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1 Polarization coherence frustration

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5 **Abstract:** From the joint analysis of polarization and coherence properties of light, a remarkable
6 concept referred to as polarization coherence frustration is introduced and analyzed. It is shown
7 that two kinds of partially polarized and partially coherent light, with different level of complexity,
8 can be distinguished and that they mathematically correspond to different equivalence classes.
9 On the one hand, light has polarization coherence properties that are not frustrated in a spatial
10 domain \mathcal{D} when there exists a configuration of local polarization devices at each location of
11 the light field that allows the maximization of the modulus of the scalar degree of coherence
12 between any couple of points in \mathcal{D} . Two conditions are shown to hold for light to be polarization
13 coherence unfrustrated and their physical interpretation are analyzed. On the other hand, if one
14 of these conditions is not verified, polarization coherence frustration occurs. These notions
15 are discussed in analogy with well-known concepts of frustration and gauge transformations
16 developed in statistical physics for spin glasses. Their relevance in the field of statistical optics is
17 demonstrated through different theoretical results and examples.

18 1. Introduction

19 Coherence and polarization are fundamental notions in optics for theoretical developments as
20 well as for experimental investigations. The analysis of these properties is more complex for
21 partially polarized light than for totally polarized light [1–10]. On the one hand, coherence is
22 generally related to the ability of light to interfere and, for totally polarized light, the modulus of
23 the scalar degree of coherence is a linear function of the interference fringes visibility [11, 12].
24 On the other hand, polarization properties are related to the amplitude repartition between the
25 different components of the transverse wave [13, 14]. It has already been shown that the analysis
26 of the coherence properties of partially polarized light can lead to remarkable behaviors such
27 as irreversibility [8, 9] or hidden symmetry [7] that are less straightforward than for totally
28 polarized light. We show in this paper that the analysis of coherence properties of partially
29 polarized light with the standard scalar degree of coherence when the interfering fields have
30 been polarized also leads to remarkable properties. More precisely, the coherence properties
31 of the field at two locations are analyzed looking at the maximal visibility of the interference
32 fringes when these fields are polarized with polarization devices. These polarization devices
33 are thus optimized in order to maximize the modulus of the scalar degree of coherence between
34 the resulting polarized fields. This analysis shows that, in a spatial domain \mathcal{D} , there exists two
35 different classes of lights: polarization coherence frustrated or unfrustrated lights. In that latter
36 case of polarization coherence unfrustrated lights, there exists a polarization device configuration
37 at each location in \mathcal{D} that maximizes the modulus of the scalar degree of coherence between fields
38 observed at any pairs of locations in \mathcal{D} . In other words, it is possible for unfrustrated lights to
39 exhibit a spatial distribution of local polarization transformations that will allow the interference
40 between any pair of locations in \mathcal{D} to be maximized simultaneously. Such a property does not
41 hold for polarization coherence frustrated lights that are introduced in this article. Furthermore,
42 we demonstrate that these properties (polarization coherence frustration or unfrustration) are
43 invariant by local non-singular polarization transformations described by deterministic Jones
44 matrices. This invariance property allows one to show that frustrated and unfrustrated lights
45 define two different mathematical classes of equivalence.

46 Polarization coherence frustration and the invariance of this property by the action of local

47 non-singular deterministic Jones matrices present interesting analogies with the concepts of
 48 frustration and gauge transformation in spin glasses [15, 16]. These concepts have been essential
 49 to analyze complex systems [15, 16] in statistical physics. Frustration is the impossibility
 50 to minimize simultaneously the energy of interaction of each couple in a set of interacting
 51 spins [15, 16]. Gauge invariance is related to the invariance of the property of frustration (or
 52 of unfrustration) by local transformations of the local spin basis. Frustration of spin glasses
 53 is an essential characteristic to observe the complex behavior of these systems. Indeed, in the
 54 absence of frustration, disordered magnetic systems have simpler statistical properties than
 55 for frustrated disordered magnetic systems [15, 16]. It can thus be expected that coherence
 56 polarization frustration is an important notion that separates partially coherent lights in a spatial
 57 domain \mathcal{D} into two different classes. Frustrated ones, that may exhibit complex coherence
 58 properties, and unfrustrated ones that have simpler characteristics. It is shown in the following
 59 that this is indeed the case.

60 This article is organized as follows. After background definitions and a description of the
 61 action of polarization devices that maximize the modulus of the scalar degree of coherence, the
 62 notion of polarization coherence frustration is introduced. Then, the property of gauge invariance,
 63 that allows one to define the two classes of equivalence, is discussed. General conditions that
 64 are satisfied by unfrustrated lights are then described in Section 5, followed by their physical
 65 interpretation that such lights can be decomposed in a spatial domain as the sum of two totally
 66 polarized lights incoherent between them with specific conditions on their coherence properties.
 67 Several examples are finally analyzed before the conclusion of this paper.

68 2. Background

69 In the following, we shall consider transverse light fields, i.e. their electrical fields will be
 70 assumed to fluctuate in 2D. Furthermore, they will be analyzed in the space-frequency domain
 71 although an analogous analysis could be done in the space-time domain. Thus at frequency
 72 ν and at location \mathbf{r} , the field will be simply written $\overline{\mathbf{E}}(\mathbf{r})^T = (E_x(\mathbf{r}), E_y(\mathbf{r}))$ where $\overline{\mathbf{E}}^T$ is the
 73 transpose of $\overline{\mathbf{E}}$ and where $E_x(\mathbf{r}), E_y(\mathbf{r})$ are complex scalar values (i.e. in \mathbb{C}). One over-line will
 74 denote vectors and two over-lines will denote matrices, although vectors of spatial location will
 75 be noted with bold font for clarity reasons. Since the fields will be assumed complex circular
 76 and wide-sense stationary [13, 14], the coherence properties can be analyzed with the 2×2
 77 cross-spectral density matrix (CSDM) which is defined by:

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = \langle \overline{\mathbf{E}}^*(\mathbf{r}_1) \overline{\mathbf{E}}^T(\mathbf{r}_2) \rangle \quad (1)$$

78 where statistical (or ensemble) average is noted $\langle \rangle$, and where \overline{a}^* is the complex conjugate of \overline{a} .
 79 The polarization matrix is defined by: $\overline{\overline{\Gamma}}(\mathbf{r}) = \langle \overline{\mathbf{E}}^*(\mathbf{r}) \overline{\mathbf{E}}^T(\mathbf{r}) \rangle = \overline{\overline{\Omega}}(\mathbf{r}, \mathbf{r})$. In the following $\overline{\overline{\Gamma}}(\mathbf{r})$
 80 will be assumed non singular, which corresponds to the case of non totally polarized light. The
 81 particular case of totally polarized light will be discussed specially at the end of Section 4.

82 In a standard interference experiment, in order to maximize the visibility of the interference
 83 fringes, it is useful and usual to adjust the polarization characteristics and the intensities of
 84 the interfering fields. Experimentally, such modification of the polarization characteristics can
 85 be obtained with a birefringent medium, followed by a polarizer, an intensity attenuator and a
 86 polarization rotator (see Fig. 1). The set of optical components performing such action will be
 87 referred to as the "polarization device" in the following. Let us show that the obtained polarized
 88 field can then be represented by the scalar field $a(\mathbf{r}_j)$ which can be written $a(\mathbf{r}_j) = \overline{\mathbf{p}}^T(\mathbf{r}_j) \overline{\mathbf{E}}(\mathbf{r}_j)$
 89 where the action of the polarization device is mathematically described by the 2D complex
 90 vector $\overline{\mathbf{p}}(\mathbf{r}_j)$ at location \mathbf{r}_j . Indeed, the action of a birefringent medium (in the general sense)
 91 can be described by a unitary matrix $\overline{\overline{U}}(\mathbf{r})$. A linear polarizer can be represented by a matrix

92 $\bar{p}_R(\mathbf{r}) \bar{p}_R^T(\mathbf{r})$ where $\bar{p}_R(\mathbf{r})$ is a two-dimensional real vector. Thus the action of the birefringent
 93 medium followed by the polarizer transforms $\bar{E}(\mathbf{r})$ into $a(\mathbf{r}) \bar{p}_R(\mathbf{r})$ with $a(\mathbf{r}) = \bar{p}^T(\mathbf{r}) \bar{E}(\mathbf{r})$
 94 and with $\bar{p}(\mathbf{r}) = \bar{U}^T(\mathbf{r}) \bar{p}_R(\mathbf{r})$. The vector $\bar{p}(\mathbf{r})$ can represent any complex vector since $\bar{U}^T(\mathbf{r})$
 95 can be any unitary matrix. The action of a polarization rotator at location \mathbf{r}_2 allows one to get
 96 fields with parallel polarizations at locations \mathbf{r}_1 and \mathbf{r}_2 , whose interference properties can be
 97 described by those of the scalar fields $a(\mathbf{r}_1)$ and $a(\mathbf{r}_2)$. The attenuators are necessary to obtain
 98 the maximal visibility of the interference fringes. It can be noted that other combination of
 99 optical components could lead to equivalent results.

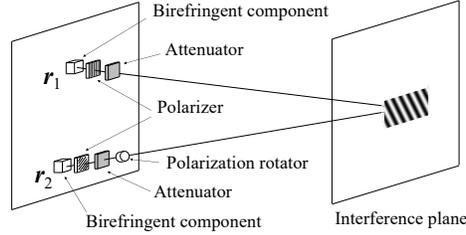


Fig. 1. Schematic representation of the considered optimization of the standard degree of coherence. The visibility of the interference fringes is directly related to the standard scalar degree of coherence.

100 The standard scalar degree of coherence $\eta(\mathbf{r}_1, \mathbf{r}_2)$ between the fields $a(\mathbf{r}_1)$ and $a(\mathbf{r}_2)$ is defined
 101 by:

$$\eta(\mathbf{r}_1, \mathbf{r}_2) = \frac{\langle a^*(\mathbf{r}_1) a(\mathbf{r}_2) \rangle}{\sqrt{\langle |a(\mathbf{r}_1)|^2 \rangle \langle |a(\mathbf{r}_2)|^2 \rangle}} \quad (2)$$

102 Then:

$$|\eta(\mathbf{r}_1, \mathbf{r}_2)|^2 = \frac{|\bar{p}^\dagger(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) \bar{p}(\mathbf{r}_2)|^2}{\bar{p}^\dagger(\mathbf{r}_1) \bar{\Gamma}(\mathbf{r}_1) \bar{p}(\mathbf{r}_1) \bar{p}^\dagger(\mathbf{r}_2) \bar{\Gamma}(\mathbf{r}_2) \bar{p}(\mathbf{r}_2)} \quad (3)$$

103 where $|x|$ is the modulus of x and where \bar{p}^\dagger is the conjugate transpose of \bar{p} .

104 The polarization devices that maximize $|\eta(\mathbf{r}_1, \mathbf{r}_2)|^2$ can be rewritten in a different form. For
 105 that purpose, let us introduce $\bar{u}(\mathbf{r}_j) = \bar{v}(\mathbf{r}_j) / \sqrt{\bar{v}^\dagger(\mathbf{r}_j) \bar{v}(\mathbf{r}_j)}$ where $\bar{v}(\mathbf{r}_j) = \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_j) \bar{p}(\mathbf{r}_j)$. The
 106 matrix $\bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_j)$ is obtained by replacing the eigenvalues λ_i of $\bar{\Gamma}(\mathbf{r}_j)$ by $\sqrt{\lambda_i}$ in the eigenvalue
 107 decomposition of this matrix. Thus:

$$|\eta(\mathbf{r}_1, \mathbf{r}_2)|^2 = |\bar{u}^\dagger(\mathbf{r}_1) \bar{M}(\mathbf{r}_1, \mathbf{r}_2) \bar{u}(\mathbf{r}_2)|^2 \quad (4)$$

108 with $\bar{u}^\dagger(\mathbf{r}) \bar{u}(\mathbf{r}) = 1$ and where:

$$\bar{M}(\mathbf{r}_1, \mathbf{r}_2) = \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_2) \quad (5)$$

109 is the normalized CSDM [17] also considered in the time domain in [3]. The matrix $\bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_j)$
 110 is the inverse matrix of $\bar{\Gamma}^{\frac{1}{2}}(\mathbf{r}_j)$. Thus, $\bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_j)$ is obtained by replacing the eigenvalues λ_i of
 111 $\bar{\Gamma}(\mathbf{r}_j)$ by $1/\sqrt{\lambda_i}$ in the eigenvalue decomposition of $\bar{\Gamma}(\mathbf{r}_j)$.

112 As a result, to maximize the visibility of the interference fringes, the optimal polarization
 113 configurations at locations \mathbf{r}_1 and \mathbf{r}_2 are thus obtained with the unit norm vectors $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_2)$
 114 and $\bar{u}_2^{opt}(\mathbf{r}_1, \mathbf{r}_2)$ that maximize:

$$C(\bar{u}_1, \bar{u}_2) = |\bar{u}_1^\dagger \bar{M}(\mathbf{r}_1, \mathbf{r}_2) \bar{u}_2| \quad (6)$$

115 with the constraint $\|\bar{u}_1\| = \|\bar{u}_2\| = 1$. The optimal polarization devices are thus $\bar{p}_j^{opt}(\mathbf{r}_1, \mathbf{r}_2) =$
 116 $c_j \bar{\Gamma}(\mathbf{r}_j)^{-\frac{1}{2}} \bar{u}_j^{opt}(\mathbf{r}_1, \mathbf{r}_2)$, where c_j are scalar normalization constants that can be introduced so
 117 that the polarization devices do not lead to an energy increase (i.e. they correspond to passive
 118 components).

119 3. Polarization coherence frustration

120 In general, $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_2)$ is not only a function of \mathbf{r}_1 but also of \mathbf{r}_2 . In that case, at location \mathbf{r}_1 , the
 121 optimal polarization device described by $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_2)$ for the couple of fields $\bar{E}(\mathbf{r}_1)$ and $\bar{E}(\mathbf{r}_2)$
 122 can be different from the polarization device described by $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_3)$ for the couple of fields
 123 $\bar{E}(\mathbf{r}_1)$ and $\bar{E}(\mathbf{r}_3)$ when $\mathbf{r}_3 \neq \mathbf{r}_2$.

124 Let us analyze simple examples before going into more details. Let us first assume that the
 125 field at three locations \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 can be written:

$$\bar{E}(\mathbf{r}_1) = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \bar{E}(\mathbf{r}_2) = \begin{pmatrix} \epsilon_1 \\ \epsilon_3 \end{pmatrix}, \bar{E}(\mathbf{r}_3) = \begin{pmatrix} \epsilon_3 \\ \epsilon_2 \end{pmatrix} \quad (7)$$

126 where ϵ_j are statistically independent random complex circular variables with $\langle \epsilon_i^* \epsilon_j \rangle = \delta_{ij}$
 127 (where δ_{ij} is the Kronecker symbol, i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ otherwise). It can be noted
 128 that circularity imposes $\langle \epsilon_j \rangle = 0$. In other words, the ϵ_j are zero mean independent and unit
 129 variance random variables. Such a configuration is schematically shown in Fig. 2.a.

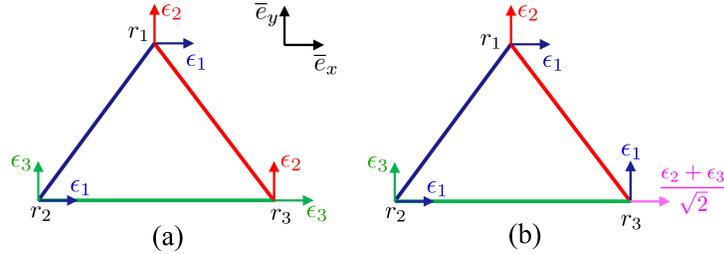


Fig. 2. Simple examples of the analysis of polarization coherence frustration between three locations. (a): example of polarization coherence frustration. (b): example of polarization coherence unfrustration. \bar{e}_x and \bar{e}_y are respectively the unit vectors of coordinates $(1, 0)^T$ and $(0, 1)^T$.

130 The optimal polarization configuration that maximizes the modulus of the scalar degree
 131 of coherence (or equivalently the visibility of interference fringes) between $\bar{E}(\mathbf{r}_1)$ and $\bar{E}(\mathbf{r}_2)$
 132 obviously corresponds to filter the fields with linear polarizers aligned along the horizontal x axis
 133 (i.e. parallel to $\bar{e}_x = (1, 0)^T$). However, the optimal polarization configuration that maximizes
 134 the modulus of the scalar degree of coherence between $\bar{E}(\mathbf{r}_1)$ and $\bar{E}(\mathbf{r}_3)$ corresponds to filter the
 135 fields with linear polarizers aligned along the vertical y axis (i.e. parallel to $\bar{e}_y = (0, 1)^T$). And
 136 finally, the optimal polarization configuration that maximizes the modulus of the scalar degree of

137 coherence between $\bar{E}(\mathbf{r}_2)$ and $\bar{E}(\mathbf{r}_3)$ corresponds to filter $\bar{E}(\mathbf{r}_2)$ with a linear polarizer aligned
 138 along the y axis and to filter $\bar{E}(\mathbf{r}_3)$ with a linear polarizer aligned along the x axis. In that latter
 139 case, a $\pi/2$ polarization rotation applied to one field is necessary to observe the interference
 fringes with maximal visibility. These results are illustrated in Fig. 3 (a), (b) and (c).

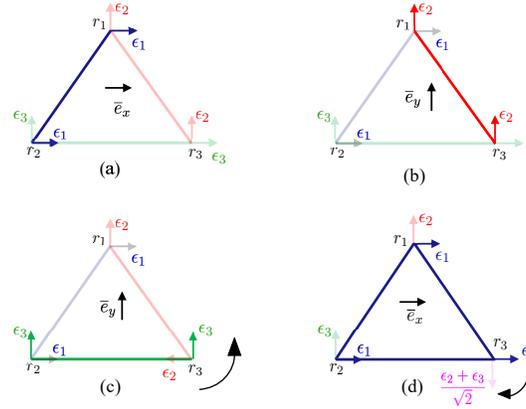


Fig. 3. Analysis of polarization coherence frustration or unfrustration between three locations of the examples of Fig. 2.a and Fig. 2.b. The $\pi/2$ arrow represents a $\pi/2$ rotation of the local polarization state (bottom right). The polarization state shown in black at the center of the triangles shows the polarization components that are made to interfere by the action of the polarization device. High contrast vertices represent maximal visibility of interference fringes with the considered polarization states. (a,b,c): frustrated case of Fig. 2.a. (d): unfrustrated case of Fig. 2.b.

140
 141 It thus appears that there does not exist a configuration for the polarization devices at each
 142 location that allows one to maximize the modulus of the scalar degree of coherence between
 143 $\bar{E}(\mathbf{r}_1)$ and $\bar{E}(\mathbf{r}_2)$, between $\bar{E}(\mathbf{r}_2)$ and $\bar{E}(\mathbf{r}_3)$ and between $\bar{E}(\mathbf{r}_1)$ and $\bar{E}(\mathbf{r}_3)$ at the same time.
 144 Such a situation leads to incompatibility between the optimal polarization configurations that
 145 maximize the modulus of the scalar degree of coherence between the fields at locations \mathbf{r}_1 and \mathbf{r}_2
 146 or at \mathbf{r}_1 and \mathbf{r}_3 or at \mathbf{r}_2 and \mathbf{r}_3 .

147 As discussed below, this situation will be referred to as "polarization coherence frustration".
 148 The situation is clearly different with (see Fig. 2.b):

$$\bar{E}(\mathbf{r}_1) = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \bar{E}(\mathbf{r}_2) = \begin{pmatrix} \epsilon_1 \\ \epsilon_3 \end{pmatrix}, \bar{E}(\mathbf{r}_3) = \begin{pmatrix} \frac{\epsilon_2 + \epsilon_3}{\sqrt{2}} \\ \epsilon_1 \end{pmatrix} \quad (8)$$

149 In that case, all scalar degrees of coherence are maximized with linear polarizers aligned along the
 150 x axis for locations \mathbf{r}_1 and \mathbf{r}_2 , and along the y axis for location \mathbf{r}_3 (followed by a $\pi/2$ polarization
 151 rotation). These results are illustrated in Fig. 3.d. As discussed below, this situation will be
 152 referred to as "polarization coherence unfrustration".

153 Such examples illustrate the case of unfrustration or frustration situations between three
 154 locations. Of course, the two above examples, as well as the concepts of frustration and of
 155 unfrustration, can be extended to the general situation of a light with continuous locations in
 156 a given spatial domain \mathcal{D} . The two above situations can be discriminated with the concept of
 157 polarization coherence frustration which we specify now. On the one hand, a light is referred to

158 as polarization coherence unfrustrated in a spatial domain \mathcal{D} if there exist a set of polarization
 159 devices that polarize the light such that, for any \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} , the polarization device at
 160 location \mathbf{r}_1 that maximizes $|\eta(\mathbf{r}_1, \mathbf{r}_2)|$ is not a function of \mathbf{r}_2 . In other words, for polarization
 161 coherence unfrustrated light, there exists a local polarization transformation that polarizes the
 162 light at each location that is optimal between any couple of locations in \mathcal{D} to ensure interference
 163 with maximum fringes visibility. On the other hand, if such a set of polarization transformations
 164 that polarize the light (such that for any \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} the polarization device at location \mathbf{r}_1
 165 that maximizes $|\eta(\mathbf{r}_1, \mathbf{r}_2)|$ is not a function of \mathbf{r}_2) does not exist, the light will be referred to as
 166 polarization coherence frustrated in \mathcal{D} .

167 Going back to the above examples, it has been shown for the example of Fig. 2.a that, at location
 168 \mathbf{r}_1 , $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_2) = \bar{e}_x$ while $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_3) = \bar{e}_y$, which shows polarization coherence frustration.
 169 However, for the example of Fig. 2.b, at location \mathbf{r}_1 , $\bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_2) = \bar{u}_1^{opt}(\mathbf{r}_1, \mathbf{r}_3) = \bar{e}_x$ which
 170 does not exhibit polarization coherence frustration.

171 4. Gauge invariance

172 4.1. General notion

173 A polarization coherence unfrustrated light in a spatial domain \mathcal{D} remains polarization coherence
 174 unfrustrated in \mathcal{D} after local linear transformation described by the action of local deterministic
 175 non-singular Jones matrices $\bar{J}(\mathbf{r})$. Thus, if $\bar{E}(\mathbf{r})$ is a polarization coherence unfrustrated light
 176 in \mathcal{D} , then $\bar{A}(\mathbf{r}) = \bar{J}(\mathbf{r}) \bar{E}(\mathbf{r})$ is also polarization coherence unfrustrated in \mathcal{D} for any set of
 177 non singular deterministic Jones matrices $\bar{J}(\mathbf{r})$ functions of \mathbf{r} . Indeed, if $\bar{E}(\mathbf{r})$ is polarization
 178 coherence unfrustrated, then there exists a vector function $\bar{u}^{opt}(\mathbf{r})$ of the spatial coordinates \mathbf{r}
 179 such that $|\bar{u}_1^\dagger \bar{M}(\mathbf{r}_1, \mathbf{r}_2) \bar{u}_2|$ is maximized with $\bar{u}_1 = \bar{u}^{opt}(\mathbf{r}_1)$ and $\bar{u}_2 = \bar{u}^{opt}(\mathbf{r}_2)$ for any \mathbf{r}_1 and
 180 \mathbf{r}_2 in \mathcal{D} . Let $\bar{p}^{opt}(\mathbf{r})$ denote the corresponding optimal polarization devices. Then, from Eq.
 181 (3):

$$\frac{|\bar{p}^\dagger(\mathbf{r}_1) \bar{\Omega}_{EE}(\mathbf{r}_1, \mathbf{r}_2) \bar{p}(\mathbf{r}_2)|^2}{\bar{p}^\dagger(\mathbf{r}_1) \bar{\Gamma}_{EE}(\mathbf{r}_1) \bar{p}(\mathbf{r}_1) \bar{p}^\dagger(\mathbf{r}_2) \bar{\Gamma}_{EE}(\mathbf{r}_2) \bar{p}(\mathbf{r}_2)} \quad (9)$$

182 is maximized with $\bar{p}^{opt}(\mathbf{r}_1)$ and $\bar{p}^{opt}(\mathbf{r}_2)$, where $\bar{\Omega}_{EE}(\mathbf{r}_1, \mathbf{r}_2)$ and $\bar{\Gamma}_{EE}(\mathbf{r})$ are respectively the
 183 CSDM and the polarization matrices of $\bar{E}(\mathbf{r})$. But the CSDM and the polarization matrices of $\bar{A}(\mathbf{r})$
 184 are respectively $\bar{\Omega}_{AA}(\mathbf{r}_1, \mathbf{r}_2) = \bar{J}^*(\mathbf{r}_1) \bar{\Omega}_{EE}(\mathbf{r}_1, \mathbf{r}_2) \bar{J}^T(\mathbf{r}_2)$ and $\bar{\Gamma}_{AA}(\mathbf{r}) = \bar{J}^*(\mathbf{r}) \bar{\Gamma}_{EE}(\mathbf{r}) \bar{J}^T(\mathbf{r})$.
 185 Let us introduce $\bar{q}^{opt}(\mathbf{r}) = \left[\bar{J}^T(\mathbf{r}) \right]^{-1} \bar{p}^{opt}(\mathbf{r})$, then:

$$\frac{|\bar{q}^\dagger(\mathbf{r}_1) \bar{\Omega}_{AA}(\mathbf{r}_1, \mathbf{r}_2) \bar{q}(\mathbf{r}_2)|^2}{\bar{q}^\dagger(\mathbf{r}_1) \bar{\Gamma}_{AA}(\mathbf{r}_1) \bar{q}(\mathbf{r}_1) \bar{q}^\dagger(\mathbf{r}_2) \bar{\Gamma}_{AA}(\mathbf{r}_2) \bar{q}(\mathbf{r}_2)} \quad (10)$$

186 is maximized with $\bar{q}^{opt}(\mathbf{r}_1)$ and $\bar{q}^{opt}(\mathbf{r}_2)$ where $\bar{q}^{opt}(\mathbf{r}_1)$ (respectively $\bar{q}^{opt}(\mathbf{r}_2)$) is a function
 187 of only \mathbf{r}_1 (respectively of only \mathbf{r}_2), which shows that $\bar{A}(\mathbf{r})$ is also a polarization coherence
 188 unfrustrated light in \mathcal{D} .

189 A corollary of this property is that polarization coherence frustrated lights in a domain \mathcal{D} remain
 190 polarization coherence frustrated in \mathcal{D} by transformations of the form $\bar{A}(\mathbf{r}) = \bar{J}(\mathbf{r}) \bar{E}(\mathbf{r})$ where
 191 $\bar{J}(\mathbf{r})$ are deterministic non-singular Jones matrices. To show this corollary, let us assume that there
 192 exists an invertible $\bar{J}(\mathbf{r})$ that transforms a polarization coherence frustrated field into a polarization
 193 coherence unfrustrated field. That would be in contradiction with the previous property that
 194 polarization coherence unfrustrated fields remains polarization coherence unfrustrated with the

195 action of local deterministic Jones matrices. Indeed, $\overline{\overline{J}}^{-1}(\mathbf{r})$ are local deterministic non-singular
 196 Jones matrices that would transform a polarization coherence unfrustrated field into a polarization
 197 coherence frustrated field.

198 For the sake of simplicity, polarization coherence frustration of light in a spatial domain \mathcal{D}
 199 will be simply referred to as frustration in the following and transformations described by local
 200 non-singular deterministic Jones matrices will be simply referred to as gauge transformations.

201 Let us illustrate the notion of gauge invariance with the two above examples of Section 3 and
 202 Fig. 2. For the situation of Fig. 2.b, it is possible to apply at location \mathbf{r}_3 the transformation
 203 $\overline{E}(\mathbf{r}_3) \rightarrow \overline{E}'(\mathbf{r}_3) = \overline{\overline{J}}_3 \overline{E}(\mathbf{r}_3)$ with:

$$\overline{\overline{J}}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (11)$$

204 and then $\overline{E}'(\mathbf{r}_3) = \epsilon_1 \overline{e}_x + \frac{\epsilon_2 + \epsilon_3}{\sqrt{2}} \overline{e}_y$. It clearly appears that the scalar fields $\overline{e}_x^T \overline{E}(\mathbf{r}_1)$, $\overline{e}_x^T \overline{E}(\mathbf{r}_2)$
 205 and $\overline{e}_x^T \overline{E}'(\mathbf{r}_3)$ are fully coherent which shows that the situation is unfrustrated. However, such a
 206 gauge transformation that would allow one to get an unfrustrated situation does not exist for the
 207 situation of Fig. 2.a, which is a consequence of the frustration.

208 4.2. Particular case of totally polarized light

209 The simple case of totally polarized light is also enlightening. Let us indeed assume that the
 210 field can be written $\overline{E}(\mathbf{r}) = \epsilon(\mathbf{r}) \overline{u}(\mathbf{r})$ in a spatial domain \mathcal{D} , where $\epsilon(\mathbf{r})$ is a circular complex
 211 random field (i.e., a circular complex random variable at each \mathbf{r}) and where $\overline{u}(\mathbf{r})$ is a deterministic
 212 spatial function with values in \mathbb{C}^2 . Thus, the scalar field $a(\mathbf{r}) = \overline{p}(\mathbf{r})^T \overline{E}(\mathbf{r})$ can be written
 213 $a(\mathbf{r}) = \alpha(\mathbf{r}) \epsilon(\mathbf{r})$ with $\alpha(\mathbf{r}) = \overline{p}(\mathbf{r})^T \overline{u}(\mathbf{r})$. Since $\alpha(\mathbf{r})$ is a deterministic quantity, the modulus
 214 of the scalar degree of coherence between $a(\mathbf{r}_1)$ and $a(\mathbf{r}_2)$ is thus independent of $\overline{p}(\mathbf{r}_1)$ and
 215 $\overline{p}(\mathbf{r}_2)$ as long as $\overline{p}(\mathbf{r}_i)$ is not orthogonal to $\overline{E}(\mathbf{r}_i)$ for $i = 1, 2$. Indeed, in that latter case, the
 216 scalar degree of coherence is undefined. This analysis shows that totally polarized lights in a
 217 spatial domain \mathcal{D} are unfrustrated since it is easy to design local polarization transformations
 218 that will align the polarization direction at all locations, and hence ensure maximum visibility of
 219 the interference fringes. As a result, polarization coherence frustration is a characteristic specific
 220 to partially polarized light, i.e., it can exist only for partially polarized lights.

221 An example of such unfrustrated light does not only correspond to uniformly totally polarized
 222 lights but also to more complex lights such as Full Poincaré beams [18], which have a spatial
 223 distribution of totally polarized states across the beam that maps the entire surface of the Poincaré
 224 sphere. In particular, in that latter case, it is possible to define a local transformation that
 225 corresponds to the Jones matrix $\overline{\overline{J}}(\mathbf{r}) = \overline{e}_x \overline{\beta}(\mathbf{r})^\dagger$, where $\overline{\beta}(\mathbf{r})$ is the local polarization vector at
 226 location \mathbf{r} , so that Full Poincaré light becomes a fully polarized and coherent field with a linear
 227 horizontal polarization. Another example corresponds to a speckle field obtained by enlightening
 228 a surface with a totally polarized and coherent field $\overline{E}(\mathbf{r})$. If the speckle field can be written
 229 $\overline{A}(\mathbf{r}) = \overline{\overline{J}}(\mathbf{r}) \overline{E}(\mathbf{r})$, with $\overline{\overline{J}}(\mathbf{r})$ deterministic, then $\overline{A}(\mathbf{r})$ is unfrustrated.

230 In the next section, a general expression for unfrustrated light is determined for non totally
 231 polarized light allowing us to exhibit general conditions that must hold to ensure unfrustration.
 232 The relevance of frustration and of the gauge transformation will then be illustrated with several
 233 simple examples in Section 7.

234 5. General conditions of unfrustration

235 The general expression of the singular value decomposition of the normalized CSDM (defined
 236 in Eq. (5)) is $\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{U}}^\dagger(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{\Lambda}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)$ where $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2)$ and $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)$ are

237 unitary matrices that can be function of both \mathbf{r}_1 and \mathbf{r}_2 and where $\overline{\overline{\Lambda}}(\mathbf{r}_1, \mathbf{r}_2)$ is a diagonal matrix
 238 with non negative values. We show in Appendix A that for light to be unfrustrated in a spatial
 239 domain \mathcal{D} , the unitary matrices $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2)$ and $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)$ must have particular properties. Indeed,
 240 for unfrustrated light, at any locations \mathbf{r}_1 and \mathbf{r}_2 , the normalized CSDM (defined in Eq. (5)) can
 241 be written (see Appendix A):

$$\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{N}}(\mathbf{r}_1) \overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}^\dagger(\mathbf{r}_2) \quad (12)$$

242 where:

$$\overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} \tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2) & 0 \\ 0 & \tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2) \end{pmatrix} \quad (13)$$

243 with the properties that in \mathcal{D} :

- 244
 245 (i) $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)| \geq |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$ for any \mathbf{r}_1 and \mathbf{r}_2 , and
 246
 247 (ii) the unitary matrix $\overline{\overline{N}}(\mathbf{r}_1)$ has to be a function of only \mathbf{r}_1 and the unitary matrix $\overline{\overline{N}}(\mathbf{r}_2)$
 248 has to be a function of only \mathbf{r}_2 .

249 Furthermore, in the above equations, $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)|$ and $|\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$ are equal to the intrinsic
 250 degrees of coherence [3], but defined in the spectral domain, and are the diagonal elements of
 251 $\overline{\overline{\Lambda}}(\mathbf{r}_1, \mathbf{r}_2)$.

252 The normalized CSDM of an unfrustrated light thus has to satisfy two conditions. The first
 253 condition is that the diagonal matrix $\overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2)$ must satisfy $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)| \geq |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$ for any
 254 \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} . The second condition is that the unitary matrix $\overline{\overline{N}}(\mathbf{r})$ must be a function of only
 255 the location \mathbf{r} in \mathcal{D} .

256 The first condition means that there must not exist locations \mathbf{r}_1 and \mathbf{r}_2 for which $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)| \geq$
 257 $|\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$ and locations \mathbf{r}_1 and \mathbf{r}_3 for which $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_3)| < |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_3)|$.

258 The second condition implies that the hermitian matrix:

$$\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{M}}^\dagger(\mathbf{r}_1, \mathbf{r}_2) \quad (14)$$

259 can be written:

$$\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{N}}(\mathbf{r}_1) \overline{\overline{Y}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_1)^\dagger \quad (15)$$

260 with $\overline{\overline{Y}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{D}}^\dagger(\mathbf{r}_1, \mathbf{r}_2)$ and hence:

$$\overline{\overline{Y}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} |\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)|^2 & 0 \\ 0 & |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|^2 \end{pmatrix} \quad (16)$$

261 Thus, for an unfrustrated light, the matrix $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2)$ is diagonal in an eigenvector basis that is
 262 independent of \mathbf{r}_2 . Since $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2)$ and $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3)$ are diagonal in the same eigenvector basis
 263 if they commute, the second condition is thus satisfied if:

$$\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3) = \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3) \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) \quad (17)$$

264 This relation can be a practical means to check if a light is frustrated or unfrustrated (see last
 265 example of Section 7).

266 It is shown in Appendix A that the two above conditions are necessary. It is immediate to
 267 verify that they are sufficient. Indeed, let introduce Eq. (12) into Eq. (6) then $C(\bar{u}_1, \bar{u}_2) =$
 268 $|\bar{u}_1^\dagger \bar{N}(\mathbf{r}_1) \bar{D}(\mathbf{r}_1, \mathbf{r}_2) \bar{N}^\dagger(\mathbf{r}_2) \bar{u}_2|$. But since it is assumed that $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)| \geq |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$, then
 269 $\bar{u}_j^{opt}(\mathbf{r}_1, \mathbf{r}_2) = \bar{N}(\mathbf{r}_j) \bar{e}_x$, which is a function of only \mathbf{r}_j and which thus shows that the light is
 270 unfrustrated.

271 In conclusion, the two above conditions are necessary and sufficient conditions for the light to
 272 be unfrustrated. It is frustrated if it does not satisfy at least one of these conditions.

273 6. Mutual incoherence separability

274 6.1. Physical interpretation of unfrustration

275 In this section, we provide a physical insight of the above conditions for unfrustrated light in a
 276 spatial domain \mathcal{D} . Eq. (12) implies (see Appendix A) that the normalized CSDM can be written:

$$\bar{M}(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2) \bar{n}_1(\mathbf{r}_1) \bar{n}_1^\dagger(\mathbf{r}_2) + \tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2) \bar{n}_2(\mathbf{r}_1) \bar{n}_2^\dagger(\mathbf{r}_2) \quad (18)$$

277 where $\bar{n}_1(\mathbf{r})$ is the vector defined with the first column of $\bar{N}(\mathbf{r})$ and $\bar{n}_2(\mathbf{r})$ is the vector defined
 278 with the second column of the unitary matrix $\bar{N}(\mathbf{r})$ and where $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)| \geq |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$ for all
 279 \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} . Since $\bar{n}_1(\mathbf{r})$ and $\bar{n}_2(\mathbf{r})$ are vectors defined with the columns of $\bar{N}(\mathbf{r})$, they are of
 280 unit norm and orthogonal, i.e. $\bar{n}_1^\dagger(\mathbf{r}) \bar{n}_1(\mathbf{r}) = \bar{n}_2^\dagger(\mathbf{r}) \bar{n}_2(\mathbf{r}) = 1$ and $\bar{n}_1^\dagger(\mathbf{r}) \bar{n}_2(\mathbf{r}) = 0$. Furthermore,
 281 $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)|$ and $|\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$ are equal to the intrinsic degrees of coherence and are thus invariant
 282 by non-singular deterministic Jones transformations (gauge transformations) of $\bar{E}(\mathbf{r})$ [3].

283 As a consequence if $\bar{E}(\mathbf{r})$ is an unfrustrated light, then there exists a local polarization and
 284 intensity transformation $\bar{A}(\mathbf{r}) = \bar{J}(\mathbf{r}) \bar{E}(\mathbf{r})$ so that the normalized CSDM of $\bar{A}(\mathbf{r})$ can be written
 285 in \mathcal{D} :

$$\bar{M}_{AA}(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2) \bar{e}_x \bar{e}_x^\dagger + \tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2) \bar{e}_y \bar{e}_y^\dagger \quad (19)$$

286 with $|\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)| \geq |\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)|$. Thus, the scalar field $a_x(\mathbf{r}_1)$ and $a_x(\mathbf{r}_2)$, where $a_x(\mathbf{r}) = \bar{e}_x^\dagger \bar{A}(\mathbf{r})$,
 287 has a scalar degree of coherence equal to $\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)$, and the scalar field $a_y(\mathbf{r}_1)$ and $a_y(\mathbf{r}_2)$,
 288 where $a_y(\mathbf{r}) = \bar{e}_y^\dagger \bar{A}(\mathbf{r})$ has a scalar degree of coherence equal to $\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)$.

289 Let us now analyze the consequences on the CSDM $\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2)$ in the case of an unfrustrated
 290 light in \mathcal{D} . Since $\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) = \bar{\Gamma}^{\frac{1}{2}}(\mathbf{r}_1) \bar{M}(\mathbf{r}_1, \mathbf{r}_2) \bar{\Gamma}^{\frac{1}{2}}(\mathbf{r}_2)$, it can be written:

$$\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2) \bar{n}'_1(\mathbf{r}_1) \bar{n}'_1{}^\dagger(\mathbf{r}_2) + \tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2) \bar{n}'_2(\mathbf{r}_1) \bar{n}'_2{}^\dagger(\mathbf{r}_2) \quad (20)$$

291 where $\bar{n}'_1(\mathbf{r}) = \bar{\Gamma}^{\frac{1}{2}}(\mathbf{r}) \bar{n}_1(\mathbf{r})$ and $\bar{n}'_2(\mathbf{r}) = \bar{\Gamma}^{\frac{1}{2}}(\mathbf{r}) \bar{n}_2(\mathbf{r})$. Let $I_1(\mathbf{r}) = [\bar{n}'_1(\mathbf{r})]^\dagger \bar{n}'_1(\mathbf{r}) =$
 292 $\bar{n}_1^\dagger(\mathbf{r}) \bar{\Gamma}(\mathbf{r}) \bar{n}_1(\mathbf{r})$ and $I_2(\mathbf{r}) = [\bar{n}'_2(\mathbf{r})]^\dagger \bar{n}'_2(\mathbf{r}) = \bar{n}_2^\dagger(\mathbf{r}) \bar{\Gamma}(\mathbf{r}) \bar{n}_2(\mathbf{r})$. Then $\bar{\psi}(\mathbf{r}) = \frac{1}{\sqrt{I_1(\mathbf{r})}} \bar{n}'_1(\mathbf{r})$
 293 and $\bar{\phi}(\mathbf{r}) = \frac{1}{\sqrt{I_2(\mathbf{r})}} \bar{n}'_2(\mathbf{r})$ are unit norm vectors. Thus, in \mathcal{D} , the CSDM of an unfrustrated light
 294 can be written:

$$\begin{aligned} \bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) = & \sqrt{I_1(\mathbf{r}_1)I_1(\mathbf{r}_2)} \tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2) \bar{\psi}(\mathbf{r}_1) \bar{\psi}^\dagger(\mathbf{r}_2) \\ & + \sqrt{I_2(\mathbf{r}_1)I_2(\mathbf{r}_2)} \tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2) \bar{\phi}(\mathbf{r}_1) \bar{\phi}^\dagger(\mathbf{r}_2) \end{aligned} \quad (21)$$

295 The physical interpretation of this property is simple. An unfrustrated light in a spatial domain
 296 \mathcal{D} can be represented as the sum of two totally polarized lights that are incoherent between them.

297 Furthermore, the first necessary condition for the light to be unfrustrated is that one of these
 298 totally polarized lights must have a modulus of the scalar degree of coherence that is always (i.e.
 299 $\forall \mathbf{r}_1$ and \mathbf{r}_2 in \mathcal{D}) greater than or equal to the other. This property is a motivation to denominate
 300 the first condition as the "coherence domination condition".

301 The second necessary condition for the light to be unfrustrated is that $\overline{\overline{N}}(\mathbf{r})$ is only a function
 302 of \mathbf{r} in \mathcal{D} . This condition is indeed sufficient in order to be able to describe the light as the sum
 303 of two partially coherent but totally polarized lights that are incoherent between them. This
 304 property is a motivation to denominate the second condition the "mutual incoherence separability
 305 condition". It is however not sufficient to guarantee that the light is unfrustrated since the
 306 coherence domination condition must also be fulfilled.

307 6.2. Interpretation of mutual incoherence separability

308 If the mutual incoherence separability condition is satisfied in \mathcal{D} , and independently that the
 309 coherence domination condition is satisfied or not, the CSDM of the light can be separated into
 310 two totally polarized lights in \mathcal{D} that are incoherent between them (see Eq. (21)). Thus, Eq. (21)
 311 is a beam decomposition in \mathcal{D} that can be deduced from the normalized CSDM. In that case,
 312 let us assume that at each location \mathbf{r} in \mathcal{D} , the field is projected on the polarization state $\overline{\varphi}(\mathbf{r})$,
 313 defined such that $\overline{\varphi}^T(\mathbf{r})\overline{\varphi}(\mathbf{r}) = 0$. Then $a_1(\mathbf{r}) = \overline{\varphi}^\dagger(\mathbf{r})\overline{E}(\mathbf{r})$ is a scalar field with a scalar degree
 314 of coherence equal to $\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2)$. Analogously, if the field is projected on the polarization state
 315 $\overline{\xi}(\mathbf{r})$, defined such that $\overline{\xi}^T(\mathbf{r})\overline{\psi}(\mathbf{r}) = 0$, then $a_2(\mathbf{r}) = \overline{\xi}^\dagger(\mathbf{r})\overline{E}(\mathbf{r})$ is a scalar field with a scalar
 316 degree of coherence equal to $\tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2)$. Moreover, $a_1(\mathbf{r})$ and $a_2(\mathbf{r})$ are scalar fields that are
 317 incoherent between them, i.e. $\langle a_1^*(\mathbf{r}_1)a_2(\mathbf{r}_2) \rangle = 0$ for all \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} . Such a polarization
 318 coherence filtering is not possible if the light does not satisfy the mutual incoherence separability
 319 condition. If the light is unfrustrated, in which case the coherence domination condition is also
 320 fulfilled, such a polarization coherence filtering amounts to extracting a field proportional to the
 321 most coherent part of the field in \mathcal{D} .

322 Reciprocally, if it is possible to extract with two local non parallel polarization filtering
 323 devices (i.e. with polarizations states that are not proportional) two scalar fields that are
 324 incoherent between them in a spatial domain \mathcal{D} , then the light satisfies the mutual incoherence
 325 separability condition. Indeed, let $\overline{\varphi}(\mathbf{r})$ and $\overline{\xi}(\mathbf{r})$ be two non parallel local polarization states
 326 (i.e. such that there does not exist a complex scalar field $\lambda(\mathbf{r})$ such that $\overline{\varphi}(\mathbf{r}) = \lambda(\mathbf{r})\overline{\xi}(\mathbf{r})$).
 327 Let us then assume that $a_1(\mathbf{r}) = \overline{\varphi}^\dagger(\mathbf{r})\overline{E}(\mathbf{r})$ and $a_2(\mathbf{r}) = \overline{\xi}^\dagger(\mathbf{r})\overline{E}(\mathbf{r})$ are incoherent between
 328 them (i.e. $\langle a_1^*(\mathbf{r}_1)a_2(\mathbf{r}_2) \rangle = 0 \forall \mathbf{r}_1$ and \mathbf{r}_2 in \mathcal{D}). The field $\overline{A}(\mathbf{r}) = a_1(\mathbf{r})\overline{\varphi}(\mathbf{r}) + a_2(\mathbf{r})\overline{\xi}(\mathbf{r})$
 329 satisfies the incoherence separability condition. But, since $\overline{A}(\mathbf{r})$ can also be written $\overline{A}(\mathbf{r}) =$
 330 $\overline{\varphi}^\dagger(\mathbf{r})\overline{E}(\mathbf{r})\overline{\varphi}(\mathbf{r}) + \overline{\xi}^\dagger(\mathbf{r})\overline{E}(\mathbf{r})\overline{\xi}(\mathbf{r})$, then $\overline{A}(\mathbf{r}) = \overline{U}(\mathbf{r})\overline{E}(\mathbf{r})$ with $\overline{U}(\mathbf{r}) = \overline{\varphi}(\mathbf{r})\overline{\varphi}^\dagger(\mathbf{r}) + \overline{\xi}(\mathbf{r})\overline{\xi}^\dagger(\mathbf{r})$
 331 and then $\overline{E}(\mathbf{r}) = \overline{U}^{-1}(\mathbf{r})\overline{A}(\mathbf{r})$. Indeed, $\overline{U}(\mathbf{r})$ is invertible since $\overline{\varphi}(\mathbf{r})$ and $\overline{\xi}(\mathbf{r})$ are assumed non
 332 parallel. Then,

$$\overline{E}(\mathbf{r}) = a_1(\mathbf{r})\overline{\varphi}(\mathbf{r}) + a_2(\mathbf{r})\overline{\psi}(\mathbf{r}) \quad (22)$$

333 with $\overline{\varphi}(\mathbf{r}) = \overline{U}^{-1}(\mathbf{r})\overline{\varphi}(\mathbf{r})$ and $\overline{\psi}(\mathbf{r}) = \overline{U}^{-1}(\mathbf{r})\overline{\xi}(\mathbf{r})$ which shows that $\overline{E}(\mathbf{r})$ is the sum of two
 334 polarized light incoherent between them, i.e., $\overline{E}(\mathbf{r})$ verifies the mutual incoherence separability
 335 condition. Indeed, $\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = \langle \overline{E}^*(\mathbf{r}_1)\overline{E}^T(\mathbf{r}_2) \rangle$ leads to:

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = F(\mathbf{r}_1, \mathbf{r}_2)\overline{\varphi}^*(\mathbf{r}_1)\overline{\varphi}^T(\mathbf{r}_2) + G(\mathbf{r}_1, \mathbf{r}_2)\overline{\psi}^*(\mathbf{r}_1)\overline{\psi}^T(\mathbf{r}_2) \quad (23)$$

336 with $F(\mathbf{r}_1, \mathbf{r}_2) = \langle a_1^*(\mathbf{r}_1)a_1(\mathbf{r}_2) \rangle$ and $G(\mathbf{r}_1, \mathbf{r}_2) = \langle a_2^*(\mathbf{r}_1)a_2(\mathbf{r}_2) \rangle$, showing it has a form that
 337 corresponds to Eq. (21).

338 The mutual incoherence separability condition as described by Eq. (21) can be interpreted as
 339 a decomposition of the CSDM $\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2)$, with some analogies with the one discussed in [19].

340 Such decompositions are different from the one considered in [20], as the former rely on the
 341 singular value decomposition (SVD) of the normalized CSDM while the latter uses the SVD of
 342 the CSDM itself. These approaches lead to different results, as will be illustrated in the example
 343 B in the next section.

344 The previous analysis and Eq. (21) provide a practical solution to generate unfrustrated lights.
 345 Unfrustrated lights in a spatial domain \mathcal{D} can be obtained with the sum of two partially coherent
 346 but totally polarized lights. These lights have to be incoherent between them, which guarantees
 347 the mutual incoherence separability condition. The modulus of the scalar degree of coherence
 348 in \mathcal{D} of one of these lights has to be always greater than or equal to the modulus of the scalar
 349 degree of coherence of the other light in order to guarantee the coherence domination condition.

350 7. Examples

351 We provide in this section several examples that illustrate the concepts introduced above.

352 7.1. Fields defined at three locations

353 Let us first illustrate the properties of the previous sections on the examples of Section 3 and
 354 illustrated in Fig. 2. The example of Fig. 2.a leads to $\overline{\overline{\Gamma}}(\mathbf{r}_j) = \overline{\overline{I}}_d$ where $\overline{\overline{I}}_d$ is the 2D identity
 355 matrix. It is easy to see that:

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (24)$$

356 and

$$\overline{\overline{\Omega}}(\mathbf{r}_2, \mathbf{r}_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (25)$$

357 from which it can be deduced (see Eq. (14)):

$$\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{\overline{\mathcal{W}}}(\mathbf{r}_i, \mathbf{r}_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (26)$$

358 for $i = 1, 2$ and where it is recalled that $\overline{\overline{\mathcal{W}}}(\mathbf{r}_i, \mathbf{r}_j) = \overline{\overline{M}}(\mathbf{r}_i, \mathbf{r}_j) \overline{\overline{M}}^\dagger(\mathbf{r}_i, \mathbf{r}_j)$ and $\overline{\overline{M}}(\mathbf{r}_i, \mathbf{r}_j) =$
 359 $\overline{\overline{\Gamma}}(\mathbf{r}_i)^{-\frac{1}{2}} \overline{\overline{\Omega}}(\mathbf{r}_i, \mathbf{r}_j) \overline{\overline{\Gamma}}(\mathbf{r}_j)^{-\frac{1}{2}}$. Eq. (26) clearly shows that the coherence domination condition is
 360 not satisfied and thus that the light is frustrated. Indeed, the first eigenvalue of $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_j)$ is not
 361 maximal for all couple of \mathbf{r}_i and \mathbf{r}_j . Nevertheless, it can be noticed that $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_j)$ is diagonal
 362 in an eigenvector basis that is independent of the \mathbf{r}_j which illustrates that both the properties (i)
 363 and (ii) are required for the light to be unfrustrated.

364 Example of Fig. 2.b also leads to $\overline{\overline{\Gamma}}(\mathbf{r}_j) = \overline{\overline{I}}_d$. Furthermore,

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (27)$$

365

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_3) = \overline{\overline{\Omega}}(\mathbf{r}_2, \mathbf{r}_3) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (28)$$

366 From which it can be deduced (see Eq. (14)):

$$\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{\overline{\mathcal{W}}}(\mathbf{r}_i, \mathbf{r}_3) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad (29)$$

367 for $i = 1, 2$ and which clearly shows that $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_j)$ is diagonal in an eigenvector basis that is
 368 independent of \mathbf{r}_j , and the first eigenvalue of $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_j)$ is maximal for any of the considered
 369 \mathbf{r}_j . Thus both the coherence domination and the mutual incoherence separability conditions are
 370 satisfied, which is expected since the light is unfrustrated.

371 Let us now consider a third example between three locations \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 (illustrated in Fig.
 372 4) for which the mutual incoherence separability condition is not satisfied. For this example, the
 373 fields are:

$$\begin{aligned} \overline{E}(\mathbf{r}_1) &= \epsilon_1 \overline{e}_x + \epsilon_2 \overline{e}_y \\ \overline{E}(\mathbf{r}_2) &= (\epsilon_1 + \epsilon_3) \overline{e}_x + (\epsilon_2 + \epsilon_3) \overline{e}_y \\ \overline{E}(\mathbf{r}_3) &= (\epsilon_1 + \epsilon_3) \overline{e}_x + \left(\frac{1}{2} \epsilon_2 + \epsilon_3\right) \overline{e}_y \end{aligned} \quad (30)$$

374 Thus $\overline{\overline{\Gamma}}(\mathbf{r}_1) = \overline{\overline{I}}_d$, but:

$$\overline{\overline{\Gamma}}(\mathbf{r}_2) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \overline{\overline{\Gamma}}(\mathbf{r}_3) = \begin{pmatrix} 2 & 1 \\ 1 & \frac{5}{4} \end{pmatrix} \quad (31)$$

375 Furthermore, $\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{I}}_d$ and

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_3) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \overline{\overline{\Omega}}(\mathbf{r}_2, \mathbf{r}_3) = \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \quad (32)$$

376 However, since $\overline{\overline{\Gamma}}(\mathbf{r}_1) = \overline{\overline{I}}_d$ it is immediate to see (see Eq. (14)) that $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_j) = \overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_j) \overline{\overline{\Gamma}}^{-1}(\mathbf{r}_j) \overline{\overline{\Omega}}^\dagger(\mathbf{r}_1, \mathbf{r}_j)$,
 377 which leads to:

$$\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3) = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad (33)$$

378 It is easy to verify that $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3) \neq \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3) \overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2)$. Thus $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_2)$ and
 379 $\overline{\overline{\mathcal{W}}}(\mathbf{r}_1, \mathbf{r}_3)$ cannot be diagonal with the same unitary matrix $\overline{\overline{N}}(\mathbf{r}_1)$ as in Eq. (15). The mutual
 380 incoherence separability condition is thus not satisfied (i.e. Eq. (17) is not satisfied) and the light
 381 is frustrated.

382 7.2. Mean square coherent light

383 Mean square coherent light (MSC) provides a simple example of a category of unfrustrated light.
 384 Such lights can indeed be written [17]:

$$\overline{E}(\mathbf{r}) = \sqrt{I_1} \epsilon_1 \overline{f}_1(\mathbf{r}) + \sqrt{I_2} \epsilon_2 \overline{f}_2(\mathbf{r}) \quad (34)$$

385 with $I_i > 0$, $\langle \epsilon_i^* \epsilon_j \rangle = \delta_{ij}$ and where $\overline{f}_1(\mathbf{r})$ and $\overline{f}_2(\mathbf{r})$ are two deterministic functions of \mathbf{r} with
 386 values in \mathbb{C}^2 . In that case, it is easy to see [17] that $\mu_1(\mathbf{r}_1, \mathbf{r}_2) = \mu_2(\mathbf{r}_1, \mathbf{r}_2) = 1$, which shows

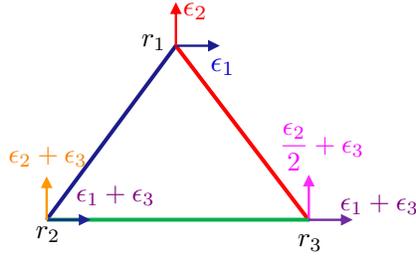


Fig. 4. Simple example for which the mutual incoherence separability condition is not satisfied.

387 that the coherence domination condition is respected. The mutual incoherence separability
 388 condition is also respected, which shows that mean square coherent lights are unfrustrated.
 389 Mean square coherent light corresponds to the particular case of unfrustrated light for which
 390 $\tilde{\mu}_1(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\mu}_2(\mathbf{r}_1, \mathbf{r}_2) = 1$. The polarization matrix is not singular at location \mathbf{r} if there does
 391 not exist a complex number λ such that $\bar{f}_1(\mathbf{r}) = \lambda \bar{f}_2(\mathbf{r})$ and, in that case, there exists a gauge
 392 transformation that leads to the field $\bar{A}(\mathbf{r}) = \epsilon_1 \bar{e}_x + \epsilon_2 \bar{e}_y$. The field $\bar{A}(\mathbf{r})$ has thus a diagonal
 393 CSDM (the polarization matrix is equal to the identity matrix as well as the normalized CSDM).

394 The field $\bar{A}(\mathbf{r})$ corresponds to uniformly and totally unpolarized light. There exists also
 395 mean square coherent lights more complex than $\bar{A}(\mathbf{r})$ such as Full Poincaré light beams of type
 396 II [21]. Another example corresponds to a speckle field obtained by enlightening a surface with a
 397 totally unpolarized and coherent field $\bar{E}(\mathbf{r}) = \epsilon_1 \bar{e}_x + \epsilon_2 \bar{e}_y$. If the speckle field can be written
 398 $\bar{A}(\mathbf{r}) = \bar{J}(\mathbf{r})\bar{E}(\mathbf{r})$, then $\bar{A}(\mathbf{r})$ is unfrustrated since $\bar{A}(\mathbf{r})$ is a gauge transformation of $\bar{E}(\mathbf{r})$ and
 399 $\bar{E}(\mathbf{r})$ is a mean square coherent light and hence unfrustrated.

400 Let us now consider another example of MSC light, which will allow us to illustrate the
 401 difference of the decomposition considered in this article in Eq. (21) with the one proposed
 402 in [20]. The field of this MSC light is assumed to be the sum of two perfectly polarized beams,
 403 incoherent between them, with non parallel polarization:

$$\bar{E}(\mathbf{r}) = f(\mathbf{r}) \epsilon_1 \bar{e}_x + \epsilon_2 \bar{e}_{\frac{\pi}{4}} \quad (35)$$

404 with $\langle \epsilon_i^* \epsilon_j \rangle = \delta_{ij}$ and $f(\mathbf{r}) \geq 0$, $\bar{e}_x = (1, 0)^T$ and $\bar{e}_{\frac{\pi}{4}} = (1, 1)^T$. Then:

$$\bar{\bar{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} 1 + f(\mathbf{r}_1)f(\mathbf{r}_2) & 1 \\ 1 & 1 \end{pmatrix} \quad (36)$$

405 Since $\bar{\bar{\Omega}}(\mathbf{r}_1, \mathbf{r}_2)$ is real and symmetric and since its eigenvalues are positive, its singular values
 406 and singular vectors are equal to its eigenvalues and eigenvectors. Its singular values are thus:

$$\lambda_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2} (2 + g_{\pm}(\mathbf{r}_1, \mathbf{r}_2)) \quad (37)$$

407 where $g_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = f(\mathbf{r}_1)f(\mathbf{r}_2) \pm \sqrt{4 + f(\mathbf{r}_1)^2 f(\mathbf{r}_2)^2}$. The normalized singular vectors are:

$$\bar{w}_{j,\pm}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{4 + g_{\pm}^2(\mathbf{r}_1, \mathbf{r}_2)}} \begin{pmatrix} g_{\pm}(\mathbf{r}_1, \mathbf{r}_2) \\ 2 \end{pmatrix} \quad (38)$$

408 Thus, the polarization vectors obtained with this decomposition are equal at locations \mathbf{r}_1 and
 409 \mathbf{r}_2 (i.e., $\bar{w}_{1,\pm}(\mathbf{r}_1, \mathbf{r}_2) = \bar{w}_{2,\pm}(\mathbf{r}_1, \mathbf{r}_2)$) for this example but they are different from \bar{e}_x and $\bar{e}_{\frac{\pi}{4}}$.
 410 Moreover, the polarization obtained with this decomposition at \mathbf{r}_1 varies if \mathbf{r}_2 varies which is not
 411 the case of \bar{e}_x and $\bar{e}_{\frac{\pi}{4}}$.

412 Let us now analyze the same situation from the point of view of the decomposition considered
 413 in this article in Eq. (21) based on the SVD of the normalized CSDM. It can indeed be performed
 414 from the only knowledge of the CSDM and the polarization matrices at \mathbf{r}_1 and \mathbf{r}_2 without
 415 requiring the knowledge of Eq. (35). Since $\bar{\Gamma}(\mathbf{r}) = \bar{\Omega}(\mathbf{r}, \mathbf{r})$, a direct calculus allows one to see
 416 with Eq. (36) that:

$$\bar{\Gamma}(\mathbf{r}) = f^2(\mathbf{r}) \bar{e}_x \bar{e}_x^T + \bar{e}_{\frac{\pi}{4}} \bar{e}_{\frac{\pi}{4}}^T = \bar{J}(\mathbf{r}) \bar{J}(\mathbf{r})^T \quad (39)$$

417 with the Jones matrix:

$$\bar{J}(\mathbf{r}) = \begin{pmatrix} f(\mathbf{r}) & 1 \\ 0 & 1 \end{pmatrix} \quad (40)$$

418 that transforms the field $\bar{E}(\mathbf{r})$ into the field $\bar{A}(\mathbf{r}) = \bar{J}^{-1}(\mathbf{r}) \bar{E}(\mathbf{r})$. It can be noticed that $\bar{J}^{-1}(\mathbf{r})$
 419 acts as a gauge transformation. A direct calculus shows that the CSDM of $\bar{A}(\mathbf{r})$ is equal to the
 420 identity matrix (i.e. $\langle \bar{A}^*(\mathbf{r}_1) \bar{A}^T(\mathbf{r}_2) \rangle = \bar{I}_d$), and so do the polarization matrices at locations
 421 \mathbf{r}_1 and \mathbf{r}_2 (since the polarization matrix is $\langle \bar{A}^*(\mathbf{r}) \bar{A}^T(\mathbf{r}) \rangle$ and is thus also equal to \bar{I}_d). Thus
 422 $\bar{A}(\mathbf{r}) = \epsilon_1 \bar{e}_x + \epsilon_2 \bar{e}_y$ with $\bar{e}_x = (1, 0)^T$ and $\bar{e}_y = (0, 1)^T$. But $\bar{E}(\mathbf{r}) = \bar{J}(\mathbf{r}) \bar{A}(\mathbf{r})$ and then:

$$\bar{E}(\mathbf{r}) = f(\mathbf{r}) \epsilon_1 \bar{e}_x + \epsilon_2 \bar{e}_{\frac{\pi}{4}} \quad (41)$$

423 which allows one to recover the model of Eq. (35) that corresponds to the two totally polarized
 424 and coherent fields (but totally incoherent between them) considered in this example. This simple
 425 example demonstrates the relevance of the normalized CSDM to get the decomposition that
 426 corresponds to the initial perfectly polarized lights and incoherent between them.

427 7.3. Diagonal Gaussian Schell model

428 Let us now consider the following particular diagonal Gaussian Schell model [22] in a spatial
 429 domain \mathcal{D} that corresponds to a particular transverse plane to the propagation axis for which the
 430 CSDM is:

$$\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} \omega_x(\mathbf{r}_1, \mathbf{r}_2) & 0 \\ 0 & \omega_y(\mathbf{r}_1, \mathbf{r}_2) \end{pmatrix} \quad (42)$$

431 with $\omega_s(\mathbf{r}_1, \mathbf{r}_2) = I_s e^{-\frac{\|\mathbf{r}_1\|^2 + \|\mathbf{r}_2\|^2}{4\sigma_0^2}} e^{-\frac{\|\mathbf{r}_1 - \mathbf{r}_2\|^2}{2\sigma^2}}$ with $s = x, y$. The normalized CSDM is thus:

$$\bar{M}(\mathbf{r}_i, \mathbf{r}_j) = \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_i) \bar{\Omega}(\mathbf{r}_i, \mathbf{r}_j) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_j) = e^{-\frac{\|\mathbf{r}_i - \mathbf{r}_j\|^2}{2\sigma^2}} \bar{I}_d \quad (43)$$

432 since $\bar{\Gamma}(\mathbf{r}) = \bar{\Omega}(\mathbf{r}, \mathbf{r})$ and which implies:

$$\bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}) = \begin{pmatrix} 1/\sqrt{\omega_x(\mathbf{r}, \mathbf{r})} & 0 \\ 0 & 1/\sqrt{\omega_y(\mathbf{r}, \mathbf{r})} \end{pmatrix} \quad (44)$$

433 The matrix $\bar{W}(\mathbf{r}_i, \mathbf{r}_j) = \bar{M}(\mathbf{r}_i, \mathbf{r}_j) \bar{M}^\dagger(\mathbf{r}_i, \mathbf{r}_j)$ is diagonal in an eigenvector basis that is
 434 independent of \mathbf{r}_i and \mathbf{r}_j (since $\bar{M}(\mathbf{r}_i, \mathbf{r}_j)$ is proportional to the identity matrix), the mutual

435 incoherence separability condition is thus respected. Furthermore, the first eigenvalue is equal to
 436 the second eigenvalue for any considered \mathbf{r}_i and \mathbf{r}_j which shows that the coherence domination
 437 condition is also respected. The light is thus unfrustrated.

438 Indeed, it is easy to see that $\bar{u}^\dagger \overline{\overline{M}}(\mathbf{r}_i, \mathbf{r}_j) \bar{u} = e^{-\frac{\|\mathbf{r}_i - \mathbf{r}_j\|^2}{2\sigma^2}}$ for all \bar{u} with $\bar{u}^\dagger \bar{u} = 1$. Thus, the
 439 polarization configuration $\bar{u}(\mathbf{r}) = \bar{u}_0$ for all \mathbf{r} and whatever \bar{u}_0 , maximizes the modulus of the
 440 scalar degree of coherence, and this is valid for any triplet of locations.

441 7.4. Frustrated light with Gaussian Degrees of Coherence

442 An example of frustrated light in a spatial domain \mathcal{D} that corresponds to a particular transverse
 443 plane to the propagation axis is observed with the sum of two fields with anisotropic coherence.
 444 Indeed, let us assume that the field is the sum of a horizontally polarized light and of a vertically
 445 polarized light incoherent between them. Let us also assume that the expression of the CSDM is
 446 still provided by Eq. (42) but with:

$$\omega_s(\mathbf{r}_1, \mathbf{r}_2) = I_s e^{-\frac{\|\mathbf{r}_1\|^2 + \|\mathbf{r}_2\|^2}{4\sigma_0^2}} e^{-\frac{(s_1 - s_2)^2}{2\sigma^2}} \quad (45)$$

447 with $s = x, y$. A development similar to the one of the previous example shows that the normalized
 448 CSDM is:

$$\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} e^{-\frac{(x_1 - x_2)^2}{2\sigma^2}} & 0 \\ 0 & e^{-\frac{(y_1 - y_2)^2}{2\sigma^2}} \end{pmatrix} \quad (46)$$

449 The matrix $\overline{\overline{W}}(\mathbf{r}_i, \mathbf{r}_j)$ (defined in Eq. (14)) is diagonal in an eigenvector basis independent of
 450 $\mathbf{r}_i, \mathbf{r}_j$ and thus the mutual incoherence separability condition is respected. However, the first
 451 eigenvalue $e^{-\frac{(x_1 - x_2)^2}{2\sigma^2}}$ is not always greater than or equal to the second eigenvalue $e^{-\frac{(y_1 - y_2)^2}{2\sigma^2}}$ which
 452 shows that the coherence domination condition is not respected. The light is thus frustrated.

453 This result can be easily illustrated by looking at the coherence at the three locations
 454 $\mathbf{r}_1 = (0, 0)^T$, $\mathbf{r}_2 = (\sqrt{2}\sigma, 0)^T$ and $\mathbf{r}_3 = (0, \sqrt{2}\sigma)^T$. Indeed, the maximal coherence between \mathbf{r}_1
 455 and \mathbf{r}_2 is obtained with projection of the field on the vertical y polarization. However, between
 456 \mathbf{r}_1 and \mathbf{r}_3 , the field has to be projected on the horizontal x polarization in order to maximize the
 457 modulus of the standard degree of coherence.

458 7.5. Sum of three Hermite-Gaussian lights

459 With this last example, the considered field, in a spatial domain \mathcal{D} that corresponds to a particular
 460 transverse plane to the propagation axis, is the sum of three Hermite-Gaussian lights which are
 461 statistically incoherent between them. Setting: $G_{nm}(\mathbf{r}) = x^n y^m e^{-\frac{x^2 + y^2}{2\sigma^2}}$ where $\mathbf{r} = (x, y)^T$, let
 462 us assume that the total field is defined with:

$$\bar{E}(\mathbf{r}) = \epsilon_0 G_{00}(\mathbf{r}) \bar{e}_0 + \epsilon_1 G_{10}(\mathbf{r}) \bar{e}_x + \epsilon_2 G_{01}(\mathbf{r}) \bar{e}_y \quad (47)$$

463 where $\langle \epsilon_i \epsilon_j^* \rangle = \delta_{ij}$ and where $\bar{e}_0^T = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then:

$$\overline{\overline{\Omega}}(\mathbf{r}_1, \mathbf{r}_2) = e^{-\frac{\|\mathbf{r}_1\|^2 + \|\mathbf{r}_2\|^2}{2\sigma^2}} \begin{pmatrix} \frac{1}{2} + x_1 x_2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} + y_1 y_2 \end{pmatrix} \quad (48)$$

464 and $\overline{\overline{\Gamma}}(\mathbf{r}) = \overline{\overline{\Omega}}(\mathbf{r}, \mathbf{r})$.

465 Let us show that the mutual incoherence separability condition is not respected for the three
 466 locations $\mathbf{r}_1 = (1, 1)^T$, $\mathbf{r}_2 = (1, 0)^T$ and $\mathbf{r}_3 = (0, 1)^T$. In that case one obtains (where the symbol
 467 \propto means proportional to):

$$\bar{\Gamma}(\mathbf{r}_1) \propto \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \bar{\Gamma}(\mathbf{r}_2) \propto \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \bar{\Gamma}(\mathbf{r}_3) \propto \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad (49)$$

468 and $\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_2) \propto \bar{\Gamma}(\mathbf{r}_2)$ and $\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_3) \propto \bar{\Gamma}(\mathbf{r}_3)$. The relation $\bar{\mathcal{W}}(\mathbf{r}_i, \mathbf{r}_j) = \bar{M}(\mathbf{r}_i, \mathbf{r}_j) \bar{M}^\dagger(\mathbf{r}_i, \mathbf{r}_j)$
 469 (see Eqs. (14) and (5)) implies $\bar{\mathcal{W}}(\mathbf{r}_i, \mathbf{r}_j) = \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_i) \bar{\Omega}(\mathbf{r}_i, \mathbf{r}_j) \bar{\Gamma}^{-1}(\mathbf{r}_j) \bar{\Omega}(\mathbf{r}_j, \mathbf{r}_i) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_i)$. But
 470 since $\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_j) \propto \bar{\Gamma}(\mathbf{r}_j)$ for $j = 2, 3$ then:

$$\bar{\mathcal{W}}(\mathbf{r}_1, \mathbf{r}_j) \propto \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_j, \mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \quad (50)$$

471 It has been seen that to satisfy the mutual incoherence separability condition it is necessary that
 472 $\bar{\mathcal{W}}(\mathbf{r}_1, \mathbf{r}_2) \bar{\mathcal{W}}(\mathbf{r}_1, \mathbf{r}_3) = \bar{\mathcal{W}}(\mathbf{r}_1, \mathbf{r}_3) \bar{\mathcal{W}}(\mathbf{r}_1, \mathbf{r}_2)$ (see Eq. (17)). Thus Eq. (17) is satisfied if:

$$\bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_2, \mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_3, \mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \quad (51)$$

473 is equal to:

$$\bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_3, \mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_2, \mathbf{r}_1) \bar{\Gamma}^{-\frac{1}{2}}(\mathbf{r}_1) \quad (52)$$

474 It thus implies that

$$\bar{\Omega}(\mathbf{r}_2, \mathbf{r}_1) \bar{\Gamma}^{-1}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_3, \mathbf{r}_1) = \bar{\Omega}(\mathbf{r}_3, \mathbf{r}_1) \bar{\Gamma}^{-1}(\mathbf{r}_1) \bar{\Omega}(\mathbf{r}_2, \mathbf{r}_1) \quad (53)$$

475 In the present case $\bar{\Omega}(\mathbf{r}_j, \mathbf{r}_1) = \bar{\Omega}(\mathbf{r}_1, \mathbf{r}_j)$ for $j = 2, 3$ and $\bar{\Omega}(\mathbf{r}_1, \mathbf{r}_j) \propto \bar{\Gamma}(\mathbf{r}_j)$ for $j = 2, 3$. Thus
 476 the separability condition would imply

$$\bar{\Gamma}(\mathbf{r}_2) \bar{\Gamma}^{-1}(\mathbf{r}_1) \bar{\Gamma}(\mathbf{r}_3) \propto \bar{\Gamma}(\mathbf{r}_3) \bar{\Gamma}^{-1}(\mathbf{r}_1) \bar{\Gamma}(\mathbf{r}_2) \quad (54)$$

477 A simple direct calculus shows that it is not the case and thus the mutual incoherence separability
 478 condition is not satisfied. The considered sum of the three Hermite-Gaussian lights is thus
 479 polarization coherence frustrated.

480 In summary, the above examples illustrate the situations of unfrustrated light, frustrated light
 481 due to the non respect of the coherence domination condition and frustrated light due to the non
 482 respect of the mutual incoherence separability condition.

483 8. Conclusion

484 We have shown in this article that there exists two classes of partially polarized and partially
 485 coherent lights in a spatial domain \mathcal{D} : polarization coherence frustrated and polarization
 486 coherence unfrustrated lights. Each of these classes of equivalence is stable by the "gauge
 487 transformations" that correspond to the application of linear local transformations described by
 488 non singular deterministic Jones matrices. These properties and the proposed denomination are
 489 in analogy with the concepts of frustration and gauge invariance in statistical physics and spin
 490 glasses. As a result of these properties of light, practical manipulation of polarization coherence
 491 unfrustrated lights are expected to be easier than for polarization coherence frustrated lights
 492 which have more complex coherence properties.

493 On the one hand, it has been shown that to be unfrustrated in a spatial domain \mathcal{D} the light has
 494 to satisfy two different conditions. Indeed, the light has to be the sum of two totally polarized
 495 light in \mathcal{D} that are incoherent between them and one of these lights has to have a modulus of the
 496 scalar (i.e. standard) degree of coherence always greater than or equal to the other in \mathcal{D} . The
 497 decomposition method of the light as the sum of two totally polarized light incoherent between
 498 them can be obtained with a singular value decomposition of the normalized CSDM. It has been
 499 shown that a polarization coherence filtering allows one to extract a field proportional to the most
 500 coherent part of the field in that case of unfrustrated light. The maximum value of the modulus
 501 of the degree of coherence that can be extracted this way can be directly obtained by determining
 502 the value of the largest intrinsic degree of coherence between different locations.

503 On the other hand, polarization coherence frustrated light in a spatial domain \mathcal{D} can appear
 504 when one or both of these conditions are not satisfied. In that case, such a polarization coherence
 505 filtering to extract a field proportional to the most coherent part of the field is not possible.

506 In terms of practical application of these concepts, the ability to verify the non frustration of
 507 the light, and to design the appropriate local gauge transformations can be of interest for optimal
 508 light focusing for instance. The faculty to extract a maximally coherent field from an unfrustrated
 509 light through appropriate polarization filtering can also be of interest for some applications.

510 There exists several perspectives for this work. In particular, it will be interesting to analyze
 511 the frustration properties of several general models of coherence used in standard coherence
 512 theoretical studies and to analyze the evolution of unfrustration or frustration with propagation
 513 for these models. It will be useful to better investigate new practical implications of polarization
 514 coherence frustration and in particular experimental demonstrations to illustrate this concept are
 515 motivating perspectives. Finally, this concept could be generalized to light for which the fields
 516 can fluctuate in 3D.

517 9. Appendix A

518 The singular value decomposition of the matrix $\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2)$ can be written:

$$\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2)^\dagger \begin{pmatrix} \mu_S(\mathbf{r}_1, \mathbf{r}_2) & 0 \\ 0 & \mu_I(\mathbf{r}_1, \mathbf{r}_2) \end{pmatrix} \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \quad (55)$$

519 where $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2)$ and $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)$ are unitary matrices and where $\mu_S(\mathbf{r}_1, \mathbf{r}_2)$ and $\mu_I(\mathbf{r}_1, \mathbf{r}_2)$ are
 520 the intrinsic degrees of coherence introduced in [3] defined in the spectral domain, that
 521 satisfy $\mu_S(\mathbf{r}_1, \mathbf{r}_2) \geq \mu_I(\mathbf{r}_1, \mathbf{r}_2) \geq 0$. As defined in section 3, the light is unfrustrated in a
 522 domain \mathcal{D} if there exists $\overline{\overline{n}}_1(\mathbf{r})$ such that $\overline{\overline{n}}_1(\mathbf{r})^\dagger \overline{\overline{n}}_1(\mathbf{r}) = 1$ and $|\overline{\overline{n}}_1(\mathbf{r}_1)^\dagger \overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_2)|$ is
 523 maximal for any \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} . If $\mu_S(\mathbf{r}_1, \mathbf{r}_2) > \mu_I(\mathbf{r}_1, \mathbf{r}_2)$, the optimal configuration leads
 524 to $|\overline{\overline{n}}_1(\mathbf{r}_1)^\dagger \overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_2)| = \mu_S(\mathbf{r}_1, \mathbf{r}_2)$, and this property implies that $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_2) =$
 525 $e^{i\varphi_v(\mathbf{r}_1, \mathbf{r}_2)} \overline{\overline{e}}_x$ and $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_1) = e^{i\varphi_u(\mathbf{r}_1, \mathbf{r}_2)} \overline{\overline{e}}_x$, where $\overline{\overline{e}}_x = (1, 0)^T$, since $\mu_S(\mathbf{r}_1, \mathbf{r}_2)$
 526 is the maximal value of the modulus of the scalar degree of coherence [3]. If $\mu_S(\mathbf{r}_1, \mathbf{r}_2) =$
 527 $\mu_I(\mathbf{r}_1, \mathbf{r}_2)$, the choices of $\overline{\overline{n}}_1(\mathbf{r}_1)$ and $\overline{\overline{n}}_1(\mathbf{r}_2)$ such that $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_2) = e^{i\varphi_v(\mathbf{r}_1, \mathbf{r}_2)} \overline{\overline{e}}_x$ and
 528 $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_1) = e^{i\varphi_u(\mathbf{r}_1, \mathbf{r}_2)} \overline{\overline{e}}_x$ are still possible to maximize the modulus of the scalar degree
 529 of coherence.

530 Let us introduce the vector $\overline{\overline{n}}_2(\mathbf{r})$ such that $\overline{\overline{n}}_2(\mathbf{r})^\dagger \overline{\overline{n}}_2(\mathbf{r}) = 1$ and $\overline{\overline{n}}_2(\mathbf{r})^\dagger \overline{\overline{n}}_1(\mathbf{r}) = 0$. One has
 531 $\overline{\overline{n}}_2(\mathbf{r}_2)^\dagger \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)^\dagger \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_2) = \overline{\overline{n}}_2(\mathbf{r}_2)^\dagger \overline{\overline{n}}_1(\mathbf{r}_2) = 0$ and also $\overline{\overline{n}}_2(\mathbf{r}_2)^\dagger \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)^\dagger \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_2) =$
 532 $e^{i\varphi_v(\mathbf{r}_1, \mathbf{r}_2)} \overline{\overline{n}}_2(\mathbf{r}_2)^\dagger \overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2)^\dagger \overline{\overline{e}}_x = 0$. Then, necessarily $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_2(\mathbf{r}_2) = e^{i\phi_v(\mathbf{r}_1, \mathbf{r}_2)} \overline{\overline{e}}_y$ with
 533 $\overline{\overline{e}}_y = (0, 1)^T$. In other words $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_2) = \overline{\overline{D}}_v(\mathbf{r}_1, \mathbf{r}_2)$ with $\overline{\overline{N}}(\mathbf{r}) = [\overline{\overline{n}}_1(\mathbf{r}), \overline{\overline{n}}_2(\mathbf{r})]$ and
 534 where $\overline{\overline{D}}_v(\mathbf{r}_1, \mathbf{r}_2)$ is the diagonal matrix with diagonal values $e^{i\varphi_v(\mathbf{r}_1, \mathbf{r}_2)}$ and $e^{i\phi_v(\mathbf{r}_1, \mathbf{r}_2)}$. An

535 analogous development leads to $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_1) = \overline{\overline{D}}_u(\mathbf{r}_1, \mathbf{r}_2)$ where $\overline{\overline{D}}_u(\mathbf{r}_1, \mathbf{r}_2)$ is the diagonal
 536 matrix with diagonal values $e^{i\varphi_u(\mathbf{r}_1, \mathbf{r}_2)}$ and $e^{i\phi_u(\mathbf{r}_1, \mathbf{r}_2)}$. Then $\overline{\overline{V}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{D}}_v(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_2)^{-1}$
 537 and $\overline{\overline{U}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{D}}_u(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_1)^{-1}$ and thus:

$$\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \left[\overline{\overline{N}}(\mathbf{r}_1)^{-1} \right]^\dagger \overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_2)^{-1} \quad (56)$$

538 with:

$$\overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} \tilde{\mu}_S(\mathbf{r}_1, \mathbf{r}_2) & 0 \\ 0 & \tilde{\mu}_I(\mathbf{r}_1, \mathbf{r}_2) \end{pmatrix} \quad (57)$$

539 where $\tilde{\mu}_S(\mathbf{r}_1, \mathbf{r}_2) = \mu_S(\mathbf{r}_1, \mathbf{r}_2) e^{i[\varphi_v(\mathbf{r}_1, \mathbf{r}_2) - \varphi_u(\mathbf{r}_1, \mathbf{r}_2)]}$ and $\tilde{\mu}_I(\mathbf{r}_1, \mathbf{r}_2) = \mu_I(\mathbf{r}_1, \mathbf{r}_2) e^{i[\phi_v(\mathbf{r}_1, \mathbf{r}_2) - \phi_u(\mathbf{r}_1, \mathbf{r}_2)]}$.
 540 It is easy to verify that $\overline{\overline{N}}(\mathbf{r}) \overline{\overline{N}}(\mathbf{r})^\dagger = \overline{\overline{N}}(\mathbf{r})^\dagger \overline{\overline{N}}(\mathbf{r}) = \overline{\overline{I}}_d$, where $\overline{\overline{I}}_d$ is the identity matrix in 2D,
 541 and thus $\overline{\overline{N}}(\mathbf{r})^{-1} = \overline{\overline{N}}(\mathbf{r})^\dagger$. Thus:

$$\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\overline{N}}(\mathbf{r}_1) \overline{\overline{D}}(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{N}}(\mathbf{r}_2)^\dagger \quad (58)$$

542 which leads to Eq. (12). Furthermore, since $\overline{\overline{N}}(\mathbf{r}) = [\overline{\overline{n}}_1(\mathbf{r}), \overline{\overline{n}}_2(\mathbf{r})]$, it is easy to see that
 543 $\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2)$ can also be written:

$$\overline{\overline{M}}(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\mu}_S(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_1(\mathbf{r}_1) \overline{\overline{n}}_1^\dagger(\mathbf{r}_2) + \tilde{\mu}_I(\mathbf{r}_1, \mathbf{r}_2) \overline{\overline{n}}_2(\mathbf{r}_1) \overline{\overline{n}}_2^\dagger(\mathbf{r}_2) \quad (59)$$

544 with $|\tilde{\mu}_S(\mathbf{r}_1, \mathbf{r}_2)| \geq |\tilde{\mu}_I(\mathbf{r}_1, \mathbf{r}_2)|$ for any \mathbf{r}_1 and \mathbf{r}_2 in \mathcal{D} and where $|\tilde{\mu}_S(\mathbf{r}_1, \mathbf{r}_2)|$ and $|\tilde{\mu}_I(\mathbf{r}_1, \mathbf{r}_2)|$
 545 are the intrinsic degrees of coherence of the light for the couple of locations \mathbf{r}_1 and \mathbf{r}_2 .

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