



HAL
open science

Generalized exchange operators for a system of spin-1 particles

Michel L. Rouleux, Charlie Jeudy

► **To cite this version:**

Michel L. Rouleux, Charlie Jeudy. Generalized exchange operators for a system of spin-1 particles. Integrable Systems and Quantum Symmetries (ISQS-28), Cestmir Burdik, Jul 2024, Prague (CZ), Czech Republic. hal-04769849

HAL Id: hal-04769849

<https://cnrs.hal.science/hal-04769849v1>

Submitted on 6 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Generalized exchange operators for a system of spin-1 particles

Charlie Jeudy & Michel Rouleux

November 6, 2024

Aix-Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
charlie.jeudy@etu.univ-amu.fr, rouleux@univ-tln.fr

Abstract

The irreps $(\text{SU}(2), \mathcal{H}, U)$ of $\text{SU}(2)$ of dimension $(2S + 1)^N$, i.e. operators acting on the space $\mathcal{H} = \mathcal{H}_N = \mathbf{C}^{(2S+1)^N}$ of N identical particles with spin S , are described by Clebsch-Gordan decomposition into inequivalent irreps. In the special case $S = 1/2$, Dirac [1] discovered that there is another rep given by $(\mathcal{S}(N), \mathcal{H}, V)$ where $\mathcal{S}(N)$ is the permutation group. Thus, the standard “linear” Hamiltonian, or Heisenberg interaction Hamiltonian $H_0 = \sum_{1 \leq i < j \leq N} \vec{S}_i \cdot \vec{S}_j$, where $\vec{\sigma}_i = 2\vec{S}_i$ is the vector of Pauli matrices, can be interpreted as the sum of the “Exchange Operators” P_{ij} between particles i and j . Schrödinger [3] generalized to higher spin numbers S the Exchange Operator $P_{ij} = P_S(\vec{S}_i \cdot \vec{S}_j)$ as a polynomial of degree $2S$ in $\vec{S}_i \cdot \vec{S}_j$. This we call the P -representation. There is another rep induced by the one particle permutation of states operators \tilde{Q}_α , which we call the Q -rep. Our main purpose is to write some physical Hamiltonians for a few particles in the P - or Q -rep and compute their spectrum. The simplest case where there are as many particles as available states for the spin operator along the z -axis, i.e. $N = 2S + 1 = 3$, see Weyl [4] or Hamermesh [5]. Finally, we consider the relationship between permutations and rotation invariance when $S = 1/2$ and $S = 1$.

1 Rotational degeneracy and multiplets : a review

The situation is best understood for spin $1/2$, which fortunately encompasses a great variety of particles. However spin 1 particles, such as the gauge W_\pm

bosons, not to speak of atoms [16], [17] play also an important role.

Recall that the irreducible representations (irreps) $(\text{SU}(2), \mathcal{H}, U)$ of $\text{SU}(2)$ of dimension $(2S + 1)^N$, i.e. operators acting on the space $\mathcal{H} = \mathbf{C}^{(2S+1)^N}$ of N identical particles with spin S , are described by Clebsch-Gordan decomposition into inequivalent irreps defined inductively by $D \otimes D' = \bigoplus_{\rho} m_{\rho} D^{(\rho)}$. The group $\text{SU}(2)$ enters as a rotational symmetry group of the system.

Characters of these irreps are well known [6].

The easiest case is met for spin $S = 1/2$, where $\text{SU}(2)$ acts directly on the N particles wave function $u = u_1 \otimes \cdots \otimes u_N$ by $A \mapsto U(A) : (u \mapsto Au_1 \otimes \cdots \otimes Au_N)$. This gives the unitary rep $(\text{SU}(2), \mathcal{H}, U)$.

The permutation group $\mathcal{S}(N)$, acting as $\tau \mapsto V(\tau) : (u \mapsto (u_{\tau(1)} \otimes \cdots \otimes u_{\tau(N)}))$ enters as another symmetry group, because the particles are identical. This gives the rep $(\mathcal{S}(N), \mathcal{H}, V)$. Correspondence between reps of $\text{SU}(2)$ and $\mathcal{S}(N)$ is called Schur-Weyl duality.

We have $V(\tau)U(A) = U(A)V(\tau)$. The reps $(\mathcal{S}(N), \mathcal{H}, V)$ and $(\text{SU}(2), \mathcal{H}, U)$ are mapped onto each other by Schur-Weyl duality and are equal up to a multiple of identity, see [13] Satz 2.47.

This was discovered by Dirac [1], [2], independently of the theory of group representations set up a bit earlier by Weyl [4], and expressed in terms of the so-called *exchange operator* between spin-1/2 particles 1 and 2 defined as

$$P(12) = \frac{1}{2}(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

where $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma_1^x \otimes \sigma_2^x + \sigma_1^y \otimes \sigma_2^y + \sigma_1^z \otimes \sigma_2^z$ (Pauli matrices), and $\vec{\sigma}_i = 2\vec{S}_i$ is the vector $(\sigma_i^x, \sigma_i^y, \sigma_i^z)$ is the vector of Pauli matrices at site i .

We recognize the characteristic action of Exchange Operators given by the rep $P(12)$,

$$P(12)|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, \quad P(12)|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle, \quad P(12)|\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle, \quad P(12)|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

so that $(P(12))^2 = 1$.

Following [6], we call *class operators* of a finite group those elements of the corresponding group algebra that one obtained by summing all the group elements that belong to the same conjugacy class -they are also known as “conjugacy class sums”. Thus the standard (linear) interaction Hamiltonian for N identical spin 1/2 particles (i.e. Heisenberg model with complete graph interaction, including “self interaction”, and suitably normalized), is the class operator associated with transpositions $(ij) \in \mathcal{S}(N)$:

$$H_0(1/2, N) = \sum_{1 \leq i < j \leq N} P(ij) \quad (2)$$

Here we have identified $\tau \in \mathcal{S}(N)$ with $V(\tau)$. Dirac also allowed for orbital momentum. In total space $L^2(\mathbf{R}^{3N}) \otimes \mathcal{H}$, the wave function is invariant under simultaneous exchange of position and spin variables, so the exchange operator is relative to the spin variables alone. For simplicity we consider throughout Hamiltonians depending on spin variables only.

More generally, let \vec{S} be the vector of spin- S matrices, generators of the spin representation of $\mathfrak{su}(2)$ of dimension $2S+1$. For $S = 1/2$, $\vec{S} = \frac{1}{2}\vec{\sigma}$ are 1/2 times Pauli matrices, for $S = 1$, S^x, S^y, S^z are linear combinations of Gell-Mann matrices. Thus we define the standard linear interaction Hamiltonian for N identical spin S particles as

$$H_0(S, N) = C(N) + 2 \sum_{1 \leq i < j \leq N} \vec{S}_i \cdot \vec{S}_j \quad (3)$$

where the multiple of identity $C(N)$ is a sum of self-interaction term $\vec{S}_i^2 = \vec{S}_i \cdot \vec{S}_i$ as in (2).

The spectrum of $H_0(S, N)$ is computed according to the following ‘‘Multiplet Method’’ in a suitable CSCO (so-called *un-coupled* or *coupled basis*), which makes use of Clebsch-Gordan decomposition ([7], [6], [8]). Recall that the coupled basis for N spin particles is given by $b = ((\vec{S}_i^2)_{1 \leq i \leq N}, J^z, \vec{J}^2)$, where $\vec{J} = \sum_{i=1}^N \vec{S}_i = (J^x, J^y, J^z)$, and $\vec{J}^2 = (\sum_{i=1}^N \vec{S}_i)^2$. The quantum numbers are (S, M) , S runs over half-integers and

$$\begin{aligned} J^z |M, S, (S_i)_{1 \leq i \leq N}\rangle &= M |M, S, (S_i)_{1 \leq i \leq N}\rangle, \quad -S \leq M \leq S \\ \vec{J}^2 |M, S, (S_i)_{1 \leq i \leq N}\rangle &= S(S+1) |M, S, (S_i)_{1 \leq i \leq N}\rangle \end{aligned} \quad (4)$$

The correlation between particles arises when computing $\vec{J}^2 - N\vec{S}^2 = 2 \sum_{i < j} \vec{S}_i \cdot \vec{S}_j$, so that we can directly deduce the spectrum of H_0 from this of \vec{J}^2 .

For $N = 2$ electrons of spin $1/2$, $\vec{S}^2 = \vec{\sigma}^2/4 = 3/4$, and composition of angular momentum shows that \vec{J}^2 has eigenvalues $s \in \{1/2 + 1/2 = 1, 1/2 - 1/2 = 0\}$, so that $\vec{S}_1 \cdot \vec{S}_2 = (\vec{J}^2 - 3/2)/2$, has eigenvalues λ with multiplicity $[\cdot]$: $\{\lambda = -3/4[1], \lambda = 1/4[3]\}$. For $N = 3$, \vec{J}^2 has eigenvalues $s \in \{1/2 + 1/2 + 1/2 = 3/2, 1/2 + 1/2 - 1/2 = 1/2\}$, so that $\vec{S}_1 \cdot \vec{S}_2$ has eigenvalues λ with multiplicity $[\cdot]$: $\{\lambda = -3/4[4], \lambda = 3/4[4]\}$ and the *sum rule* gives again $1/2 \otimes 1/2 \otimes 1/2 = 1/2 \oplus 1/2 \oplus 3/2$.

For $S = 1$, $N = 3$ we can split off \mathbf{C}^{27} into eigenspaces of $H_0(1, 3)$, their dimension is given by the sum rule:

$$1 \otimes 1 \otimes 1 = (0 \oplus 1 \oplus 2) \otimes 1 = 1 \oplus 0 \oplus 1 \oplus 2 \oplus 1 \oplus 2 \oplus 3$$

namely 1 scalar (multiplicity 1), 3 spins 1 (multiplicity 9), 2 spin 2 (multiplicity 10) and 1 spin 3 (multiplicity 7). For larger (S, N) the Multiplet method can be implemented in many ways (Young diagrams, graphs, ...)

However, for $S \geq 1$, $H_0(S, N)$ can no longer be interpreted *a priori* as a class operator for the action of $\mathcal{S}(N)$. So we need another algebraic framework, to possibly reinterpret this Hamiltonian, as well as other rotation invariant (physical) ones.

2 Exchange operators, P -rep and related Hamiltonians for higher order spin systems

Dirac exchange operators were generalized by Schrödinger [3], for higher spin systems of N particles of spin S , in the form $P_S(\vec{S}_i \cdot \vec{S}_j)$ where \vec{S} is the vector of spin- S matrices, $\vec{S} = \frac{1}{2}\vec{\sigma}$ for $S = 1/2$, and $P_S(x)$ a polynomial of degree $2S$, given by the Ansatz, see also H.A. Brown [9]:

$$P_S(x) = (-1)^{2S} \left(1 + \sum_{p=1}^{2S} \frac{(-2)^p}{(p!)^2} \prod_{q=1}^p (x - x_q) \right) = \sum_{n=0}^{2S} A_n x^n \quad (5)$$

where $x_q = \frac{1}{2}q(q-1) - S(S+1)$, $q = 1, 2, \dots, 2S+1$, are the eigenvalues of x , and $P_S(x_q) = (-1)^{2S+q-1}$. We can then form the multi-particle (non-linear) Hamiltonian (Schrödinger Hamiltonian)

$$H_S(S, N) = \sum_{\langle i, j \rangle} P_S(\vec{S}_i \cdot \vec{S}_j) \quad (6)$$

Since $H_S(S, N)$ depends only on the scalar products $\vec{S}_i \cdot \vec{S}_j$, it trivially commutes with the total rotations of \mathbf{R}^3 (moving particles all together).

Such Hamiltonians generalizing Heisenberg model, are commonly discussed in Statistical Mechanics. See also [10] for non linear Ising models, using instead polynomials $Q_S(S_i^z)$ which are constructed along the same pattern as $P_S(\vec{S}_i \cdot \vec{S}_j)$.

By composing exchange operators $P_S(\vec{S}_i \cdot \vec{S}_j)$ relative to transpositions, we easily get :

Proposition 1: *Let $\mathcal{H} = \mathbf{C}^{(2S+1)^N}$, there exists a unitary rep $(\mathcal{S}(N), \mathcal{H}, V_S)$ of dimension $(2S+1)^N$, defined on generators (ij) of $\mathcal{S}(N)$ by $V_S : (ij) \mapsto V_S(ij) = P_S(\vec{S}_i \cdot \vec{S}_j)$.*

We denote $V_S((12)(23)) = \text{Pol}_S(123)$, etc. the corresponding operator valued polynomials. Thus $\text{Pol}_S(123)$ is of degree 3 with respect to $\vec{S}_i \cdot \vec{S}_j$.

The Class- k Operator associated with $(\mathcal{S}(N), \mathcal{H}, V_S)$ is defined by summing over all g_k elements belonging to the class k . This gives

$$C_k = \sum_{\ell=1}^{g_k} \text{Pol}_S(\tau_\ell)^{(k)}$$

where τ_ℓ ranges over the g_k Young diagrams. Example with $N = 3$: $g_0 = 1$ (identity element) $g_1 = 3$ (transpositions), $g_2 = 2$ (circular permutations). In particular for $(S, N) = (1, 3)$, $\text{Pol}_S(123)$ permutes circularly all spins of particles 1,2,3, namely $\text{Pol}_S(123)|\uparrow\uparrow\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle$, $\text{Pol}_S(123)|\downarrow\uparrow\uparrow\rangle = |\uparrow\uparrow\downarrow\rangle$, etc. We have $\text{Pol}_S(123)^3 = \text{Id}$.

This we call the P -representation. As in the spin-1/2 case (Dirac) we can retrieve Schrödinger Hamiltonian $H_S(S, N)$ as the C_1 class operator in the P -rep. Recall that there is some basis of \mathbf{C}^3 where the CSOC-I (see [6], p.53) rep matrices $D(C_i)$ of all class operators for $\mathcal{S}(3)$ are diagonal, in particular $D(C_1)$ has eigenvalues $(-3, 0, 3)$, see [6], p.56.

The spectrum of $H_S(1, 3)$ cannot be computed by the Multiplet Method as for the linear Hamiltonian $H_0(1, 3)$. For 3 particles of spin 1 however we get, with the help of Mathematica (Symbolic Calculus on tensor products) the following:

Proposition 2: For $(S, N) = (1, 3)$, Schrödinger Hamiltonian $C_1 = H_S(1, 3)$ (sum over transpositions) has the same eigenvalues $\rho \in \{-3, 0, 3\}$ as those constituting the CSOC-I (see [6], p.53) in the group space of $\mathcal{S}(3)$. Consequently, we have the decomposition into irreducible spaces \mathcal{L}_ρ

$$\mathbf{C}^{27} = 8\mathcal{L}_0 \oplus 10\mathcal{L}_3 + \mathcal{L}_{-3} \quad (7)$$

where \mathcal{L}_0 has dimension 2, \mathcal{L}_3 and \mathcal{L}_{-3} have dimension 1. In particular, the character in the P -representation associated with C_1 is $\chi_P(C_1) = \text{Tr}(C_1) = \text{Tr}(H_S(1, 3)) = 27$.

Note that m_ρ factors are deduced by dividing multiplicities of each eigenvalues ρ by the dimension of the irreps $D^{(\rho)}$.

3 State permutation group, Q - and O -reps

Instead of exchanging particles, we can permute their states. This gives another representation, which was pointed out by [4], [5]. It is most easily understood when there are as many particles as available states for the spin

operator along some given direction, i.e. $N = 2S + 1$, see [6], p.69.

1) *The Q -rep*

Definition 4: We call \tilde{Q} -representation of $\mathcal{S}(2S + 1)$ the permutation representation $(\mathcal{S}(2S + 1), \mathbf{C}^{2S+1}, \tilde{Q})$ defined on transpositions (ij) , i.e. $\tilde{Q}(ij)$ is the $(2S + 1) \times (2S + 1)$ transposition matrix on the one particle space, that exchanges the spin state i and j . We call Q -representation the product representation $(\mathcal{S}(2S + 1), \mathcal{H}, Q)$, $\mathcal{H} = \mathbf{C}^{(2S+1)^N}$ the N -lift of \tilde{Q} , defined on transpositions (ij) by $Q(ij) = \tilde{Q}(ij) \otimes \dots \otimes \tilde{Q}(ij)$ (N times).

When $S = 1$,

$$\tilde{Q}(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{Q}(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{Q}(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the operators $\tilde{Q}_{123} = \tilde{Q}_{12}\tilde{Q}_{23}$ and $\tilde{Q}_{132} = \tilde{Q}_{12}\tilde{Q}_{13}$ are associated with cycles.

Actually the permutation matrices $\tilde{Q}(ij)$ span a Lie algebra in the algebra of 3×3 matrices, which is idempotent of order 3, i.e. an iterated commutator of order 3 and beyond is spanned by $\tilde{Q}(\alpha)$, $[\tilde{Q}(\alpha), \tilde{Q}(\beta)]$ and $[\tilde{Q}(\alpha), [\tilde{Q}(\beta), \tilde{Q}(\gamma)]]$ only.

The P - and Q -reps are not equivalent, as shows the case $(S, N) = (1/2, 2)$. Namely the exchange operator $P(12) = P_S(S_1 \cdot S_2)$ (in the canonical basis) is given by (1), and its diagonal basis is the coupled basis, while the matrix of permutation of states in Q -representation, associated with the bi-cycle (14)(23), is

$$Q(12) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The diagonal basis for $Q(12)$ is Bell's state basis

$$b_1 = \{(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)\} \oplus \left\{ \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle), \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right\}$$

We can check that the linear Hamiltonian is not in the Q -rep, namely for any (c_0, c_1) , $H_0(1/2, 2) = P(12) \neq c_0 1 + c_1 Q(12)$.

At least in case of $(S, N) = (1/2, N)$ and $(S, N) = (1, 3)$ it is easy to see that the $Q(ij)$ span a Lie algebra L of operators on the total space \mathcal{H} and that the Cayley table for $[Q(\alpha), Q(\beta)]$ is identical to this of $[\tilde{Q}(\alpha), \tilde{Q}(\beta)]$.

Moreover since that the CSCO of the P - and Q -reps are the same in this case (see [6], p.70), we expect that Schrödinger Hamiltonian P_S , as well as other Hamiltonians, belongs to L .

2) The O -rep

We can also apply the *generator lifting* for \tilde{Q} operators. For $\sigma \in \mathcal{S}(2S+1)$, define for $i = 1, \dots, N$

$$O_i = 1^{\otimes(i-1)} \otimes \tilde{Q}(\sigma) \otimes 1^{\otimes(N-i)}$$

which span a Lie algebra by acting only on particle i . Thus for two spin-1/2 particles, we get two commuting operators O_1, O_2 , with $O_1^2 = O_2^2 = 1$, and then form $O_1 O_2$ which belong to its universal enveloping algebra \mathcal{L} in Q -rep. It turns out that the linear Hamiltonian $H_0(1/2, 2)$ belongs to \mathcal{L} , namely $H_0(1/2, 2) = \frac{1}{2}(1 - O_1 - O_2 + O_1 O_2)$, and $H_0(1/2, 3) = \frac{3}{2}(1 - O_1 O_2 O_3)$. However, given a Hamiltonian H , checking the condition $H \in \mathcal{L}$ requires to solve for many c_k , and the situation of course worsens for higher spin and particle numbers.

This procedure generalizes to N particles of spin S , but it is difficult to identify the Hamiltonians obtained this way which are rotation invariant.

It is interesting to investigate how the permutation group $\mathcal{S}(N)$ and $SU(2)$ are related in the *Universal Enveloping Algebra* (UEA), see [15]. The motivation comes from the fact that, for any semi-simple Lie group, we can define a quadratic invariant J^2 by using the appropriate Killing form so that for any $g \in G$, $[J^2, g] = 0$. For a physical system where we define the action of a rep $\rho(G)$, this is associated with a conserved quantity under the symmetry transformations given by G . In particular $J^2 = \vec{J}^2$ in our case with $G = SU(2)$. These elements, however, do not belong to the Lie algebra itself but rather to the UEA. We have seen that $J^2 \sim H_0(S, N)$ (Heisenberg Hamiltonian) and $H_S \sim C_1$ (Schrödinger Hamiltonian), So the C_1 class operator plays the same role as J^2 in determining the irreps of $\mathcal{S}(N)$ and $SU(2)$. Therefore, we are interested in whether we can identify them directly within a specific UEA. However, the challenge is to construct an algebra where both J^2 and C_1 exist within one of its subspaces, see Sect.5.

4 Relationship between permutations and rotational invariance

We can consider the cartesian product $(SU(2))^N$ the full rotation (semi-simple) group of the system, acting independently on each particle, and

associated with direct sum $\mathfrak{su}(2)_N = \bigoplus^N(\mathfrak{su}(2))$. A *partial rotation*, denoted $U(R_i)$, is an operator that rotates the i -th particle only, while leaving the other particles unaffected. By construction, this forms a normal subgroup of $(SU(2))^N$. The group of *total rotations* (same rotation applied on each particle) is a subgroup of $(SU(2))^N$. Its Lie algebra contains the set $(\vec{S}_i, \vec{S}_i^2)_{1 \leq i \leq N}$ defined in (4).

Our first result is to express $P_S(\vec{S}_i \cdot \vec{S}_j)$ in terms of the coupled basis.

When $N = 2$, for all S , $P_S(\vec{S}_i \cdot \vec{S}_j)$ is (trivially) diagonal in the coupled basis, or in the basis diagonalizing \vec{J}^2 . For N particles of spin S we can check by Symbolic Calculus for “small” values of (S, N) the following:

Proposition 5: *For $S = 1/2, 1$ we have:*

1) $P_S(\vec{S}_i \cdot \vec{S}_j)$ commutes with all elements of the coupled basis (CSCO-II of $SU(2)$), see [6], p.79:

$$[P_S(\vec{S}_i \cdot \vec{S}_j), \vec{J}^2] = [P_S(\vec{S}_i \cdot \vec{S}_j), \vec{J}^z] = [P_S(\vec{S}_i \cdot \vec{S}_j), \vec{S}_k^2] = 0$$

2) Moreover, $P_S(\vec{S}_i \cdot \vec{S}_j)$ commutes with $SU(2)$, more precisely with the (global) rotations $U(A) : (u_1, \dots, u_N) \mapsto (Au_1 \otimes \dots \otimes Au_N)$. When $S = 1$, A can be identified (locally) with an element of $SO(3)$.

We complete the coupled basis b to $\bar{b} = \{(\vec{S}_i^2)_{1 \leq i \leq N}, \vec{J}^2, \vec{J}^z, \overline{J}_z\}$, which constitute the CSCO-III of $SU(2)$ with \overline{J}_z being one generator of the intrinsic group, see [6], p.79. If $[P_S(\vec{S}_i \cdot \vec{S}_j), \overline{J}_z] = 0$, then we could infer from Dirac theorem on joint spectrum of commuting operators that $P_S(\vec{S}_i \cdot \vec{S}_j) = F_{ij}((\vec{S}_i^2)_{1 \leq i \leq N}, \vec{J}^2, \vec{J}^z, \overline{J}_z)$ where F_{ij} can be chosen as a polynomial (Lagrange interpolation polynomial in $N + 2$ variables). Possibly also we can remove J^z from the variables due to the invariance of $\vec{S}_i \cdot \vec{S}_j$ under rotation generated by $SU(2)$.

Next we investigate the commutation relations between partial rotations and exchange operators. Together with the global rotations above, we consider 3 types of partial rotations. Let $A \in SO(3)$

1) Parot(k) : $(u_1, \dots, u_N) \mapsto (u_1 \otimes \dots \otimes Au_k \otimes u_{k+1} \otimes \dots \otimes u_N)$ (rotating only particle k).

2) BiParot(ij) : $(u_1, \dots, u_N) \mapsto (u_1 \otimes \dots \otimes Au_i \otimes u_{i+1} \otimes \dots \otimes Au_j \otimes \dots \otimes u_N)$ (rotating only particles i and j)

3) Gear(ij) : $(u_1, \dots, u_N) \mapsto (u_1 \otimes \dots \otimes Au_i \otimes u_{i+1} \otimes \dots \otimes A^{-1}u_j \otimes \dots \otimes u_N)$ (rotating only particles i and j in opposite sense).

We have :

Proposition 6: For $S = 1/2, 1$ we have:

- 1) $[P_S(\vec{S}_i \cdot \vec{S}_j), \text{Parot}(k)] = 0$ for all i, j, k .
- 2) $[P_S(\vec{S}_i \cdot \vec{S}_j), \text{BiParot}(ij)] = 0$ for all i, j .

Next we list some Casimir operators (still for small (S, N)) :

Proposition 7: For $S = 1/2, 1$ there holds:

- 1) $\text{Parot}(k)$ is a normal subgroup of $G = \text{SU}(2) \times \dots \times \text{SU}(2)$.
- 2) $\text{BiParot}(ij)$ is a normal subgroup of $\text{SU}(2) \approx \{(A, \dots, A) : A \in \text{SU}(2)\}$.
- 3) For all k , Casimir operator $\text{Parot}(k) = S_k^2 \in \mathfrak{b}$
- 4) For all (i, j) , $(S_i + S_j)^2$ is a Casimir operator for $\text{BiParot}(i, j)$, but $(S_i + S_j)^2 \notin \text{Span}\{\mathfrak{b}\}$.
- 5) For all (i, j) , $(S_i - S_j)^2$ is a Casimir operator for $\text{Gear}(i, j)$, but $(S_i - S_j)^2 \notin \text{Span}\{\mathfrak{b}\}$.

Now we examine the relationship between the permutation group $\mathcal{S}(N)$ and some enveloping algebra.

5 Class operators C_n and universal enveloping algebra $U(\mathfrak{su}(2)_N)$

We denote by $U(\mathfrak{g})$ the *Universal Enveloping Algebra* UEA of a Lie algebra \mathfrak{g} , defined as the associative algebra of polynomials in elements of \mathfrak{g} , where \mathfrak{g} is the Lie algebra of a semi-simple Lie group. Consider here $G = G_1 = \text{SU}(2)$ and its Lie algebra $\mathfrak{g}_1 = \mathfrak{su}(2)$, which is also the Lie algebra of $\text{SO}(3)$.

Thanks to Poincaré–Birkhoff–Witt theorem, a basis of $U(\mathfrak{su}(2))$ can be constructed from the infinite set of elements $\{J_x^n J_y^m J_z^\ell\}_{n,m,\ell \in \mathbf{N}_0}$.

The UEA of a semi-simple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where \mathfrak{g}_1 and \mathfrak{g}_2 are simple, is thus given by the isomorphism $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$. So for $G = (\text{SU}(2))^N$ (direct product), and $\mathfrak{g} = \bigoplus^N \mathfrak{su}(2)$ (direct sum), a basis of $U(\mathfrak{su}(2)_N)$ is given by $\{\prod_{i=1}^N (J_x^n J_y^m J_z^\ell)_i\}_{n,m,\ell \in \mathbf{N}_0}$.

We may ask the following question: Can the group space of $\mathcal{S}(N)$ be mapped to a space of polynomials (e.g. the family of Schrödinger polynomials indexed by the quantum number S), valued in $U(\mathfrak{su}(2)_N)$, with the additional property that the class operators (for the action of $\mathcal{S}(N)$) belong to the centralizer of the group of total rotations (identified with $\text{SU}(2)$)? In particular, $C_1 \equiv J^2$, see Proposition 7.

These class operators can be represented as exchange operators (P -rep),

are all invariant under rotations, and commute with Heisenberg Hamiltonian. They would define quantum numbers characterizing collective property of spin systems, and we can also study their link with *generalized Casimir operators* of $\mathfrak{su}(2)_N$

Note also that there is another one particle irrep $(\text{SU}(2), \mathcal{H}_{2S}, W)$ of $\text{SU}(2)$ of dimension $2S + 1$, where \mathcal{H}_{2S} is the space of homogeneous polynomials of degree $2S$ of 2 complex variables, namely $W(A)p(Z) = p(AZ)$, $Z = {}^t(x, y)$. Taking tensor products this lifts to the $(2S + 1)^N$ dimensional rep

$$W_N(A)(p_1 \otimes \cdots \otimes p_N)(Z_1, \cdots, Z_N) = p_1(AZ_1) \cdots p_N(AZ_N)$$

We obtain in the same way another rep of $\mathcal{S}(N)$, permuting the variables Z_i .

6 A sub-representation of Lie algebra (L, \mathcal{H}, Q) and its dual space

Recall the Q -rep is a representation of the Lie algebra L of (lifted) permutation matrices. We restrain here L to a smaller class \widehat{L} , and examine the spectral properties of the Hamiltonians it contains. We expect $H_S(S, N) \in \widehat{L}$, as well as other physical Hamiltonians. Commutators in the Q -rep will play the role of polynomials in the P -rep. To fix the ideas we assume $(S, N) = (1, 3)$.

Consider first the Lie algebra \widetilde{L} (called the “small” or “parent” algebra) of all operators on \mathbf{C}^3 generated by the \widetilde{Q}_α , and recall from Sect.3.1 that \widetilde{L} is idempotent of order 3, and its lift L is spanned by the $Q_\alpha = \widetilde{Q}_\alpha \otimes \cdots \otimes \widetilde{Q}_\alpha$ is such that Cayley Tables for commutators in L coincide with this in \widetilde{L} .

For $\alpha = (ij)$ or $\alpha = (ijk)$ we choose ε_α according to the rules of a group algebra, i.e. $\varepsilon_\alpha = 1$ if α is a permutation, and $\varepsilon_\alpha = -1$ if α is a cycle, and we set

$$J_1 = \sum_{\alpha} \varepsilon_{\alpha}^3 Q_{\alpha} = Q_{12} + Q_{13} + Q_{23} - Q_{123} - Q_{132} \quad (8)$$

$$J_2 = 1, \quad J_3 = [J_1, [J_1, J_1]']' = \sum' \varepsilon_{\alpha}^3 \varepsilon_{\beta}^3 \varepsilon_{\gamma}^3 [Q_{\alpha}, [Q_{\beta}, Q_{\gamma}]]$$

where $[J_1, J_1]' = \sum_{\alpha < \beta} \varepsilon_{\alpha}^3 \varepsilon_{\beta}^3 [Q_{\alpha}, Q_{\beta}]$, and \sum' means a sum over some ordered subset α, β, γ , which reduces in the present case to $(\alpha, \beta, \gamma) = ((12), (13), (23))$, and $J_3 = 2[Q_{12}, [Q_{13}, Q_{23}]]$. Thus J_1 is the first class operator C_1 of the group algebra. We check that $[J_1, J_3] = 0$, so that $L_1 = \text{Vect}(J_1, J_2, J_3)$ is a commutative, unitary 3-D Lie algebra. In the parent algebra the corresponding class operators take the form $\widetilde{J}_1 = \widetilde{J}_2 = \text{Id}$, and $\widetilde{J}_3 = 2[\widetilde{Q}_{12}, [\widetilde{Q}_{13}, \widetilde{Q}_{23}]] =$

$2(\tilde{Q}_{13} - \tilde{Q}_{12})$. The spectral values λ with multiplicity $[m]$ of any Hamiltonian $H = c_1 J_1 + c_3 J_3 \in L_1$ is readily computed from Symbolic Calculus, ($\lambda = -3c_1[4], \lambda = 3c_1[5], \lambda = -4\sqrt{3}c_3[9], \lambda = 4\sqrt{3}c_3[9]$) but none of these Hamiltonians (up to a constant) is of the form $H_0(1, 3)$ or $H_S(1, 3)$ and probably not rotation invariant.

To remedy this fact, we introduce the dual space L_1^* (for the canonical Hermitian product $\langle A, B \rangle = \text{Tr}AB^*$) defined in the following way. In the space of all real symmetric 3×3 matrices, we construct a dual family \tilde{q}_β of the symmetric part of the \tilde{Q}_α 's, consisting of 4 terms $\tilde{q}_{12}, \tilde{q}_{13}, \tilde{q}_{23}, \tilde{q}_{123} = \tilde{q}_{132}$, which are real symmetric matrices. It is thus only defined modulo some 2-D subspace $\text{Vect}(\tilde{M}_1, \tilde{M}_2)$, \tilde{M}_1, \tilde{M}_2 real symmetric. Then we consider the lift q_α of the \tilde{q}_α 's, and for $\varepsilon = -1$ or 0 the dual class operators $k_1 = q_{12} + q_{13} + q_{13} + \varepsilon(q_{123} + q_{132})$, $k_2 = [k_1, k_1]'$ and $k_3 = [k_1, [k_1, k_1]]'$ as in (8). Up to a change of basis $(k_1, k_2, k_3) \mapsto (j_1, j_2, j_3)$ we can fulfill the orthogonality conditions $\langle j_\alpha, J_\beta \rangle = \text{Tr}(j_\alpha J_\beta) = \delta_{\alpha, \beta}$, also when modifying the \tilde{q}_α 's by a linear combination of matrices \tilde{M}_1, \tilde{M}_2 . Thus we get the dual space L_1^* of L_1 . It remains to show that L_1^* is also a Lie algebra, and that $L_1 \oplus L_1^*$ becomes a Lie bi-algebra. We expect L_1^* to be also commutative, so that $L_1 \oplus L_1^*$ identifies with a Euclidean phase-space from which we can retrieve $\text{SU}(2)$ (or rather $\text{SO}(3)$ locally) by integration. Now we can form $H = c_1 J_1 + c_2 J_2 + c_3 J_3 + d_1 j_1 + d_2 j_2 + d_3 j_3$ and try to characterize those which are rotational invariant. Thus (examining the multiplicity of eigenvalues) we strongly expect that Schrödinger Hamiltonian) $H_S(1, 3) \in \hat{L} = L_1 + L_1^*$, and that there are other Hamiltonians in $L_1 + L_1^*$ whose spectrum is integer valued as well, up to trivial factors. This reflects the fact that all irreps of $\mathcal{S}(N)$ are rational.

Acknowledgements: We thank Walter Aschbacher, Robert Coquereaux, Janos Polonyi, Laurent Raymond and Hervé Ricard for useful discussions. The first Author acknowledges financial support from QuantEdu-France during his Master 2 Internship at Aix-Marseille Université.

References

- [1] Dirac P 1929 Quantum Mechanics of Many-Electrons Systems, Proc. R.Soc.Lond. A 1923, p.714-733.
- [2] Dirac P 1958 *Quantum Mechanics* 4th. ed (Oxford Univ. Press).
- [3] Schrödinger E 1941 Proc. R.Irish Acad. **47** 39.

- [4] Weyl H 1946 *The Classical Groups* (Princeton Univ. Press)
- [5] Hamermesch M 1962 *Group Theory and its Applications to Physical Problems* (Addison-Wesley, Reading Mass)
- [6] Jin-Quan Chen 1989 *Group Representation Theory for Physicists* (World Scientific)
- [7] Cohen Tannoudji C, Diu B and Laloë F 2000 *Mécanique Quantique I, II* 2nd ed. (Editions CNRS)
- [8] Rose M 1957 *Elementary Theory of Angular Momentum* (John Wiley and Sons)
- [9] Brown H.A 1972 American J. of Phys. 40, p.1696-1697
- [10] Joseph R.I 1976 Phys. Rev. B **13** 9, p.4043-4045.
- [11] Per-Olov Löwdin 1955 Phys. Review, **97** 6, p.1474-1489.
- [12] Rouleux M 2019 Multimode entanglement for fermions. arxiv.org/abs/1910.04790
- [13] Jost R 1973 *Quantenmechanik II. Vorlesungen*. ETH Zürich Abteilung für Mathematik und Physik.
- [14] Kirillov A 1976 *Eléments de la Théorie des Représentations* (Mir, Moscou)
- [15] Fulton W and Harris J 1991 *Representation Theory* Springer.
- [16] Pavlishchuk V, Kolotilov S, Addison A, Prushan M, Schollmeyer D, Thompson L, Weyhermüller T and Goreshnik E 2003 Dalton Trans., 2003, 1587–1595
- [17] Ananikian N, Babujyan H, Burdik C, Papoyan V, and Ananikyan L 2024 This Conference.