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# Dynkin isomorphism theorems revisited

Nathalie Eisenbaum

**Abstract** Dynkin’s isomorphism Theorem as well as its derivatives, are identities in law providing the law of the sum of two independent processes. Most of the time these two processes are respectively the local time process of a Markov process and a permanental process. In spite of their various applications, these identities still remain mysterious in the sense that there is no natural reason for summing these two independent processes. We will present an explanation available in particular for the so-called ”generalized Ray-Knight Theorem”.

*Keywords* Markov process; local time; Gaussian free field; permanental process; isomorphism theorem; loop measure; loop soup.

*MSC 2020 subject classification:* 60E07; 60G15; 60J55; 60J25

## 1 Introduction

The most known version of Dynkin’s isomorphism Theorem [4] is a relation in law connecting the local time process of a transient symmetric Markov process with finite Green function to a Gaussian centered process with this Green function as covariance. More precisely for a transient Markov process  $X$  with state space  $E$ , admitting a finite symmetric Green function (0-potential density)  $g$  with respect to some reference measure, denote by  $(L_\infty^{(a,b)}(x), x \in E)$  the total accumulated local time process of  $X$  starting from  $a$  and killed at its last visit to  $b$ . Let  $(\eta_x, x \in E)$  be a centered Gaussian process with covariance  $(g(x, y), (x, y) \in E \times E)$ , independent of  $X$ , one has then:

$$\left(\frac{1}{2}\eta_x^2 + L_\infty^{(a,b)}(x), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\eta_x^2, x \in E\right) \text{ under } \mathbb{IE}\left[\frac{\eta_a\eta_b}{g(a, b)}, \cdot\right] \quad (1.1)$$

Several identities in law of the same type have been derived from this original identity. Among them, we mention its extension to transient non symmetric Markov processes [13], an unconditioned version [5], and the so-called generalized second Ray-Knight Theorem [16] from which a version for occupation times of random interlacements of

continuous symmetric Markov processes on graphs has been deduced [23]. We also mention that this last version [23] has been refined [24], [20], [2] to provide identities relating the sign of the Gaussian process to the loop clusters of the Markov process. Paths continuity is a necessary condition to obtain these identities involving the sign of the Gaussian process.

Dynkin's type isomorphism theorems turn out to be useful to transfer properties from the Gaussian processes to the local times (see e.g. Marcus and Rosen's book [21]) and to solve various questions on covering or percolation (see e.g. [3] and [1]).

Nevertheless, as was already pointed by Marcus and Rosen in p.3 of their book [21], one does not really understand the meaning of these identities in law. The main reason of this mystery is that they are precisely identities "in law". There are several kinds of proof of these identities (we describe some of them shortly in section 2) but in the general case none goes beyond this equality "in law" and none gives a natural reason for summing this two independent processes. The aim of this note is to present such a reason for some Dynkin type isomorphism theorems. We actually show that each of these identities in law can be seen as an illustration of a richer a.s. identity with an obvious meaning.

These results are presented in section 3. Section 2 is a preliminary section which reminds some Dynkin type identities and the connected known results used in section 3.

## 2 Dynkin type identities

We briefly remind in chronological order, some of Dynkin type identities together with the nature of their proof. The considered Markov processes are assumed to be strong Markov processes with right continuous left limits paths. One starts by the so-called "Dynkin's isomorphism Theorem" (1.1). Dynkin's intuition of the first isomorphism theorem came from the expression of the joint moments of the total accumulated local times of a transient symmetric Markov process. Comparing with the expression of the joint moments of the associated squared Gaussian vector  $(\eta_x^2, x \in E)$ , he obtained his identity with combinatorial arguments.

The so-called "generalized second Ray-Knight Theorem" requires a recurrent Markov process  $X$  such that  $g_{T_a}$  the Green function of  $X$  killed at  $T_a$ , the first hitting time of a point  $a$  of its space state  $E$ , is symmetric with respect to some reference measure. Denote by  $(L_{\tau_r}^x, x \in E)$  the local time process of  $X$  at time  $\tau_r$ , where  $\tau_r = \inf\{s > 0 : L_s^a > r\}$ . Denote by  $\mathbb{P}_a$  the probability under which  $X$  starts at  $a$ . Let  $(\eta_x, x \in E)$  be a centered Gaussian process with covariance  $(g_{T_a}(x, y), (x, y) \in E^2)$ , independent of  $X$ . One has under  $\mathbb{P}_a$

$$\left(\frac{1}{2}\eta_x^2 + L_{\tau_r}^x, x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2r})^2, x \in E\right) \quad (2.1)$$

The proof of the generalized second Ray-Knight Theorem [16] is simpler to handle because it is based on a direct computation of Laplace transforms.

At this point, authors started to try to characterize the Gaussian processes which were involved in the isomorphism theorems. The first motivation was to catch the properties that could be transferred via these identities from the Gaussian processes to the local time process. But a characterization would maybe also lead to another kind of explanation of the existence of these identities. In [7], [6] and [11], the property of infinite divisibility appears as a characteristic property of the squared Gaussian processes involved in these identities. More precisely a centered Gaussian process  $(\eta_x, x \in E)$  admits the Green function of a transient Markov process for covariance iff  $((\eta_x + r)^2, x \in E)$  is infinitely divisible for every real  $r$ .

In [8], we have shown that every nonnegative infinitely divisible process generates a Dynkin type isomorphism theorem. Still, the reason why in the case of an infinitely divisible squared Gaussian process, the process involved in the corresponding identity should be a local time process was not clear.

In the mean time, we could relax the assumption of symmetry for the Green function [13] and could extend (1.1) in the special case when  $a = b$  to obtain:

$$\left(\frac{1}{2}\phi_x + L_\infty^{(a,a)}(x), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\phi_x, x \in E\right) \text{ under } \mathbb{E}\left[\frac{\phi_a}{\mathbb{E}[\phi_a]}, \cdot\right] \quad (2.2)$$

where the process  $(\phi_x, x \in E)$  is a so-called ‘‘permanental process’’ with kernel  $g$  the Green function of  $X$  and index  $1/2$ .

We remind that a nonnegative process  $(\varphi_x, x \in E)$  is a permanental process with index  $\beta > 0$  and kernel  $(u(x, y), (x, y) \in E \times E)$  if its finite dimensional Laplace transforms are given by:

$$\mathbb{E}[\exp\{-\frac{1}{2} \sum_{i=1}^n \alpha_i \varphi_{x_i}\}] = [\det(I + \alpha U)]^{-\beta}, \quad (2.3)$$

for every  $\alpha_1, \dots, \alpha_n \geq 0$ ,  $x_1, \dots, x_n$  in  $E$ , where  $I$  is the  $n \times n$  identity matrix,  $\alpha$  is the diagonal matrix with entries  $\alpha_i$ ,  $1 \leq i \leq n$  and  $U = (u(x_i, x_j))_{1 \leq i, j \leq n}$ .

The proof of (2.2) makes use of the computation of the finite-dimensional Laplace transforms of  $L_\infty^{(a,a)}$  and of the existence of permanental processes with kernel  $g$ . From (2.2) we could immediately relax the assumption of symmetry in (2.1) to obtain under  $\mathbb{P}_a$ :

$$\left(\frac{1}{2}\phi_x, x \in E | \phi_a = 0\right) + \left(L_{\tau_r}^x, x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\phi_x, x \in E | \frac{1}{2}\phi_a = r\right), \quad (2.4)$$

where  $\phi$  is a permanental process, independent of  $X$ , with index  $1/2$  and with kernel  $g_{T_a} + 1$ , with  $g_{T_a}$  the Green function of  $X$  killed at its first hitting time of  $a$ .

Note that  $(\phi_x, x \in E | \phi_a = 0)$  is a permanental process with kernel  $g_{T_a}$  [13]. Besides when  $X$  is symmetric:  $(\phi | \phi_a = 2r) \stackrel{(\text{law})}{=} (\eta + \sqrt{2r})^2$ , with  $\eta$  centered Gaussian process with covariance  $g_{T_a}$ .

Moreover, we showed [13] that the property of infinite divisibility characterizes the permanental processes admitting the Green function of a transient Markov process for kernel. We mention that (2.2) is available for  $\phi$  permanental process with kernel  $g$  and index  $\beta > 0$  whatever the value of  $\beta$ .

The property of infinite divisibility of a nonnegative process  $(\psi_x, x \in E)$  is equivalent to the existence of a Lévy measure  $\nu$  on  $\mathbb{R}_+^E$  such that

$$(\psi_x, x \in E) \stackrel{(\text{law})}{=} \left( \sum_{y \in \chi} y(x), x \in E \right),$$

where  $\chi$  is a Poisson point process (i.e. a Poisson random measure) on  $\mathbb{R}_+^E$  with intensity measure  $\nu$ . For a complete presentation of the notion of Lévy measure of infinitely divisible processes, we recommend Rosinski's paper [22] ([17] presents illustrations in the case of positive infinitely divisible processes). In the sequel we will write PPP for Poisson point process (e.g.  $\chi = \text{PPP}(\nu)$ ).

The interest of the knowledge of the Lévy measure of the permanental processes admitting for kernel the Green function of a transient Markov process was hence clear. In case of a discrete state space and a symmetric transient time continuous Markov chain, the first identification of the Lévy measure  $\nu$  of  $(\frac{1}{2}\eta_x^2, x \in E)$  is done in [19]. This result is then extended in [18] to any transient Markov process  $X$  with continuous finite Green function  $g$  (without assumption of symmetry) with respect to a reference measure  $m$ . In this general framework,  $\nu$  denotes the Lévy measure of  $\frac{1}{2}\psi$  for  $\psi$  permanental process with kernel  $g$  and index 1/2. The measure  $\nu$  is supported by the set of measurable paths  $y$  from  $E$  to  $\mathbb{R}_+$ . There was an attempt to give the expression of  $\nu$  in [11] but as explained in [9] we actually computed the Lévy measure of  $(L_\infty^{(a,a)}(x), x \in E)$ . In [9] we have established that the Lévy measure of  $L_\infty^{(a,a)}$  is  $2\nu(dy)1_{\{y(a)>0\}}$ . Besides in [9], we do not need the continuity of the Green function to obtain the expression of  $\nu$  established in [18].

The measure  $\nu$  equals the law of the occupation times under the so-called loop measure. More precisely, let  $\mu$  be defined by

$$\mu(F) = \int_E \mathbb{P}_x \left[ \int_0^\infty \frac{1}{t} F \circ k_t d_t L_t^x \right] dm(x), \quad (2.5)$$

where  $k_t$  is the killing operator at time  $t$  defined on a path  $(z(t), t \geq 0)$  by  $k_t z(s) = z(s)$  if  $s < t$  and  $k_t z(s) = \Delta$  if  $s \geq t$  ( $\Delta \notin E$ ), and  $(L_t^x, x \in E, t \geq 0)$  is the local time process of  $X$  such that for any  $x, y$  in  $E$ :  $\mathbb{E}_x[L_\infty^y] = g(x, y)$ . One denotes by  $d_t L_t^x$  the nonnegative measure generated by the increasing process  $(L_t^x, t \geq 0)$ .

We mention that when  $X$  has transition densities, there is an alternative expression of  $\mu$  in terms of bridge measures (see [18]). We also mention a first appearance of the loop measure in [25].

The measure  $\mu$  is supported by a set of right continuous with left limits  $E$ -valued paths  $\gamma$  with finite length  $t(\gamma)$  such that  $\gamma(0) = \gamma(t(\gamma)_-) = \gamma(t(\gamma))$  i.e. a set of "loops" (see [18]). A loop  $\gamma$  in the support of  $\mu$  admits occupation times  $(\ell(\gamma)(x), x \in E)$ . Moreover they satisfy:

$$\int_0^{t(\gamma)} f(\gamma_s) ds = \int_E \ell(\gamma)(x) f(x) m(dx),$$

for every bounded nonnegative with compact support function  $f$ .  
The measure  $\nu$  is then given by

$$\nu(F) = \frac{1}{2}\mu(F(\ell)) = \frac{1}{2} \int_E \mathbb{P}_x \left[ \int_0^\infty \frac{1}{t} F(L_t) d_t L_t^x \right] dm(x). \quad (2.6)$$

The satisfying point with (2.6) is that  $\frac{1}{2}\psi$  appears then as the field of occupation times of PPP( $\frac{1}{2}\mu$ ). Indeed, one has:

$$\left( \frac{1}{2}\psi_x, x \in E \right) \stackrel{(\text{law})}{=} \left( \sum_{y \in \text{PPP}(\nu)} y(x), x \in E \right) \stackrel{(\text{law})}{=} \left( \sum_{\gamma \in \text{PPP}(\frac{1}{2}\mu)} \ell(\gamma)(x), x \in E \right). \quad (2.7)$$

In [18], the isomorphism theorem (2.2) is then presented as a simple application of the Palm formula to PPP( $\frac{1}{2}\mu$ ) and some computations under the loop measure. Note that this proof of the isomorphism theorem boils down to deduce it from the infinite divisibility of  $\psi$ . Although (2.2) looks then as the law of the sum of two independent occupation time processes, the use of the Palm formula to obtain it prevents from a convincing interpretation.

Taking up the chronological description of some of the Dynkin type isomorphism theorems, we present now Sznitman's identity in its simplest version. Let  $(\eta_x, x \in \mathbb{Z}^d)$  be a centered Gaussian process with covariance the Green function of the time continuous simple random walk  $X$  on  $\mathbb{Z}^d$  ( $d \geq 3$ ), then for every  $r > 0$ :

$$\left( \frac{1}{2}\eta_x^2 + L^r(x), x \in \mathbb{Z}^d \right) \stackrel{(\text{law})}{=} \left( \frac{1}{2}(\eta_x + \sqrt{2r})^2, x \in \mathbb{Z}^d \right), \quad (2.8)$$

where  $L^r$  is the field of the occupation times of the random interlacements of  $X$  at level  $r$ , independent of  $\eta$ . The random interlacements at level  $r$  is a PPP with intensity  $r\kappa$  where  $\kappa$  is a measure on the doubly-infinite nearest neighbors trajectories on  $\mathbb{Z}^d$  modulo time-shift (see [23] for a full description).

Sznitman deduced (2.8) from (2.1) (see also [19]) by first considering the vertex sets of an increasing sequence  $(K_n)_{n \geq 0}$  of connected compact sets of the graph  $\mathbb{Z}^d$  (e.g. for  $n \in \mathbb{N}$ ,  $K_n = \{k \in \mathbb{Z} : -n \leq k \leq n\}^d$ ) and identifying the boundary of the compact set  $K_n$  to a point  $a_n$ , to obtain a sequence of recurrent Markov processes. For each  $n$ , (2.1) is available and (2.8) is obtained by sending  $n$  to  $\infty$ .

In [14], we have shown that (2.8) is actually (2.1) applied to a recurrent Markov process  $(Y_t, t \geq 0)$  with state space  $\mathbb{Z}^d \cup \{\delta\}$  ( $\delta \notin \mathbb{Z}^d$ ) such that  $(L^r(x), x \in \mathbb{Z}^d)$  has the same law as the local time process of  $Y$  starting from  $\delta$ , at the first time the local time at  $\delta$  is greater than  $r$ .

We have established an identity analogue to (2.8) for any transient Markov process  $X$  admitting local times [14] [15]. We will detailed the obtained identity in section 3.3.

Recently, we have established [9] the following Dynkin type isomorphism theorem for any transient Markov process  $X$  admitting a finite Green function  $g$ . Denote by  $g_{T_a}$

the Green function of  $X$  killed at the first time  $X$  hits  $a$  element of its state space  $E$  and by  $L_\infty^{(a,a)}$  the process of the total accumulated local times of  $X$  starting from  $a$ , up to its last visit to  $a$ . Let  $\Psi_g$  and  $\Psi_{g_{T_a}}$  be two permanental processes with index 1 and respective kernels  $g$  and  $g_{T_a}$ , independent of  $X$  (see the definition of permanental processes (2.3)). One has then:

$$\left(\frac{1}{2}\Psi_{g_{T_a}}(x) + L_\infty^{(a,a)}(x), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_g(x), x \in E\right). \quad (2.9)$$

To establish (2.9) we took inspiration from a general decomposition of nonnegative infinitely divisible processes [9]. We have shown [9] that (2.1) and more generally (2.4) are immediate consequences of (2.9). Actually (2.9) can generate several other identities. But as we will emphasize in the next section, now our interest in (2.9) is that it allows to catch a key to visualize some of these identities as consequences of the splitting of loop soups into independent pieces.

### 3 Interpretations of some Dynkin type identities

#### 3.1 Identity (2.9)

We now present an interpretation of (2.9) which will allow to better understand the content of some others Dynkin type isomorphism theorems.

Consider the loop measure  $\mu$ , defined by (2.5), of a transient Markov process  $X$  with a finite Green function  $g$  with respect to a measure  $m$ . Select a point  $a$  in the state space to write:  $\mu(d\gamma) = \mu(d\gamma)1_{\{a \in \gamma\}} + \mu(d\gamma)1_{\{a \notin \gamma\}}$ . One has:

$$\text{PPP}(\mu) \stackrel{\text{a.s.}}{=} \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) \cup \text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}). \quad (3.1)$$

Note that the above equality is just the splitting of  $\text{PPP}(\mu)$  according to  $a$  belongs to the loop or not. It is not an equality in law. Moreover the two components are independent since they do not intersect.

Then one takes the occupation times field on both sides of (3.1) to obtain an equality which is not an equality in law (we abbreviate occupation times field by O.T.):

$$\text{O.T of } \text{PPP}(\mu) \stackrel{\text{a.s.}}{=} \text{O.T of } \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) + \text{O.T of } \text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}), \quad (3.2)$$

with the two occupation times fields on the right hand side independent.

**Proposition 3.1** *The translation in terms of law of (3.2) is the identity (2.9).*

**Proof** One knows thanks to (2.7) that: O.T of  $\text{PPP}(\mu) \stackrel{(\text{law})}{=} \frac{1}{2}\Psi_g$ , where  $\Psi_g$  is a permanental process with index 1 and kernel  $g$ . Besides we have established in [9] that: O.T of  $\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) \stackrel{(\text{law})}{=} L_\infty^{(a,a)}$ , and

$$\text{O.T of } \text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \stackrel{(\text{law})}{=} \frac{1}{2}\Psi_{g_{T_a}}, \quad (3.3)$$

where  $\Psi_{g_{T_a}}$  is a permenal process with index 1 and kernel  $g_{T_a}$  the Green function of  $X$  killed at  $T_a$ .  $\square$

**Remark 3.2** Note that (3.1) is richer than (3.2) and can give rise to many other identities. As an example, for  $\Lambda$  compact subset of  $E$ , denote by  $N(\Lambda)$  the number of loops of PPP( $\mu$ ) included in  $\Lambda$ . The Poisson variable  $N(\Lambda)$  can not be expressed in terms of occupation times of the points in  $\Lambda$  since loops intersecting  $\Lambda$  contribute to these quantities. Assume that  $a \notin \Lambda$ , then one obtains:

$$(\text{O.T of PPP}(\mu)|N(\Lambda)) \stackrel{\text{a.s.}}{=} \text{O.T of PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) + (\text{O.T of PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}|N(\Lambda)),$$

with the two occupation times fields on the right hand side independent.

### 3.2 Generalized second Ray-Knight Theorem

In [9], we have shown that the generalized second Ray-Knight Theorem (2.1), and its extension to the non-symmetric case (2.4), are immediate consequences of (2.9). But a posteriori one can also directly justify their existence by using random sets of paths which are conditioned loop soups.

Consider a recurrent Markov process  $X$  with state space  $E$ , admitting a local time process  $(L_t^x, x \in E, t \geq 0)$  with respect to a reference measure  $m$ . We fix a point  $a$  in  $E$ , and use the notation of (2.4). Set  $\tau_r = \inf\{t > 0 : L_t^a > r\}$  and  $\tau_e = \inf\{t > 0 : L_t^a > \mathbf{e}\}$ , where  $\mathbf{e}$  is an exponential variable with mean 1, independent of  $X$ .

Let  $\mu$  be the loop measure of  $X$  killed at  $\tau_e$ , hence

$$\mu(F) = \int_E \mathbb{P}_x \left[ \int_0^{\tau_e} \frac{1}{t} F \circ k_t d_t L_t^x \right] dm(x). \quad (3.4)$$

One writes:  $\mu(d\gamma) = \mu(d\gamma)1_{\{a \notin \gamma\}} + \mu(d\gamma)1_{\{a \in \gamma\}}$ . One has:

$$\text{PPP}(\mu(d\gamma)) \stackrel{\text{a.s.}}{=} \text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \cup \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}),$$

the two components on the right hand being independent.

Let  $\mathcal{L}(a)$  be the occupation time of  $a$  by PPP( $\mu(d\gamma)$ ):  $\mathcal{L}(a) = \sum_{\gamma \in \text{PPP}(\mu)} \ell(\gamma)(a)$ . Note that  $\mathcal{L}(a)$  is also equal to the occupation time of  $a$  by PPP( $\mu(d\gamma)1_{\{a \in \gamma\}}$ ), and is hence independent of PPP( $\mu(d\gamma)1_{\{a \notin \gamma\}}$ ). One has:

$$(\text{PPP}(\mu(d\gamma)), \mathcal{L}(a)) \stackrel{\text{a.s.}}{=} (\text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \cup \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}, \mathcal{L}(a)),$$

and hence:

$$\begin{aligned} (\text{PPP}(\mu(d\gamma)) | \mathcal{L}(a)) &\stackrel{\text{a.s.}}{=} (\text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \cup \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}} | \mathcal{L}(a))) \\ &= (\text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \cup (\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}} | \mathcal{L}(a))) \end{aligned}$$



This implies for almost every  $r \geq 0$

$$(\text{PPP}(\mu) | \mathcal{L}(a) = r) \stackrel{\text{a.s.}}{=} \text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \cup (\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) | \mathcal{L}(a) = r) \quad (3.5)$$

with the two random sets on the right hand side independent.

We then take the occupation time field on both sides of (3.5) to obtain:

$$\begin{aligned} \text{O.T of } (\text{PPP}(\mu) | \mathcal{L}(a) = r) &\stackrel{\text{a.s.}}{=} \text{O.T of } \text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \\ &+ \text{O.T of } (\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) | \mathcal{L}(a) = r) \end{aligned} \quad (3.6)$$

**Proposition 3.3** *The translation in terms of law of (3.6) is*

$$\left(\frac{1}{2}\Psi \mid \frac{1}{2}\Psi(a) = r\right) \stackrel{(\text{law})}{=} \frac{1}{2}\Psi_{g_{T_a}} + L_{\tau_r} \text{ under } \mathbb{P}_a \quad (3.7)$$

where  $L_{\tau_r}$  is the local time process of  $X$  at time  $\tau_r$ , and  $\Psi$  and  $\Psi_{g_{T_a}}$  are permanental process with index 1 and respective kernels  $g_{T_a} + 1$  and  $g_{T_a}$ , with  $\Psi_{g_{T_a}}$  independent of  $L_{\tau_r}$ .

Before proving Proposition 3.3, we show how to obtain the second Ray-Knight Theorem (2.1) from (3.7). In the symmetric case (3.7) is:

$$\frac{1}{2}\|\varphi + \sqrt{r}\|^2 \stackrel{(\text{law})}{=} \frac{1}{2}\|\varphi\|^2 + L_{\tau_r} \text{ under } \mathbb{P}_a \quad (3.8)$$

where  $\varphi = (\eta, \tilde{\eta})$  is a 2-dimensional centered Gaussian process independent of  $X$ , with  $\eta$  and  $\tilde{\eta}$  two iid centered Gaussian processes with covariance  $g_{T_a}$  ( $\varphi + \sqrt{r} = (\eta + \sqrt{r}, \tilde{\eta} + \sqrt{r})$ ).

Actually (3.8) has exactly the same content as the generalized second Ray-Knight Theorem (2.1). More generally (3.7) has the same content as (2.4). Indeed we know [9] that  $\Psi_{g_{T_a}} \stackrel{(\text{law})}{=} (\Psi \mid \Psi(a) = 0)$ . We use then the following additivity property of permanental processes with index 1/2 (see Remark 2.5.1 in [13]). For  $(\phi_x, x \in E)$  and  $(\tilde{\phi}_x, x \in E)$  two independent permanental process with index 1/2 and kernel  $g_{T_a} + 1$ , one has:

$$(\phi \mid \phi_a = p) + (\tilde{\phi} \mid \tilde{\phi}_a = \tilde{p}) \stackrel{(\text{law})}{=} (\phi \mid \phi_a = q) + (\tilde{\phi} \mid \tilde{\phi}_a = \tilde{q}) \stackrel{(\text{law})}{=} (\Psi \mid \Psi_a = p + \tilde{p})$$

for every nonnegative  $p, \tilde{p}, q, \tilde{q}$  such that  $p + \tilde{p} = q + \tilde{q}$ .

One has in particular:

$$\left(\frac{1}{2}\Psi \mid \frac{1}{2}\Psi(a) = r\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\phi \mid \frac{1}{2}\phi_a = r\right) + \left(\frac{1}{2}\tilde{\phi} \mid \frac{1}{2}\tilde{\phi}_a = 0\right)$$

and obviously:

$$\left(\frac{1}{2}\Psi \mid \frac{1}{2}\Psi(a) = 0\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\phi \mid \frac{1}{2}\phi_a = 0\right) + \left(\frac{1}{2}\tilde{\phi} \mid \frac{1}{2}\tilde{\phi}_a = 0\right)$$

Together these two identities with (3.7) give (2.4):

$$\left(\frac{1}{2}\phi \mid \frac{1}{2}\phi_a = r\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\phi \mid \frac{1}{2}\phi_a = 0\right) + L_{\tau_r} \text{ under } \mathbb{P}_a$$

In particular when  $X$  is symmetric, one has:  $\left(\frac{1}{2}\phi \mid \frac{1}{2}\phi_a = r\right) \stackrel{(\text{law})}{=} \frac{1}{2}(\eta + \sqrt{2r})^2$ , which gives the generalized second Ray Knight Theorem of [16].

**Proof** To prove Proposition 3.3, it is sufficient to establish the following identities:

- (i) O.T. of  $(\text{PPP}(\mu(d\gamma)) \mid \mathcal{L}(a) = r) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi \mid \frac{1}{2}\Psi(a) = r\right)$ ,  
where  $\Psi$  is a permenantal process with index 1 and kernel  $g_{T_a} + 1$ .
- (ii) O.T. of  $\text{PPP}(\mu(d\gamma)1_{\{a \notin \gamma\}}) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_{g_{T_a}}(x), x \in E\right)$ ,  
where  $\Psi_{g_{T_a}}$  is a permenantal process with index 1 and kernel  $g_{T_a}$ .
- (iii) O.T. of  $(\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) \mid \mathcal{L}(a) = r) \stackrel{(\text{law})}{=} (L_{\tau_r}^x, x \in E)$  under  $\mathbb{P}_a$ .

The computation of the law of the occupation time field of a random set of finite length paths  $\mathcal{R}$  boils down to the computation of the Laplace transform of  $F(\mathcal{R})$  for any functional  $F$  given by  $F(\gamma) = \int_0^{t(\gamma)} f(\gamma(s))ds$ , where  $f$  is nonnegative function on  $E$  (and  $F(\mathcal{R}) = \sum_{\gamma \in \mathcal{R}} F(\gamma)$ ). Denote by  $\chi_r$  the conditioned loop soup  $(\text{PPP}(\mu) \mid \mathcal{L}(a) = r)$ . One has:  $\mathbb{E}[\exp\{-F(\chi_r)\}] = \mathbb{E}[\exp\{-F(\text{PPP}(\mu))\} \mid \mathcal{L}(a) = r]$ , hence one can claim that

$$\text{O.T. of } (\text{PPP}(\mu) \mid \mathcal{L}(a) = r) \stackrel{(\text{law})}{=} (\text{O.T. of } \text{PPP}(\mu) \mid \mathcal{L}(a) = r).$$

We already know that the field of occupation time of  $\text{PPP}(\mu)$  has the law of a permenantal process with index 1 and kernel  $g_{\tau_e}$ , the Green function of  $X$  killed at  $\tau_e = \inf\{t > 0 : L_t^a > \mathbf{e}\}$ . As it has been shown in [12]:  $g_{\tau_e} = g_{T_a} + 1$ . Hence one has: O.T. of  $\text{PPP}(\mu) \stackrel{(\text{law})}{=} \frac{1}{2}\Psi$ , which implies:

$$\text{O.T. of } (\text{PPP}(\mu) \mid \mathcal{L}(a) = r) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi \mid \frac{1}{2}\Psi(a) = r\right).$$

Similarly:

$$\text{O.T. of } (\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) \mid \mathcal{L}(a) = r) \stackrel{(\text{law})}{=} (\text{O.T. of } \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) \mid \mathcal{L}(a) = r).$$

Since:  $(\text{O.T. of } \text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}), \mathcal{L}(a)) \stackrel{(\text{law})}{=} (L_{\tau_e}^{(a,a)}, \mathbf{e}) \stackrel{\text{a.s.}}{=} (L_{\tau_e}, \mathbf{e})$  under  $\mathbb{P}_a$ , one obtains

$$\text{O.T. of } (\text{PPP}(\mu(d\gamma)1_{\{a \in \gamma\}}) \mid \mathcal{L}(a) = r) \stackrel{(\text{law})}{=} L_{\tau_r} \text{ under } \mathbb{P}_a$$

Hence (i) and (iii) are established.

To obtain (ii) one first notes that  $X$  killed at  $\tau_e$  and then killed at its first hitting time of  $a$  is  $X$  killed at its first hitting time of  $a$ . One uses then (3.3) to conclude.  $\square$

### 3.3 Sznitman's identity

Sznitman wrote the identity (2.8) in a larger framework than continuous time simple symmetric random walks on  $\mathbb{Z}^d$ . Indeed in [23]  $\mathbb{Z}^d$  can be replaced by the vertex set of any locally finite, connected, transient weighted graph  $G$ . The corresponding continuous time random walks all share the following continuity property: a jump between two vertices  $x$  and  $y$  is allowed if  $[x, y]$  is an edge of  $G$ . Lupu [19] extended then (2.8) to Brownian motion of such graphs  $G$ . In [14] [15] we could extend Sznitman's identity to a much more general framework. For sake of clarity we limit ourselves to the symmetric case.

Let  $X$  be a transient Markov process with state space  $E$ , a measurable metric space, admitting a finite symmetric Green function  $g$  with respect to a reference measure  $m$  such that:

- either  $X$  is a standard process,
- or  $E$  is discrete.

For such an  $X$ , there exists a positive function  $q$  on  $E$  such that  $\tilde{m}(dx) = q(x)m(dx)$  is finite and  $\int_E g(x, x)\tilde{m}(dx) < \infty$ , and there exists a recurrent symmetric Markov process  $(Y_t)_{t \geq 0}$  with state space  $E \cup \{a^*\}$  ( $a^* \notin E$ ) admitting a local time process on  $E$ ,  $(L_t^x(Y), x \in E \cup \{a^*\}, t \geq 0)$  satisfying for every  $f$  with compact support in  $E$ :  $\int_0^t f(Y_s)ds = \int_E L_t^x(Y)f(x)\tilde{m}(dx)$ , and for every  $b$  in  $E$ :

$$(L_\infty^x(X), x \in E | X_0 = b) \stackrel{(\text{law})}{=} (L_{\tilde{T}_{a^*}}^x(Y), x \in E | Y_0 = b) \quad (3.9)$$

where  $\tilde{T}_{a^*}$  is the first hitting time of  $a^*$  by  $Y$ .

Moreover for every  $r > 0$ , one has under  $\mathbb{P}[\cdot | Y_0 = a^*]$ :

$$\left(\frac{1}{2}\eta_x^2 + L_{\tilde{\tau}_r}^x(Y), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2r})^2, x \in E\right) \quad (3.10)$$

where  $\tilde{\tau}_r = \inf\{t \geq 0 : L_t^{a^*}(Y) > r\}$  and  $(\eta_x, x \in E)$  is a centered Gaussian process with covariance  $g$  independent of  $Y$ .

To establish (3.10) it has been shown (Remark 2.7 and Lemma 2.8 in [15]) that for any  $x, y$  in  $E \cup \{a^*\}$ :

$$\mathbb{E}[L_{\tilde{\tau}_e}^y(Y) | Y_0 = x] = g(x, y) + 1. \quad (3.11)$$

where  $\tilde{\tau}_e = \inf\{t \geq 0 : L_t^{a^*}(Y) > e\}$ .

Besides one has extended in [10] the definition of random interlacements to such processes  $X$  under the additional assumption that  $X$  admits a weak dual with respect to  $m$ . We have showed that  $(L^r(x), x \in E)$  the field of occupation times at level  $r$  of the random interlacements of  $X$  satisfies:

$$(L^r(x), x \in E) \stackrel{(\text{law})}{=} (L_{\tilde{\tau}_r}^x(Y), x \in E) \text{ under } \mathbb{P}[\cdot | Y_0 = a^*]$$

which gives the following extension of Sznitman's identity (2.8):

$$\left(\frac{1}{2}\eta_x^2 + L^r(x), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2r})^2, x \in E\right). \quad (3.12)$$

We would like to show now that (3.10) (and hence (3.12)) can be obtained by splitting some loop soup. The part of the point  $a$  in section 3.2 should be played by  $a^*$  the additional point. One extends the measure  $\tilde{m}$  to  $E \cup \{a^*\}$  by setting:  $m^*(dx) = \tilde{m}(dx) + \delta_{a^*}(dx)$ . In view of (3.9), one can set:  $g(x, a^*) = g(a^*, x) = 0, \forall x \in E \cup \{a^*\}$ . We consider the process  $(Y_t, t \geq 0)$  killed at  $\tilde{\tau}_{\mathbf{e}}$  where  $\tilde{\tau}_{\mathbf{e}} = \inf\{t \geq 0 : L_t^{a^*}(Y) > \mathbf{e}\}$ , with  $\mathbf{e}$  an exponential variable with mean 1 independent of  $Y$ . In general  $Y$  killed at  $\tilde{\tau}_{\mathbf{e}}$  does not satisfy the assumptions required by Theorem 3.1 in [18] nor the ones of Theorem 1.5 in [9]. We will make use instead of the results established in [9] for a general nonnegative infinitely divisible process (Theorems 1.1 and 1.2 in [9]).

Let  $\tilde{\mu}$  be the measure on the set of measurable  $E \cup \{a^*\}$ -valued paths  $\gamma$  with finite length  $t(\gamma)$  such that  $\gamma(0) = \gamma(t(\gamma))$  defined by:

$$\tilde{\mu}(F) = \int_{E \cup \{a^*\}} (g(y, y) + 1) \mathbb{E}\left[\frac{F(Y_s, 0 \leq s \leq \lambda(y))}{\int_{E \cup \{a^*\}} L_{\lambda(y)}^x(Y) m^*(dx)} \mid Y_0 = y\right] m^*(dy) \quad (3.13)$$

where  $\lambda(y) = \sup\{t \in [0, \tilde{\tau}_{\mathbf{e}}] : Y_t = y\}$ .

A loop  $\gamma$  in the support of  $\tilde{\mu}$  admits a field  $(\ell(\gamma)(x), x \in E \cup \{a^*\})$  of occupation times. One writes:  $\tilde{\mu}(d\gamma) = \tilde{\mu}(d\gamma)1_{\{a^* \notin \gamma\}} + \tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}$ . One has:

$$\text{PPP}(\tilde{\mu}(d\gamma)) \stackrel{\text{a.s.}}{=} \text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \notin \gamma\}}) \cup \text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}),$$

the two components on the right hand being independent since they do not intersect.

Let  $\mathcal{L}(a^*)$  be the occupation time of  $a^*$  by  $\text{PPP}(\tilde{\mu}(d\gamma))$ :  $\mathcal{L}(a^*) = \sum_{\gamma \in \text{PPP}(\tilde{\mu})} \ell(\gamma)(a^*)$ . Note that  $\mathcal{L}(a^*)$  is also equal to the occupation time of  $a^*$  by  $\text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}})$ , and is hence independent of  $\text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \notin \gamma\}})$ . This implies for almost every  $r \geq 0$ :

$$(\text{PPP}(\tilde{\mu}) \mid \mathcal{L}(a^*) = r) \stackrel{\text{a.s.}}{=} \text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \notin \gamma\}}) \cup (\text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}) \mid \mathcal{L}(a^*) = r) \quad (3.14)$$

with the two random sets on the right hand side independent.

We then take the occupation time field on  $E$  on both sides of (3.14) to obtain:

$$\begin{aligned} \text{O.T of } (\text{PPP}(\tilde{\mu}) \mid \mathcal{L}(a^*) = r) &\stackrel{\text{a.s.}}{=} \text{O.T of } \text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \notin \gamma\}}) \\ &+ \text{O.T of } (\text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}) \mid \mathcal{L}(a^*) = r). \end{aligned} \quad (3.15)$$

**Proposition 3.4** *Assume that either  $E$  is discrete or the Green function  $g$  is continuous. Then the translation in terms of law of (3.15) is*

$$\left(\frac{1}{2}\Psi \mid \frac{1}{2}\Psi(a^*) = r\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_g(x) + L_{\tilde{\tau}_r}^x(Y), x \in E\right) \text{ under } \mathbb{P}[\mid Y_0 = a^*] \quad (3.16)$$

where  $\Psi$  and  $\Psi_g$  are permanental processes with index 1 and respective kernels  $g + 1$  and  $g$ , and  $\Psi_g$  is independent of  $L_{\tilde{\tau}_r}(Y)$ .

By Corollary 2.5 in [13], one knows that:  $(\Psi_g(x), x \in E) \stackrel{(\text{law})}{=} (\Psi(x), x \in E | \Psi(a^*) = 0)$ . One shows then exactly as in section 3.2 that (3.16) has the same content as (3.10) and (3.12).

**Proof** To establish Proposition 3.4 it is sufficient to prove the following identities:

- (i) O.T. of  $(\text{PPP}(\tilde{\mu}(d\gamma)) | \mathcal{L}(a^*) = r) \stackrel{(\text{law})}{=} (\frac{1}{2}\Psi(x), x \in E | \frac{1}{2}\Psi(a^*) = r)$ ,  
where  $\Psi$  is a permanental process with index 1 and kernel  $g + 1$ .
- (ii) O.T. of  $(\text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \notin \gamma\}}) \stackrel{(\text{law})}{=} (\frac{1}{2}\Psi_g(x), x \in E)$ ,  
where  $\Psi_g$  is a permanental process with index 1 and kernel  $g$ .
- (iii) O.T. of  $(\text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}) | \mathcal{L}(a^*) = r) \stackrel{(\text{law})}{=} (L_{\tilde{\tau}_r}^x(Y), x \in E)$  under  $\mathbb{P}[\cdot | Y_0 = a^*]$ .

The permanental process  $(\Psi_x, x \in E \cup \{a^*\})$  with index 1 and kernel  $(g(x, y) + 1, (x, y) \in (E \cup \{a^*\})^2)$  is nonnegative and infinitely divisible. In order to use Theorem 1.2 in [9],  $\Psi$  would have to be stochastically continuous which is a priori irrelevant for a process indexed by  $E \cup \{a^*\}$ . Looking closer at the proof of Theorem 1.2 [9], we see that we need that given  $\int_{E \cup \{a^*\}} \Psi_x m^*(dx) = 0$ ,  $\Psi$  must admit an identically equal to 0 version. But:  $\int_{E \cup \{a^*\}} \Psi_x m^*(dx) = \int_E \Psi_x m(dx) + \Psi_{a^*}$ , hence given  $\int_{E \cup \{a^*\}} \Psi_x m^*(dx) = 0$ , one has:  $\int_E \Psi_x m(dx) = 0$  and  $\Psi_{a^*} = 0$ . Since  $g$  is continuous on  $E^2$ , one shows as in [9] that  $\Psi|_E$  admits an identically equal to 0 version. Consequently one can make use of Theorem 1.2 [9].

The Lévy measure  $\tilde{\nu}$  of  $(\frac{1}{2}\Psi_x, x \in E \cup \{a^*\})$  is hence provided by Theorem 1.2 [9]:

$$\tilde{\nu}(F) = \int_{E \cup \{a^*\}} \mathbb{E}\left[\frac{F(r^{(b)})}{\int_{E \cup \{a^*\}} r^{(b)}(x) m^*(dx)}\right] \mathbb{E}\left[\frac{1}{2}\Psi_b\right] m^*(db)$$

where for every  $b \in E \cup \{a^*\}$ , the law of  $r^{(b)}$  is characterized by :

$$\frac{1}{2}\Psi + r^{(b)} \stackrel{(\text{law})}{=} \frac{1}{2}\Psi \text{ under } \mathbb{E}\left[\frac{\Psi_b}{\mathbb{E}[\Psi_b]}, \cdot\right]. \quad (3.17)$$

On the other hand for any  $b$  in  $E \cup \{a^*\}$ , one writes (2.2) for  $Y$  killed at  $\tilde{\tau}_e$  under  $\mathbb{P}[\cdot | Y_0 = b]$ :

$$\left(\frac{1}{2}\Psi_x + L_{\lambda^{(b)}}^x(Y), x \in E \cup \{a^*\}\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_x, x \in E \cup \{a^*\}\right) \text{ under } \mathbb{E}\left[\frac{\Psi_b}{\mathbb{E}[\Psi_b]}, \cdot\right] \quad (3.18)$$

The identity (2.2) is available in this case since  $Y$  satisfies the Markov property at inverse local times and  $\tilde{\tau}_e$ , and one has (3.11) (see [15]). Hence one obtains that the

process  $r^{(b)}$  involved in (3.17) has the same law as  $L_{\lambda(b)}(Y)$  under  $\mathbb{P}[\cdot | Y_0 = b]$ . One obtains:

$$\tilde{\nu}(F) = \int_{E \cup \{\delta\}} (g(y, y) + 1) \mathbb{E}\left[\frac{F(L_{\lambda(b)}^x(Y), x \in E \cup \{a^*\})}{\int_{E \cup \{\delta\}} L_{\lambda(y)}^x(Y) m^*(dx)} \mid Y_0 = b\right] m^*(dy).$$

Consequently:  $\tilde{\nu}(F) = \tilde{\mu}(F(\ell(\gamma)))$ , where for a loop  $\gamma$  in the support of  $\tilde{\mu}$ ,  $(\ell(\gamma)(x), x \in E \cup \{a^*\})$  denotes the occupation time of  $\gamma$ .

One obtains:

$$\begin{aligned} \left(\frac{1}{2}\Psi_x, x \in E \cup \{a^*\}\right) &\stackrel{(\text{law})}{=} \left(\sum_{w \in \text{PPP}(\tilde{\nu})} w(x), x \in E \cup \{a^*\}\right) \\ &\stackrel{(\text{law})}{=} \left(\sum_{\gamma \in \text{PPP}(\tilde{\mu})} \ell(\gamma)(x), x \in E \cup \{a^*\}\right) \end{aligned} \quad (3.19)$$

According to (3.19):

$$\text{O.T. of } \text{PPP}(\tilde{\mu}(d\gamma) | \mathcal{L}(a^*) = r) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_x, x \in E \mid \frac{1}{2}\Psi_{a^*} = r\right)$$

Note that:  $\tilde{\mu}(F(\gamma)1_{a^* \notin \gamma}) = \tilde{\mu}(F(\gamma)1_{\ell(\gamma)(a^*)=0})$ . Hence thanks to Theorem 1.2 in [9], one has:

$$\text{O.T. of } \text{PPP}(\tilde{\mu}(d\gamma)1_{a^* \notin \gamma}) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_x, x \in E \mid \frac{1}{2}\Psi_{a^*} = 0\right)$$

(i) and (ii) are hence established.

Choosing  $b = a^*$  in (3.18), one has under  $\mathbb{P}[\cdot | Y_0 = a^*]$ :

$$\left(\frac{1}{2}\Psi_x + L_{\tilde{\tau}_e}^x(Y), x \in E \cup \{a^*\}\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}\Psi_x, x \in E \cup \{a^*\}\right) \text{ under } \mathbb{E}\left[\frac{\Psi_{a^*}}{\mathbb{E}[\Psi_{a^*}]}, \cdot\right]$$

Knowing that the Lévy measure of  $\frac{1}{2}\Psi$  is  $\tilde{\nu}$ , one easily obtains from the above identity that the Lévy measure of  $L_{\tilde{\tau}_e}(Y)$  under  $\mathbb{P}[\cdot | Y_0 = a^*]$  is  $\tilde{\nu}(y)1_{\{y(a^*) > 0\}}$  (just compute the finite dimensional Laplace transforms on both sides). Since:

$$\text{O.T. of } \text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}) = \left(\sum_{w \in \text{PPP}(\tilde{\nu}(dy)1_{\{y(a^*) > 0\}})} w(x), x \in E \cup \{a^*\}\right)$$

one obtains:

$$\text{O.T. of } \text{PPP}(\tilde{\mu}(d\gamma)1_{\{a^* \in \gamma\}}) \stackrel{(\text{law})}{=} (L_{\tilde{\tau}_e}^x(Y), x \in E \cup \{a^*\}) \text{ under } \mathbb{P}[\cdot | Y_0 = a^*]$$

which leads to (iii).  $\square$

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