure:

$$u_2(t) = \begin{cases} \bar{u}_j(t) & \text{if } t \in [\tau_j, (\tau_j + k_0\rho_\xi) \wedge T_2] \\ u_1(t - k_0\rho_\xi + t_1 - t_2) & \text{if } t \in (\tau_j + k_0\rho_\xi, (\tau_j + \delta_0) \wedge T_2] \end{cases}$$

379 for some control functions $\bar{u}_j : [\tau_j, (\tau_j + k_0\rho_\xi) \wedge T_2] \rightarrow U$, $j = 1, \dots, M$.

380 Theorem 4.1 gives also a nonanticipative map $\Phi : \mathcal{V}([t_2, T], \zeta_2) \rightarrow \mathcal{V}([t_1, T], \zeta_1)$ (related to the
 381 control system $\dot{z} = f_2(t, z, v)$), such that, for any admissible process (z_2, v_2) on $[t_2, T]$ with
 382 $z_2(t_2) = \zeta_2$, we have an admissible process $(z_1, v_1 = \Phi(v_2))$ on $[t_1, T]$ satisfying

$$\|z_2(\cdot + t_2 - t_1) - z_1(\cdot)\|_{L^\infty(t_1, T_1)} \leq K_0(|\zeta_1 - \zeta_2| + |t_1 - t_2|) \quad (41)$$

383 and, again since (40) is in force, we have

$$v_1(t) = \begin{cases} \bar{v}_i(t) & \text{if } t \in [\sigma_i, (\sigma_i + k_0\rho_\zeta) \wedge T_1] \\ v_2(t - k_0\rho_\zeta + t_2 - t_1) & \text{if } t \in (\sigma_i + k_0\rho_\zeta, (\sigma_i + \delta_0) \wedge T_1] \end{cases}$$

384 for some $\bar{v}_i : [\sigma_i, (\sigma_i + k_0\rho_\zeta) \wedge T_1] \rightarrow V$, $i = 1, \dots, N$. Observe that the composition of the nonan-
 385 ticipative maps Φ , β_2 and Ψ provides a nonanticipative strategy $\beta_1 := \Phi \circ \beta_2 \circ \Psi : \mathcal{U}([t_1, T], \xi_1) \rightarrow$
 386 $\mathcal{V}([t_1, T], \zeta_1)$. We emphasize that the strategies Ψ and Φ provided by Theorem 4.1 are such that
 387 also the composition $\Phi \circ \beta_2 \circ \Psi (= \beta_1)$ is nonanticipative and the situation when $t_1 \neq t_2$ does not
 388 generate an issue. Indeed, it is immediate to see that the map β_1 is anticipative when $t_2 \leq t_1$.
 389 So we restrict our attention to the case when $t_1 < t_2$ (that is $T_1 := T + t_1 - t_2 < T$) and $|t_1 - t_2|$
 390 is small enough. Take two admissible controls $u_1, u_2 \in \mathcal{U}([t_1, T]; \xi_1)$ and any $\sigma \in [0, T - t_1]$. If
 391 $t_1 + \sigma \in [t_1, T_1]$ and $u_1 = u_2$ a.e. on $[t_1, t_1 + \sigma]$, then it is immediate to see that $\beta_1(u_1) = \beta_1(u_2)$
 392 a.e. on $[t_1, t_1 + \sigma]$. Suppose now that $t_1 + \sigma \in (T_1, T]$. Then the trajectories \tilde{y}_1 and \tilde{y}_2 associated
 393 respectively with $\beta_1(u_1)$ and $\beta_1(u_2)$ are such that $\tilde{y}_1(T_1) = \tilde{y}_2(T_1)$. Therefore, from Step 6 of
 394 the proof of Theorem 4.1 we know that on $[T_1, T]$ we use either a given (fixed) control u_0 (when
 395 $\tilde{y}_1(T_1) = \tilde{y}_2(T_1) \in (\text{int}A_1) \setminus (\partial A_1 + \frac{\bar{\eta}}{2}\mathbb{B})$) or a particular control \bar{u} constructed in Step 2 of the
 396 proof of Theorem 4.1 (when $\tilde{y}_1(T_1) = \tilde{y}_2(T_1) \in (\text{int}A_1) \cap (\partial A_1 + \frac{\bar{\eta}}{2}\mathbb{B})$). In either case, we have
 397 $\beta_1(u_1) = \beta_1(u_2)$ a.e. also on $[T_1, t_1 + \sigma]$.

398 From the definition of V^\sharp we have $V^\sharp(t_1, x_1) \geq \inf_{u \in \mathcal{U}(t_1, \xi_1)} J(t_1, x_1; u(\cdot), \beta_1(u)(\cdot)) - \varepsilon$ and, there-
 399 fore, there exists a control $\hat{u}_1 \in \mathcal{U}(t_1, \xi_1)$ such that

$$V^\sharp(t_1, x_1) \geq J(t_1, x_1; \hat{u}_1(\cdot), \beta_1(\hat{u}_1)(\cdot)) - \varepsilon. \quad (42)$$

400 Write (\hat{y}_1, \hat{u}_1) and $(\hat{z}_1, \beta_1(\hat{u}_1))$ the associated admissible process such that $\hat{y}_1(t_1) = \xi_1$ and
 401 $\hat{z}_1(t_1) = \zeta_1$. Consider also the admissible control $\hat{u}_2 = \Psi(\hat{u}_1) \in \mathcal{U}(t_2, \xi_2)$. From (38) we deduce
 402 that

$$V^\sharp(t_2, x_2) \leq J(t_2, x_2; \hat{u}_2(\cdot), \beta_2(\hat{u}_2)(\cdot)) + \varepsilon. \quad (43)$$

403 Denote by (\hat{y}_2, \hat{u}_2) and $(\hat{z}_2, \beta_2(\hat{u}_2))$ the corresponding admissible processes with $\hat{x}_2(t_2) :=$
 404 $(\hat{y}_2(t_2), \hat{z}_2(t_2)) = x_2 := (\xi_2, \zeta_2)$. In view of (42) and (43) it follows that

$$\begin{aligned} V^\sharp(t_2, x_2) - V^\sharp(t_1, x_1) &\leq J(t_2, x_2; \hat{u}_2(\cdot), \beta_2(\hat{u}_2)(\cdot)) - J(t_1, x_1; \hat{u}_1(\cdot), \beta_1(\hat{u}_1)(\cdot)) + 2\varepsilon \\ &= \int_{t_2}^T L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) dt - \int_{t_1}^T L(t, \hat{x}_1(t), \hat{u}_1(t), \beta_1(\hat{u}_1)(t)) dt \\ &\quad + g(\hat{x}_2(T)) - g(\hat{x}_1(T)) + 2\varepsilon \\ &= \int_{t_2}^{T_2} L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) dt + \int_{T_2}^T L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) dt \\ &\quad - \int_{t_2}^{T_2} L(t+t_1-t_2, \hat{x}_1(t+t_1-t_2), \hat{u}_1(t+t_1-t_2), \beta_1(\hat{u}_1)(t+t_1-t_2)) dt \\ &\quad - \int_{T_1}^T L(t, \hat{x}_1(t), \hat{u}_1(t), \beta_1(\hat{u}_1)(t)) dt + g(\hat{x}_2(T)) - g(\hat{x}_1(T)) + 2\varepsilon \end{aligned} \quad (44)$$

405 From (39) (resp. (41)) which is valid for \hat{y}_1 and \hat{y}_2 (resp. \hat{z}_1 and \hat{z}_2) we obtain

$$\begin{aligned} \|\hat{y}_1(\cdot + t_1 - t_2) - \hat{y}_2(\cdot)\|_{L^\infty(t_2, T_2)} &\leq K_0 (|\xi_1 - \xi_2| + |t_2 - t_1|). \\ \text{(resp. } \|\hat{z}_2(\cdot + t_2 - t_1) - \hat{z}_1(\cdot)\|_{L^\infty(t_1, T_1)} &\leq K_0 (|\zeta_1 - \zeta_2| + |t_2 - t_1|). \end{aligned} \quad (45)$$

406 In particular we deduce that

$$|(\hat{y}_2(T) - \hat{z}_2(T)) - (\hat{y}_1(T) - \hat{z}_1(T))| \leq \sqrt{2}c_0|t_2 - t_1| + \sqrt{2}K_0 (|\xi_1 - \xi_2| + |\zeta_1 - \zeta_2| + |t_1 - t_2|). \quad (46)$$

407 In addition, since $|L| \leq c_0$ along the reference trajectories, we have

$$\left| \int_{T_2}^T L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) dt - \int_{T_1}^T L(t, \hat{x}_1(t), \hat{u}_1(t), \beta_1(\hat{u}_1)(t)) dt \right| \leq 2c_0|t_2 - t_1|. \quad (47)$$

408 It remains to provide an estimate of the term

$$\begin{aligned} \Delta &:= \left| \int_{t_2}^{T_2} L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) dt \right. \\ &\quad \left. - \int_{t_2}^{T_2} L(t+t_1-t_2, \hat{x}_1(t+t_1-t_2), \hat{u}_1(t+t_1-t_2), \beta_1(\hat{u}_1)(t+t_1-t_2)) dt \right| \\ &\leq \left| \int_{t_2}^{T_2} [L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) - L(t, \hat{x}_1(t+t_1-t_2), \hat{u}_1(t+t_1-t_2), \beta_1(\hat{u}_1)(t+t_1-t_2))] dt \right| \\ &\quad + \int_{t_2}^{T_2} [\eta(t+t_1-t_2) \vee \eta(t) - \eta(t+t_1-t_2) \wedge \eta(t)] dt \\ &\leq \left| \int_{t_2}^{T_2} [L(t, \hat{x}_2(t), \hat{u}_2(t), \beta_2(\hat{u}_2)(t)) - L(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2), \beta_1(\hat{u}_1)(t+t_1-t_2))] dt \right| \\ &\quad + \int_{t_2}^{T_2} |L(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2), \beta_1(\hat{u}_1)(t+t_1-t_2)) \\ &\quad - L(t, \hat{x}_1(t+t_1-t_2), \hat{u}_1(t+t_1-t_2), \beta_1(\hat{u}_1)(t+t_1-t_2))| dt + \eta(T)|t_1 - t_2| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_{t_2}^{T_2} [L(t, \hat{x}_2(t), \Psi(\hat{u}_1)(t), \beta_2(\hat{u}_2)(t)) - L(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2), \beta_2(\hat{u}_2)(t))] dt \right| \\
&+ \left| \int_{t_2}^{T_2} [L(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2), \beta_2(\hat{u}_2)(t)) - L(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2), \Phi(\beta_2(\hat{u}_2))(t+t_1-t_2))] dt \right| \\
&+ \int_{t_2}^{T_2} k_L(t) |(\hat{y}_2(t), \hat{z}_2(t)) - (\hat{y}_2(t+t_1-t_2), \hat{z}_2(t+t_1-t_2))| dt + \eta(T)|t_1-t_2| \\
&\leq \eta(T)|t_1-t_2| + \sqrt{2}K_0K_L (|\xi_1 - \xi_2| + |\zeta_1 - \zeta_2| + |t_1-t_2|) \\
&+ \sum_{j=1}^M \left| \int_{\tau_j}^{(\tau_j+\delta_0)\wedge T_2} L_1(t, \hat{x}_2(t), \Psi(\hat{u}_1)(t)) dt - \int_{\tau_j}^{(\tau_j+\delta_0)\wedge T_2} L_1(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2)) dt \right| \\
&+ \sum_{i=1}^N \left| \int_{\sigma_i}^{(\sigma_i+\delta_0)\wedge T_2} L_2(t, \hat{x}_2(t), \beta_2(\hat{u}_2)(t)) dt - \int_{\sigma_i}^{(\sigma_i+\delta_0)\wedge T_2} L_2(t, \hat{x}_2(t), \Phi(\beta_2(\hat{u}_2))(t+t_1-t_2)) dt \right|
\end{aligned}$$

410 Introducing $K_L := \int_0^T k_L(t)dt$, we have

$$\begin{aligned}
&\left| \int_{\tau_j}^{(\tau_j+\delta_0)\wedge T_2} [L_1(t, \hat{x}_2(t), \Psi(\hat{u}_1)(t)) - L_1(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2))] dt \right| \\
&\leq \left| \int_{\tau_j}^{\tau_j+\delta_0-k_0\rho_\xi} L_1(t+k_0\rho_\xi, \hat{x}_2(t+k_0\rho_\xi), \hat{u}_1(t+t_1-t_2)) dt \right. \\
&\quad \left. - \int_{\tau_j+k_0\rho_\xi}^{\tau_j+\delta_0} L_1(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2)) dt \right| + 2c_0k_0\rho_\xi \\
&\leq \int_{\tau_j}^{\tau_j+\delta_0-k_0\rho_\xi} |L_1(t+k_0\rho_\xi, \hat{x}_2(t+k_0\rho_\xi), \hat{u}_1(t+t_1-t_2)) - L_1(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2))| dt \\
&+ \left| \int_{\tau_j}^{\tau_j+\delta_0-k_0\rho_\xi} L_1(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2)) dt - \int_{\tau_j+k_0\rho_\xi}^{\tau_j+\delta_0} L_1(t, \hat{x}_2(t), \hat{u}_1(t+t_1-t_2)) dt \right| \\
&\quad + 2c_0k_0\rho_\xi \\
&\leq 4c_0k_0\rho_\xi + \sqrt{2}c_0K_Lk_0\rho_\xi + \eta(T)k_0\rho_\xi
\end{aligned}$$

411 and

$$\begin{aligned}
&\left| \int_{\sigma_i}^{(\sigma_i+\delta_0)\wedge T_2} [L_2(t, \hat{x}_2(t), \beta_2(\hat{u}_2)(t)) - L_2(t, \hat{x}_2(t), \Phi(\beta_2(\hat{u}_2))(t+t_1-t_2))] dt \right| \\
&\leq \left| \int_{\sigma_i+k_0\rho_\zeta}^{\sigma_i+\delta_0} L_2(t, \hat{x}_2(t), \beta_2(\hat{u}_2)(t)) - \int_{\sigma_i}^{\sigma_i+\delta_0-k_0\rho_\zeta} L_2(t+k_0\rho_\zeta, \hat{x}_2(t+k_0\rho_\zeta), \beta_2(\hat{u}_2)(t)) dt \right| \\
&\quad + 2c_0k_0\rho_\zeta \\
&\leq \left| \int_{\sigma_i+k_0\rho_\zeta}^{\sigma_i+\delta_0} L_2(t, \hat{x}_2(t), \beta_2(\hat{u}_2)(t)) dt - \int_{\sigma_i}^{\sigma_i+\delta_0-k_0\rho_\zeta} L_2(t, \hat{x}_2(t), \beta_2(\hat{u}_2)(t)) dt \right| \\
&\quad + 2c_0k_0\rho_\zeta \\
&+ \int_{\sigma_i}^{\sigma_i+\delta_0-k_0\rho_\zeta} |L_2(t, \hat{x}_2(t), \beta_2(\hat{u}_2)(t)) - L_2(t+k_0\rho_\zeta, \hat{x}_2(t+k_0\rho_\zeta), \beta_2(\hat{u}_2)(t))| dt \\
&\leq 4c_0k_0\rho_\zeta + \sqrt{2}c_0K_Lk_0\rho_\zeta + \eta(T)k_0\rho_\zeta.
\end{aligned}$$

412 Using (40), we finally obtain the estimate :

$$\Delta \leq \left(\sqrt{2}K_0K_L + 2M_0k_0(4c_0 + \sqrt{2}c_0K_L + \eta(T))k_0(1 + \eta(T))K + \eta(T) \right) \times (|\xi_1 - \xi_2| + |\zeta_1 - \zeta_2| + |t_1 - t_2|)$$

413 with $M_0 \geq M \vee N$ and $K \geq \int_0^T k_f(t)dt$.

Exchanging the role of $V^b(t_1, x_1)$ and $V^b(t_2, x_2)$ in the inequality above, we obtain

$$|V^b(t_1, x_1) - V^b(t_2, x_2)| \leq K^b(|t_1 - t_2| + |x_1 - x_2|),$$

414 for some constant $K^b > 0$, which depends only on the data of the problem, confirming the
415 proposition statement.

416 □

417 6 Solutions of the Hamilton-Jacobi-Isaacs equations

418 **Proposition 6.1 (Dynamic Programming Principle)** *Let $A_1 \subset \mathbb{R}^{n_1}$ and $A_2 \subset \mathbb{R}^{n_2}$ be*
419 *closed nonempty sets. Assume (H1)-(H4), (BV) and (IPC). For any $(t_0, x_0 = (y_0, z_0)) \in$*
420 *$[0, T] \times A_1 \times A_2$ and for all $\sigma \in (0, T - t_0]$ we have*

$$V^b(t_0, x_0) = \inf_{\alpha \in S_U(t_0, x_0)} \sup_{v \in \mathcal{V}(t_0, y_0)} \left\{ \int_{t_0}^{t_0 + \sigma} L(t, x[t_0, x_0; \alpha(v), v](t), \alpha(v)(t), v(t)) dt + \right. \\ \left. + V^b(t_0, x[t_0, x_0; \alpha(v), v](t_0 + \sigma)) \right\}, \quad (48)$$

421 and

$$V^\sharp(t_0, x_0) = \sup_{\beta \in S_V(t_0, x_0)} \inf_{u \in \mathcal{U}(t_0, z_0)} \left\{ \int_{t_0}^{t_0 + \sigma} L(t, x[t_0, x_0; u, \beta(u)](t), u(t), \beta(u)(t)) dt + \right. \\ \left. + V^\sharp(t_0, x[t_0, x_0; u, \beta(u)](t_0 + \sigma)) \right\}. \quad (49)$$

422 Proposition 6.1 can be proved adopting standard arguments already employed for the state
423 constraints free case (cf. [5], [22]). Therefore its proof is omitted.

424 **Theorem 6.2** *Let $A_1 \subset \mathbb{R}^{n_1}$ and $A_2 \subset \mathbb{R}^{n_2}$ be closed nonempty sets. Assume that conditions*
425 *(H1)-(H6), (BV) and (IPC) are satisfied. Then, the lower value function V^b and the upper value*
426 *function V^\sharp are viscosity solution on $[0, T] \times A_1 \times A_2$ of the HJI equation (7).*

427 **Proof.** We show here only that V^b is a viscosity solution of the HJI equation (7). The proof
428 for V^\sharp is similar so we omit it.

429 (i) Recall that $n = n_1 + n_2$. Take any $(t_0, x_0 = (y_0, z_0)) \in [0, T] \times A_1 \times A_2$ and $\varphi \in C^1(\mathbb{R}, \mathbb{R}^n)$
430 such that $V^b - \varphi$ has a local minimum at $(t_0, (y_0, z_0))$ (relative to $[0, T] \times A_1 \times A_2$). We
431 can assume that $\varphi(t_0, (y_0, z_0)) = V^b(t_0, (y_0, z_0))$, and that there exists $r > 0$ such that
432 $V^b(t, (y, z)) \geq \varphi(t, (y, z))$ for all $(t, (y, z)) \in ((t_0, (y_0, z_0)) + r\mathbb{B}) \cap ([0, T] \times A_1 \times A_2)$.
433 Suppose, by contradiction, that there exists $\theta > 0$ such that

$$(H_{in})^*(t_0, (y_0, z_0), \nabla_{y,z}\varphi(t_0, (y_0, z_0))) - \nabla_t\varphi(t_0, (y_0, z_0)) \leq -\theta.$$

From the definition of the upper semicontinuous envelope, we have

$$\begin{aligned}
& (H_{in})^*(t_0, (y_0, z_0), \nabla_{y,z}\varphi(t_0, (y_0, z_0))) \\
& \geq \limsup_{y \xrightarrow{\text{int}A_1} y_0} H_{in}(t_0, (y, z_0), \nabla_{y,z}\varphi(t_0, (y, z_0))) \\
& = \limsup_{y \xrightarrow{\text{int}A_1} y_0} \inf_{(e_2, \ell_2) \in G_2(t_0, (y, z_0))} \sup_{(e_1, \ell_1) \in G_1(t_0, (y, z_0))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2] \quad (50) \\
& = \inf_{(e_2, \ell_2) \in G_2(t_0, (y_0, z_0))} \sup_{(e_1, \ell_1) \in Q_1(t_0, (y_0, z_0))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2].
\end{aligned}$$

where $(p_y, p_z) := \nabla_{y,z}\varphi(t_0, (y_0, z_0))$.

Then we can select $(\tilde{e}_2, \tilde{\ell}_2) \in (\text{co } f_2, L_2)(t_0^+, (y_0, z_0), V)$ such that $\tilde{e}_2 \in \text{int}T_{A_2}(z_0)$

$$\sup_{(e_1, \ell_1) \in Q_1(t_0, (y_0, z_0))} [-e_1 \cdot p_y - \tilde{e}_2 \cdot p_z - \ell_1 - \tilde{\ell}_2] - \nabla_t \varphi(t_0, (y_0, z_0)) \leq -\theta.$$

Using the stability properties of the interior of Clarke tangent cone and the arguments of the proof of Theorem 4.1 (see Remark 4.2), we can find an admissible control $\tilde{v}_0 \in \mathcal{V}(t_0, z_0)$ and $\sigma_0 \in (0, T - t_0)$ such that, for every strategy $\alpha \in S_U(t_0, x_0)$, we have

$$\int_{t_0}^{t_0 + \sigma_0} \left[\mathcal{H}(s, \tilde{x}(s), \nabla_x \varphi(s, \tilde{x}(s)), \alpha(\tilde{v})(s), \tilde{v}(s)) - \nabla_t \varphi(s, \tilde{x}(s)) \right] ds \leq -\frac{\theta}{2},$$

where $\tilde{x}(s) := x[t_0, x_0; \alpha(\tilde{v}), \tilde{v}](s)$.

Now, applying the Dynamic Programming Principle and standard arguments (cf. the proof of [12, Theorem 4.3]) we arrive at a contradiction.

- (ii) Let $(t_0, x_0 = (y_0, z_0))$ be a local maximum (relative to $[0, T] \times A_1 \times A_2$) for $(V^b - \varphi)$, $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ and there exists $r > 0$ such that $V^b(t, (y, z)) \leq \varphi(t, (y, z))$ for all $(t, (y, z)) \in ((t_0, (y_0, z_0)) + r\mathbb{B}) \cap ([0, T] \times A_1 \times A_2)$. Suppose that, by contradiction, there exists $\theta > 0$ such that

$$(H_{in})_*(t_0, (y_0, z_0), \nabla_{y,z}\varphi(t_0, (y_0, z_0))) - \nabla_t \varphi(t_0, (y_0, z_0)) \geq \theta.$$

Then from the definition of the lower semicontinuous envelope

$$\begin{aligned}
& (H_{in})_*(t_0, (y_0, z_0), \nabla_{y,z}\varphi(t_0, (y_0, z_0))) \\
& \leq \liminf_{z \xrightarrow{\text{int}A_2} z_0} H_{in}(t_0, (y_0, z), \nabla_{y,z}\varphi(t_0, (y_0, z))) \\
& = \liminf_{z \xrightarrow{\text{int}A_2} z_0} \inf_{(e_2, \ell_2) \in G_2(t_0, (y_0, z))} \sup_{(e_1, \ell_1) \in G_1(t_0, (y_0, z))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2] \\
& = \inf_{(e_2, \ell_2) \in Q_2(t_0, (y_0, z_0))} \sup_{(e_1, \ell_1) \in G_1(t_0, (y_0, z_0))} [-e_1 \cdot p_y - e_2 \cdot p_z - \ell_1 - \ell_2]
\end{aligned}$$

where $(p_y, p_z) := \nabla_{y,z}\varphi(t_0, (y_0, z_0))$.

It follows that we can choose $(\tilde{e}_1, \tilde{\ell}_1) \in (\text{co } f_1, L_1)(t_0^+, (y_0, z_0), U)$ such that $\tilde{e}_1 \in \text{int}T_{A_1}(y_0)$ and

$$\inf_{(e_2, \ell_2) \in Q_2(t_0, (y_0, z_0))} [-\tilde{e}_1 \cdot p_y - e_2 \cdot p_z - \tilde{\ell}_1 - \ell_2] - \nabla_t \varphi(t_0, (y_0, z_0)) \geq \theta,$$

Remark 4.2 tells us that associated with the initial data (t_0, y_0) and the vector $\tilde{e}_1 \in \text{co } f_1(t_0^+, y_0, U) \cap \text{int}T_{A_1}(y_0)$, we can construct an admissible control $\bar{u}_0 \in \mathcal{U}(t_0, y_0)$ such that an estimate like that one in formula (13) is in force. Consider the (constant) strategy

454 $\alpha_0 \in \mathcal{S}_U(t_0, x_0)$ such that for all $v \in \mathcal{V}(t_0, z_0)$ we have $\alpha_0(v) := \bar{u}_0$. Then α_0 is clearly
 455 nonanticipative. Moreover we can find $\sigma_0 \in (0, T - t_0)$ such that, for all $v \in \mathcal{V}(t_0, z_0)$

$$\int_{t_0}^{t_0+\sigma_0} \left[\mathcal{H}(s, \tilde{x}(s), \nabla_x \varphi(s, \tilde{x}(s)), \alpha_0(v)(s), v(s)) - \nabla_t \varphi(s, \tilde{x}(s)) \right] ds \geq \theta/2,$$

456 where $\tilde{x}(s) := x[t_0, x_0; \alpha_0(v), v](s)$.

457 Invoking again the Dynamic Programming Principle and known arguments (cf. the proof
 458 of [12, Theorem 4.3]) we arrive at a contradiction.

459 □

460 Under the hypotheses (H1)–(H6), (BV) and (IPC), Proposition 5.1 ensures that the lower
 461 value function V^b and the upper value function V^\sharp are locally Lipschitz continuous on $[0, T] \times$
 462 $A_1 \times A_2$. Moreover, Theorem 6.2 establishes that V^b and V^\sharp are viscosity solution of (7) on
 463 $[0, T] \times A_1 \times A_2$. Therefore, we can summarize these results in the following theorem.

464 **Theorem 6.3** *Let $A_1 \subset \mathbb{R}^{n_1}$ and $A_2 \subset \mathbb{R}^{n_2}$ be nonempty closed sets. Assume that conditions*
 465 *(H1)–(H6), (BV) and (IPC) are satisfied. Then*

466 (i) *the lower value function V^b is locally Lipschitz continuous and is a viscosity solution of*
 467 *(7);*

468 (ii) *the upper value function V^\sharp is locally Lipschitz continuous and is a viscosity solution of*
 469 *(7).*

470 Imposing some additional assumptions we obtain the existence and uniqueness of the value
 471 for the reference differential game.

472 **Theorem 6.4** *Assume that (H1)–(H6), (BV) and (IPC) hold true. Suppose in addition that*

473 (i) *$A_1 \subset \mathbb{R}^{n_1}$, $A_2 \subset \mathbb{R}^{n_2}$, $U \subset \mathbb{R}^{m_1}$ and $V \subset \mathbb{R}^{m_2}$ are compact nonempty sets;*

474 (ii) *f_1 is locally Lipschitz continuous w.r.t. (t, y) and continuous in u ; L_1 is continuous;*

475 (iii) *f_2 is locally Lipschitz continuous w.r.t. (t, z) and continuous in v ; L_2 is continuous.*

476 *Then $V := V^b = V^\sharp$ is the unique viscosity solution on $[0, T] \times A_1 \times A_2$ of (7).*

477 **Proof of Theorem 6.4**

478 In this case, since Theorem 6.3 holds, the lower value function V^b is continuous on $[0, T] \times$
 479 $A_1 \times A_2$ and is a subsolution and a supersolution of (7). Observe that condition (iv) of Theorem
 480 7.1 (below) is satisfied as a consequence of the validity of assumption (IPC). Also, from this
 481 theorem, we deduce that for any viscosity solution W of (7), we have $W \leq V^b$ and $V^b \leq W$
 482 on $[0, T] \times A_1 \times A_2$. This proves the uniqueness of the viscosity solution of (7). Since V^\sharp is a
 483 viscosity solution of (7), we obtain that $V^b = V^\sharp$ on $[0, T] \times A_1 \times A_2$ and $V := V^b = V^\sharp$ is the
 484 unique viscosity solution of (7).

485 □

7 A Comparison Result

Theorem 7.1 *Assume that conditions (H1)-(H6) and (BV) are satisfied. Suppose also that*

- (i) $A_1 \subset \mathbb{R}^{n_1}$, $A_2 \subset \mathbb{R}^{n_2}$, $U \subset \mathbb{R}^{m_1}$ and $V \subset \mathbb{R}^{m_2}$ are compact nonempty sets;
- (ii) f_1 is locally Lipschitz continuous w.r.t. (t, y) and continuous in u ; L_1 is continuous;
- (iii) f_2 is locally Lipschitz continuous w.r.t. (t, z) and continuous in v ; L_2 is continuous;
- (iv) for all $y \in \partial A_1$, and $z \in \partial A_2$,

$$\text{int } T_{A_1}(y) \neq \emptyset, \quad \text{int } T_{A_2}(z) \neq \emptyset.$$

Consider two continuous functions $W_1, W_2 : [0, T] \times A_1 \times A_2 \rightarrow \mathbb{R}$ satisfying the following properties

- (v) $W_1(t, (y, z))$ is a viscosity subsolution of the (HJI) equation (7);
- (vi) $W_2(t, (y, z))$ is a viscosity supersolution of the (HJI) equation (7);
- (vii) $W_1(T, \cdot) = W_2(T, \cdot) (= g(\cdot))$ on $A_1 \times A_2$.

Then we obtain:

$$W_1(t, (y, z)) \leq W_2(t, (y, z)), \quad \forall (t, y, z) \in [0, T] \times A_1 \times A_2 .$$

Proof. *Step 1.* Suppose that

$$\max_{(t, (y, z)) \in [0, T] \times A_1 \times A_2} \left\{ W_1(t, (y, z)) - W_2(t, (y, z)) \right\} > 0,$$

then there exists $(t_0, (y_0, z_0)) \in [0, T] \times A_1 \times A_2$ such that

$$W_1(t_0, (y_0, z_0)) - W_2(t_0, (y_0, z_0)) > 0. \tag{51}$$

There exists a constant $M \leq 0$ such that $W_1, W_2 \geq M$ on $[0, T] \times A_1 \times A_2$ and $L - 1 \geq M$ on $[0, T] \times A_1 \times A_2 \times U \times V$. Set $c := 1 - MT (> 0)$ and define $\tilde{W}_1, \tilde{W}_2, \tilde{H}_{in}$ the following functions (obtained by a Kruzkov type transform):

$$\begin{aligned} \tilde{W}_i(s, (y, z)) &:= \frac{1}{1+s} \log \left(W_i(T-s, (y, z)) + M(T-s) - M + c \right), \quad i = 1, 2 \\ \tilde{H}_{in}(s, (y, z), w, p_t, p_y, p_z) &:= \inf_{(e_2, \ell_2) \in G_2(T-s, (y, z))} \sup_{(e_1, \ell_1) \in G_1(T-s, (y, z))} \left[(1+s)p_t \right. \\ &\quad \left. - (1+s)p_y \cdot e_1 - (1+s)p_z \cdot e_2 - (\ell_1 + \ell_2 - M)/e^{(1+s)w} \right]. \end{aligned} \tag{52}$$

Set $s_0 := T - t_0$. Observe that $\tilde{W}_1(0, (y, z)) = \tilde{W}_2(0, (y, z)) = \tilde{g}(y, z)$, with $\tilde{g}(y, z) := \log(g(y, z) - M + 1)$, and

$$\max_{(s, (y, z)) \in [0, T] \times A_1 \times A_2} \left\{ \tilde{W}_1(s, (y, z)) - \tilde{W}_2(s, (y, z)) \right\} \geq \tilde{W}_1(s_0, (y_0, z_0)) - \tilde{W}_2(s_0, (y_0, z_0)) > 0.$$

Since \tilde{W}_1, \tilde{W}_2 are continuous on the compact set $[0, T] \times A_1 \times A_2$ and $\tilde{W}_1(0, (y, z)) = \tilde{W}_2(0, (y, z))$, we can find a point $(\bar{s}, (\bar{y}, \bar{z})) \in (0, T] \times A_1 \times A_2$ such that

$$\max_{(s, (y, z)) \in [0, T] \times A_1 \times A_2} \left\{ \tilde{W}_1(s, (y, z)) - \tilde{W}_2(s, (y, z)) \right\} = \tilde{W}_1(\bar{s}, (\bar{y}, \bar{z})) - \tilde{W}_2(\bar{s}, (\bar{y}, \bar{z})) =: \alpha (> 0).$$

506 **Lemma 7.2** Suppose that the assumptions of Theorem 7.1 are satisfied. Then, \tilde{W}_1 and \tilde{W}_2 are
507 respectively viscosity subsolution and supersolution on $(0, T] \times A_1 \times A_2$ of

$$\begin{cases} W(s, (y, z)) + \tilde{H}_{in}(s, (y, z), W(s, (y, z)), \partial_{s,y,z}W(s, (y, z))) = 0 & \text{on } (0, T) \times (A_1 \times A_2) \\ W(0, (y, z)) = \tilde{g}(y, z) & \text{on } A_1 \times A_2. \end{cases} \quad (53)$$

508 **Proof.** Assume that $(s_0, (y_0, z_0)) \in (0, T] \times A_1 \times A_2$ is a local minimizer for $\tilde{W}_2 - \psi$, $\psi \in \mathcal{C}^1$
509 and $\tilde{W}_2(s, (y, z)) \geq \psi(s, (y, z))$ for all $(s, (y, z))$ in a neighbourhood of $(s_0, (y_0, z_0))$. Then $(t_0 :=$
510 $T - s_0, (y_0, z_0))$ is a local minimizer for $W_2 - \varphi$, where

$$\varphi(t, (y, z)) := e^{(1+T-t)\psi(T-t, (y, z))} - Mt + M - c.$$

511 We know that if W_2 is a supersolution of (7) then

$$H_{in}(T - s_0, (y_0, z_0), p_0) - \nabla_t \varphi(T - s_0, (y_0, z_0)) \geq 0,$$

512 where $p_0 := \nabla_{y,z} \varphi(T - s_0, (y_0, z_0))$, that is

$$\begin{aligned} & \left\{ \inf_{(e_2, \ell_2) \in G_2(t, (y_0, z_0))} [-p_z \cdot e_2 - \ell_2] \right. \\ & \left. + \sup_{(e_1, \ell_1) \in G_1(t, (y_0, z_0))} [-p_y \cdot e_1 - \ell_1] \right\} - \nabla_t \varphi(T - s_0, (y_0, z_0)) \geq 0. \end{aligned}$$

513 We deduce, writing $q_0 := \nabla_{y,z} \psi(s_0, (y_0, z_0))$, that

$$\begin{aligned} & (1 + s_0)e^{(1+s_0)\psi(s_0, (y_0, z_0))} \times \left\{ \inf_{(e_2, \ell_2) \in G_2(T-s, (y_0, z_0))} [-q_z \cdot e_2 - \ell_2] \right. \\ & \left. + \sup_{(e_1, \ell_1) \in G_1(T-s, (y_0, z_0))} [-q_y \cdot e_1 - \ell_1] \right\} \\ & + e^{(1+s_0)\psi(s_0, (y_0, z_0))} [\psi(s_0, (y_0, z_0)) + (1 + s_0)\nabla_t \psi(s_0, (y_0, z_0))] + M \geq 0. \end{aligned}$$

514 It follows that

$$\begin{aligned} & \inf_{(e_2, \ell_2) \in G_2(T-s, (y_0, z_0))} \sup_{(e_1, \ell_1) \in G_1(T-s, (y_0, z_0))} [- (1 + s)q_y \cdot e_1 \\ & - (1 + s)q_z \cdot e_2 - (\ell_1 + \ell_2 - M)/e^{(1+s)\psi(s, (y, z))}] \\ & + \psi(s_0, (y_0, z_0)) + (1 + s_0)\nabla_t \psi(s_0, (y_0, z_0)) \geq 0 \end{aligned}$$

515 and so

$$\tilde{H}_{in}(s_0, (y_0, z_0), \tilde{W}_2(y_0, z_0), \nabla_{s,y,z} \psi(s_0, (y_0, z_0))) \geq 0.$$

516 This confirms that \tilde{W}_2 is a viscosity supersolution of (53). Similar arguments show that \tilde{W}_1 is
517 a viscosity subsolution of (53). This concludes the proof of Lemma 7.2. \square

518

519

520 We continue the proof of the theorem by constructing suitable test functions.

521 *Step 2.* For $s, t \in [0, T]$, $y, y' \in A_1$, $z, z' \in A_2$ we set

$$\begin{aligned} \phi_n(s, t, y, y', z, z') & := \tilde{W}_1(s, (y, z)) - \tilde{W}_2(t, (y', z')) + n^2 \left| y - y' - \frac{1}{n} \xi_1 \right|^2 - n^2 \left| z' - z - \frac{1}{n} \bar{\xi}_2 \right|^2 \\ & - |y' - \bar{y}|^2 - |z - \bar{z}|^2 - |t - \bar{s}|^2 - n^2 |s - t|^2, \end{aligned}$$

522 where $\bar{\xi}_1 \in \text{int}T_{A_1}(\bar{y})$ and $\bar{\xi}_2 \in \text{int}T_{A_2}(\bar{z})$. Then, there exist constants $\delta \in (0, 1)$ and $\eta \in (0, 1)$
 523 such that (cf. [29])

$$y + (0, \delta](\bar{\xi}_1 + \eta\mathbb{B}) \subset \text{int}A_1, \quad \forall y \in (\bar{y} + 2\delta\mathbb{B}) \cap A_1$$

524 and

$$z + (0, \delta](\bar{\xi}_2 + \eta\mathbb{B}) \subset \text{int}A_2, \quad \forall z \in (\bar{z} + 2\delta\mathbb{B}) \cap A_2.$$

525 Since ϕ_n is continuous and $[0, T] \times A_1 \times A_2$ is a compact set, for each $n \geq 1$, there exists a
 526 maximizer $(s_n, t_n, y_n, y'_n, z_n, z'_n)$ for ϕ_n on $([0, T] \times A_1 \times A_2)^2$.

527 We know that

$$\phi_n(s_n, t_n, y_n, y'_n, z_n, z'_n) \geq \phi_n(\bar{s}, \bar{s}, \bar{y} + \frac{1}{n}\bar{\xi}_1, \bar{y}, \bar{z}, \bar{z} + \frac{1}{n}\bar{\xi}_2)$$

528 that is

$$\begin{aligned} & n^2 \left| y_n - y'_n - \frac{1}{n}\bar{\xi}_1 \right|^2 + n^2 \left| z'_n - z_n - \frac{1}{n}\bar{\xi}_2 \right|^2 + n^2 |s_n - t_n|^2 + |y'_n - \bar{y}|^2 + |z_n - \bar{z}|^2 + |t_n - \bar{s}|^2 \\ & \leq \tilde{W}_1(s_n, (y_n, z_n)) - \tilde{W}_2(t_n, (y'_n, z'_n)) - \left(\tilde{W}_1(\bar{s}, (\bar{y} + \frac{1}{n}\bar{\xi}_1, \bar{z})) - \tilde{W}_2(\bar{s}, (\bar{y}, \bar{z} + \frac{1}{n}\bar{\xi}_2)) \right). \end{aligned}$$

529 Thus, using also the fact that $0 \leq \tilde{W}_1(\bar{s}, (\bar{y}, \bar{z})) - \tilde{W}_2(\bar{s}, (\bar{y}, \bar{z})) - (\tilde{W}_1(s_n, (y_n, z_n)) - \tilde{W}_1(s_n, (y_n, z_n)))$
 530 (recall that $(\bar{s}, (\bar{y}, \bar{z}))$ is a maximizer for $\tilde{W}_1 - \tilde{W}_2$ on $[0, T] \times A_1 \times A_2$), we obtain

$$\begin{aligned} & n^2 \left| y_n - y'_n - \frac{1}{n}\bar{\xi}_1 \right|^2 + n^2 \left| z_n - z'_n - \frac{1}{n}\bar{\xi}_2 \right|^2 + n^2 |s_n - t_n|^2 + |y'_n - \bar{y}|^2 + |z_n - \bar{z}|^2 + |t_n - \bar{s}|^2 \\ & \leq \omega(|(t_n - s_n, y_n - y'_n, z_n - z'_n)|) + \omega\left(\left|\frac{1}{n}\bar{\xi}_1\right|\right) + \omega\left(\left|\frac{1}{n}\bar{\xi}_2\right|\right) \\ & \leq C, \end{aligned}$$

531 for some constant $C > 0$. Here, $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the modulus of continuity for the continuous
 532 functions \tilde{W}_1 and \tilde{W}_2 . This yields

- 533 • $|y_n - y'_n - \frac{1}{n}\bar{\xi}_1|, |z'_n - z_n - \frac{1}{n}\bar{\xi}_2|, |s_n - t_n| \leq \frac{\sqrt{C}}{n}$;
- 534 • $|y_n - y'_n| \leq \frac{1}{n}(\sqrt{C} + |\bar{\xi}_1|)$;
- 535 • $|z_n - z'_n| \leq \frac{1}{n}(\sqrt{C} + |\bar{\xi}_2|)$.

Therefore we obtain that $y_n \rightarrow \bar{y}$, $y'_n \rightarrow \bar{y}$, $z_n \rightarrow \bar{z}$, $z'_n \rightarrow \bar{z}$, $s_n \rightarrow \bar{s}$ and $t_n \rightarrow \bar{s}$ as $n \rightarrow +\infty$.
 Moreover, taking $\bar{n} \geq 1/\delta$ large enough, we also have

$$y_n \in \text{int}A_1 \quad \text{and} \quad z'_n \in \text{int}A_2, \quad \text{for all } n \geq \bar{n}.$$

536

537

538 *Step 3.* For each $n \geq \bar{n}$ we consider the maps

$$\begin{aligned} \psi_n^1(s, (y, z)) & := \tilde{W}_2(t_n, (y'_n, z'_n)) + n^2 \left| y - y'_n - \frac{1}{n}\bar{\xi}_1 \right|^2 + n^2 \left| z'_n - z - \frac{1}{n}\bar{\xi}_2 \right|^2 + n^2 |s - t_n|^2 \\ & \quad + |y'_n - \bar{y}|^2 + |z - \bar{z}|^2 + |t_n - \bar{s}|^2, \\ \psi_n^2(t, (y', z')) & := \tilde{W}_1(s_n, (y_n, z_n)) - n^2 \left| y_n - y' - \frac{1}{n}\bar{\xi}_1 \right|^2 - n^2 \left| z' - z_n - \frac{1}{n}\bar{\xi}_2 \right|^2 - n^2 |s_n - t|^2 \\ & \quad - |y' - \bar{y}|^2 + |z_n - \bar{z}|^2 + |t - \bar{s}|^2 \end{aligned}$$

539 Observe that ψ_n^1 and ψ_n^2 are \mathcal{C}^1 , $\tilde{W}_1 - \psi_n^1$ has a (local) maximum at $(s_n, (y_n, z_n))$ relative to
540 $[0, T] \times A_1 \times A_2$ and $\tilde{W}_2 - \psi_n^2$ has a (local) minimum at $(t_n, (y'_n, z'_n))$ relative to $[0, T] \times A_1 \times A_2$.
541 Bearing in mind that \tilde{W}_1 and \tilde{W}_2 are respectively viscosity subsolution and supersolution on
542 $(0, T] \times A_1 \times A_2$ of (53), and using basic properties of the lower and upper semicontinuous
543 envelopes and the regularity assumptions on f_1, f_2, L_1 and L_2 , we have

$$\begin{aligned}
& \tilde{W}_1(s_n, (y_n, z_n)) - \tilde{W}_2(t_n, (y'_n, z'_n)) \\
& \leq \inf_{v \in V} \sup_{u \in U} \left[2(1+t_n)[n^2(s_n - t_n) - (t_n - \bar{s})] \right. \\
& \quad - (1+t_n)[2n^2(y_n - y'_n - \frac{1}{n}\bar{\xi}_1) - (y'_n - \bar{y})] \cdot f_1(T - t_n, y'_n, u) \\
& \quad + (1+t_n)[2n^2(z'_n - z_n - \frac{1}{n}\bar{\xi}_2)] \cdot f_2(T - t_n, z'_n, v) \\
& \quad \left. - \frac{(L_1(T - t_n, (y'_n, z'_n), u) + L_2(T - t_n, (y'_n, z'_n), v) - M)}{e^{(1+t_n)\tilde{W}_2(t_n, (y'_n, z'_n))}} \right] \\
& - \inf_{v \in V} \sup_{u \in U} \left[2(1+s_n)[n^2(s_n - t_n)] \right. \\
& \quad - (1+s_n)[2n^2(y_n - y'_n - \frac{1}{n}\bar{\xi}_1)] \cdot f_1(T - s_n, y_n, u) \\
& \quad + (1+s_n)[2n^2(z'_n - z_n - \frac{1}{n}\bar{\xi}_2) + (z_n - \bar{z})] \cdot f_2(T - s_n, z_n, v) \\
& \quad \left. - \frac{(L_1(T - s_n, (y_n, z_n), u) + L_2(T - s_n, (y_n, z_n), v) - M)}{e^{(1+s_n)\tilde{W}_1(s_n, (y_n, z_n))}} \right].
\end{aligned}$$

544 Taking the limit as $n \rightarrow +\infty$ and extracting a subsequence if necessary (we do not relabel), we
545 deduce that, for some $\bar{u} \in U$ and $\bar{v} \in V$,

$$\begin{aligned}
0 < \frac{\alpha}{2} & \leq \tilde{W}_1(\bar{s}, (\bar{y}, \bar{z})) - \tilde{W}_2(\bar{s}, (\bar{y}, \bar{z})) \\
& \leq \frac{L_1(T - \bar{s}, (\bar{y}, \bar{z}), \bar{u}) + L_2(T - \bar{s}, (\bar{y}, \bar{z}), \bar{v}) - M}{e^{(1+\bar{s})}} \times \left(\frac{1}{e^{\tilde{W}_1(\bar{s}, (\bar{y}, \bar{z}))}} - \frac{1}{e^{\tilde{W}_2(\bar{s}, (\bar{y}, \bar{z}))}} \right).
\end{aligned}$$

546 A contradiction. It follows that $W_1 - W_2 \leq 0$ on $(t, x) \in [0, T] \times A_1 \times A_2$. This confirms the
547 theorem statement.

548 □

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