Fundamental solutions for parabolic equations and systems: universal existence, uniqueness, representation

Pascal Auscher, Khalid Baadi

FUNDAMENTAL SOLUTIONS FOR PARABOLIC EQUATIONS AND SYSTEMS: UNIVERSAL EXISTENCE, UNIQUENESS, REPRESENTATION

PASCAL AUSCHER AND KHALID BAADI

Abstract. In this paper, we develop a universal, conceptually simple and systematic method to prove well-posedness to Cauchy problems for weak solutions of parabolic equations with non-smooth, time-dependent, elliptic part having a variational definition. Our classes of weak solutions are taken with minimal assumptions. We prove the existence and uniqueness of a fundamental solution which seems new in this generality: it is shown to always coincide with the associated evolution family for the initial value problem with zero source and it yields representation of all weak solutions. Our strategy is a variational approach avoiding density arguments, a priori regularity of weak solutions or regularization by smooth operators. One of our main tools are embedding results which yield time continuity of our weak solutions going beyond the celebrated Lions regularity theorem and that is addressing a variety of source terms. We illustrate our results with three concrete applications : second order uniformly elliptic part with Dirichlet boundary condition on domains, integro-differential elliptic part, and second order degenerate elliptic part.

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1. Introduction

Linear parabolic problems have a long history. The standard method usually begins by solving the Cauchy problem, with or without source terms, and representing the solutions through what is called fundamental solution. Abstractly, starting from an initial data u_0, the evolution takes the form $\partial_t u + \mathcal{B}u = f$ where time describes an interval $(0,T)$, the sought solution u and the source f are valued in some different vector spaces and $\mathcal{B}$ is an operator that could also depend on time. Modeled after the theory of ordinary scalar differential equations, it is assumed some kind of dissipativity for $-\mathcal{B}$. One issue toward extension to non-linear equations is to not use regularity, but measurability, of the coefficients. We stick here to the linear ones. The amount of literature is too vast to be mentioned and we shall isolate only a few representative works for the sake of motivation.

In the abstract case, when $\mathcal{B}$ does not depend on time, that is the autonomous situation, the approach from semi-group of operators and interpolation has been fruitful, starting from the earlier works (see Kato's book [Kat13]) up to the criterion for maximal regularity in UMD Banach spaces (see L. Weis [Wei01] or the review by Kunstmann-Weis [KW04]). In the non-autonomous case, results can be obtained by perturbation of this theory, assuming some time-regularity; see, for example, Kato's paper [Kat61].

A specific class of such problems is when $\mathcal{B}$ originates from a sesquilinear form with coercivity assumptions. Lions developed an approach in [Lio57] which he systematized in [Lio13]. The nice thing about this approach is that there is no need for regularity with respect to time; its drawback is that it is restricted to Hilbert initial spaces.

In parallel, there has been a systematic study of concrete parabolic Cauchy problems of differential type starting when the coefficients are regular. In this case, several methods exist for constructing the fundamental solution. The most effective technique involves a parametrix in combination with the freezing point method [Fri08]. This approach simplifies the problem to one where the coefficients become independent of space, leading to explicit solutions represented by kernels $\Gamma(t,x,s,y)$ with Gaussian decay. When the coefficients are measurable (and possibly unbounded for the lower order terms), the theory of weak solutions, developed in the 1950s and 1960s, applies. This theory culminated in the book of Ladyzenskaja, Solonnikov and Ural'ceva [LSU68]. Although we shall not consider it here, the specific situation of second order parabolic equations with real, measurable, time-dependent coefficients was systematically treated by Aronson [Aro67, Aro68]: his construction of fundamental solutions and the proof of lower and upper bounds relied on regularity properties of local weak solutions by Nash [Nas58] and its extensions, and by taking limits from operators with regular coefficients. A recent result [AN24] follows this approach for non autonomous degenerate parabolic problems in the sense of A_2-weights. In contrast, not using regularity theory, the article [AE23] developed a framework for a Laplacian (and its integral powers), extending the Lions embedding theorem (see below) as a first step to obtain new results on fundamental solutions for equations with unbounded coefficients.

A natural question is whether one can develop a framework, going beyond the one of Lions, that provides us with

(1) An optimal embedding theorem with integral identities,
(2) The largest classes of weak solutions for which one obtains existence and uniqueness,
(3) Definition, existence and uniqueness of a fundamental solution.

The answer is yes. The arguments can be developed in a very abstract manner, bearing on functional calculus of positive self-adjoint operators. We next present a summary of our results as a road map. Clearly, when it comes to concrete applications, our results do not distinguish equations from systems, the nature of the elliptic part (local or non-local) and its order, boundary conditions, etc. We give three examples at the end as an illustration.

The first example deals with second-order uniformly elliptic parts under Dirichlet boundary conditions on domains. This situation may not seem original but we still give the main consequences for readers to be able to compare with literature. Our second example is parabolic equations with integro-differential elliptic part. The typical example is the fractional Laplacian and such equations arise in many fields from PDE to probability [Lio69, CS07, BJ12]. The usual theory for the fractional Laplacians yields the fundamental solution as a density of a probability measure. The fundamental solution for general integro-differential parabolic operators with kernels are considered in the literature, assuming positivity condition and pointwise bounds. We refer to the introduction of [KW23] and the references there in. In that article, a proof of the poinswise upper bound is presented. Here, we do not assume any kind of positivity and we show the existence of a fundamental solution as an evolution family of operators. At our level of generality, these operators may not have kernels with pointwise bounds. Still, this family can be used to represent weak solutions without further assumptions. In any case, it gives a universal existence result. For example, the fundamental solution used in [KW23] must be the kernel of our fundamental solution operator. The third example is for degenerate operators as in [AN24], without assuming the coefficients to be real. It directly gives existence of a fundamental solution not using local properties of solutions. In a forthcoming article, the second author will use this to give a new proof of the pointwise estimates.

2. Summary of our results

2.1. **Embeddings and integral identities.** Lions' embedding theorem [Lio57] asserts that if V and H are two Hilbert spaces such that V is densely embedded in H, itself densely embedded in $V^\star$, the dual of V in the inner product of H, we have the continuous embedding

$$L^2((0,\mathfrak{T});V)\cap H^1((0,\mathfrak{T});V^\star)\hookrightarrow C([0,\mathfrak{T}];H),$$

and absolute continuity of the map $t\mapsto \|u(t)\|_H^2$, which yields integral identities. The triple $(V,H,V^\star)$ is said to be a Gelfand triple. One of our main results is the following improvement.

Theorem 2.1. *Consider a positive, self-adjoint operator on a separable Hilbert space H. Let $I=(0,\mathfrak{T})$ be a bounded, open interval of $\mathbb{R}$. Let $u\in L^1(I;H)$ such that $Su\in L^2(I;H)$. Assume that $\partial_t u = Sf+S^\beta g$ with $f\in L^2(I;H)$ and $g\in L^{\rho'}(I;H)$, where $\beta=2/\rho\in[0,1)$ and ρ' is the conjugate Hölder exponent to ρ. Then $u\in C(\bar I,H)$ and $t\mapsto\|u(t)\|_H^2$ is absolutely continuous on $\bar I$ with, for all $\sigma,\tau\in\bar I$ such that $\sigma<\tau$, the integral identity*

$$\|u(\tau)\|_H^2-\|u(\sigma)\|_H^2=2\mathrm{Re}\int_\sigma^\tau \langle f(t),Su(t)\rangle_H+\langle g(t),S^\beta u(t)\rangle_H\,\mathrm{d}t. \tag{2.2}$$

As a consequence, $u\in L^r((0,\mathfrak{T});D(S^\alpha))$ for all $r\in(2,\infty]$ such that $\alpha=2/r$ with for a constant depending only on β,

$$\|S^\alpha u\|_{L^r((0,\mathfrak{T});H)}\lesssim \|Su\|_{L^2((0,\mathfrak{T});H)}+\|f\|_{L^2((0,\mathfrak{T});H)}+\|g\|_{L^{\rho'}((0,\mathfrak{T});H)}+\inf_{\tau\in[0,\mathfrak{T}]}\|u(\tau)\|_H. \tag{2.3}$$

Let us comment this result. Its core is the continuity and the integral identity proved in Corollary 5.13 when S is injective and Proposition 7.1 in the general case. If u had belonged to $L^2(I;H)$ then $u\in L^2(I;V)$ with V the domain of S. Here we only assume $u\in L^1(I;V)$, which is used qualitatively, and $\partial_t u$ is taken in the sense of distributions on I valued in V^*. We allow an extra second term $S^\beta g$ in the time derivative expression of u. In fact, $S^\beta g$ belongs to $SL^2(I;H)+L^1(I;H)\subset L^1(I;V^*)$ (see Proposition 5.4 when S is injective and the proof is the same otherwise). Note that it could be a finite combination of such terms and the integral identity would be modified accordingly. The L^r estimate (2.3) follows *a posteriori*: one first proves (2.3) with $r=\infty$ using (2.2), together with the interpolation inequality

$$\|S^\alpha u\|_{L^r((0,\mathfrak{T});H)}\le \|Su\|^\alpha_{L^2((0,\mathfrak{T});H)}\|u\|^{1-\alpha}_{L^\infty((0,\mathfrak{T});H)}$$

for $\alpha = \beta$ when $0 < \beta < 1$. Then we reuse this inequality for different $\alpha \in [0, 1)$. See Proposition 5.4 when S is injective together with the remark that follows it and the proof applies *verbatim* when S is not injective.

Theorem 2.1 admits versions on unbounded intervals. On the half-line $(0, \infty)$ (resp. on $\mathbb{R}$), it holds assuming that $u \in L^1((0, \mathfrak{T}); H)$ for all $\mathfrak{T} < \infty$ (resp. $u \in L^1_{\rm loc}(\mathbb{R}; H)$) when S is not injective. In this case, u is bounded in H on $(0, \infty)$ (resp. $\mathbb{R}$), see Proposition 7.1. When S is injective, the local integrability condition of u in H can be dropped using completions of V and V^* for the homogeneous norms $\|Su\|_H$ and $\|S^{-1}u\|_H$. In this case, not only u is bounded, but it also tends to zero at infinity (resp. at $\pm\infty$) in H, as shown in Proposition 5.7 and Corollary 5.10. As a consequence, we can eliminate the last term in (2.3).

Our strategy is to prove the embedding first when $I = \mathbb{R}$, proceed by restriction to $(0, \infty)$. When it comes to bounded intervals, although the condition $u \in L^1(I; H)$ does not appear in (2.3), the condition $Su \in L^2(I; H)$ alone does not suffice as an example shows. We added strong integrability in H but in fact, as the proof shows, it suffices that u exists as a distribution on I valued in $V^\star$ and there exists one t for which $u(t) \in H$. But for applications to Cauchy problems, it is more natural to have the integrability condition and we stick to that.

Our proof of the embedding already has a PDE flavor using $\partial_t u + S^2 u = S^2 u + Sf + S^\beta g$. This leads us to a thorough study of the abstract heat operator $\partial_t + S^2$ (hence the notation $\partial_t u$ rather than u') which has some interest on its own right. This is done in Sections 3, 4, 5.

2.2. **Weak solutions and Cauchy problems.** The embedding and its variants allow us to consider the largest possible class of weak solutions to abstract parabolic operators $\partial_t + \mathcal{B}$ with a time-dependent elliptic part $\mathcal{B}$ associated to a family of bounded and sesquilinear forms on the domain of S. We do not assume any time-regularity on $\mathcal{B}$ apart its weak measurability. This can be done either with estimates being homogeneous in S if we decide to work on infinite intervals, (see Section 6.1) or inhomogeneous (See Section 7.0.3; we called the elliptic operator $\tilde{\mathcal{B}}$ there).

Let us state the final result in the latter case, that is with inhomogeneous $\tilde{\mathcal{B}}$ as in Section 7.0.3, combining Theorems 7.7 and 8.1 on a finite interval $(0, \mathfrak{T})$. We fix $\rho \in (2, \infty)$ and set $\beta = 2/\rho$. Given an initial condition $a \in H$ and source terms $f \in L^2((0, \mathfrak{T}); H)$ and $g \in L^{\rho'}((0, \mathfrak{T}); H)$, $h \in L^1((0, \mathfrak{T}); H)$ we wish to solve the Cauchy problem

$$\begin{cases} \partial_t u + \tilde{\mathcal{B}} u = Sf + S^\beta g + h \ \text{ in } \mathcal{D}'((0, \mathfrak{T}); \mathrm{D}), \\ u(0) = a \ \text{ weakly in D}, \end{cases} \tag{2.4}$$

where D is a Hausdorff topological dense subspace of the domain of S, equipped with the graph norm, that is, a core of $D(S)$. The first equation is thus understood in the weak sense against test functions in $\mathcal{D}((0, \mathfrak{T}); \mathrm{D})$. The meaning of the second equation is by taking the limit $\langle u(t), \tilde{a}\rangle_H \to \langle a, \tilde{a}\rangle_H$ for all $\tilde{a} \in \mathrm{D}$ along a sequence converging to 0.

Theorem 2.5. *There exists a unique weak solution $u \in L^1((0, \mathfrak{T}); H)$ with $Su \in L^2((0, \mathfrak{T}); H)$ to the problem (2.4). Moreover, $u \in C([0, \mathfrak{T}]; H) \cap L^r((0, \mathfrak{T}); D(S^\alpha))$ for all $r \in [2, \infty)$ with $\alpha = 2/r$, and we have the estimate*

$$\sup_{t\in[0,\mathfrak{T}]} \|u(t)\|_H + \|S^\alpha u\|_{L^r((0,\mathfrak{T});H)} \le C(\|f\|_{L^2((0,\mathfrak{T});H)} + \|g\|_{L^{\rho'}((0,\mathfrak{T});H)} + \|h\|_{L^1((0,\mathfrak{T});H)} + \|a\|_H),$$

where C is a constant independent of f, g, h and a. In addition, we can write the energy equality corresponding to the absolute continuity of $t \mapsto \|u(t)\|_H^2$.

Theorem 7.7 also contains a variant on the interval $(0, \infty)$ (case (b) there) where we replace S by the operator $\tilde{S} = (S^2 + 1)^{1/2}$ and we also obtain decay of the solution at ∞ while the class of uniqueness is $u \in L^1((0, \mathfrak{T}); H)$ for all $\mathfrak{T} < \infty$ with $Su \in L^2((0, \infty); H)$.

We note that we consider classes of solutions in $L^1((0, \mathfrak{T}); H)$ rather than $L^\infty((0, \mathfrak{T}); H)$ as is customary. This condition suffices to obtain a priori continuity in time by Theorem 2.1. There is a similar theorem for the backward parabolic adjoint operator $-\partial_s + \tilde{\mathcal{B}}^*$ with final condition $\tilde{a}$ at $\mathfrak{T}$.

The estimates and the energy equality are a consequence of Theorem 2.1. Uniqueness relies on the energy equality. Existence is obtained by restriction from constructions first on $\mathbb{R}$ and then on $(0, \infty)$.

The role of the core D of $D(S)$ is in fact irrelevant here and is only for the purpose of having a weak formulation with a small space of test functions. It can be equivalently replaced by $D(S)$ itself (see Theorem 8.1 and its proof). However, for concrete partial differential equations where D can be taken as a space of smooth and compactly supported functions of the variable x, we can work with smooth and compactly supported functions of the variables (t, x).

Homogeneous variants on $(0, \infty)$ and $\mathbb{R}$ will be found in the text (see Section 6), where we had to develop an appropriate theoretical functional framework in Sections 3, 4, 5. Actually, we start the proof by implementing a result of Kaplan (see Lemma 6.4) proving the invertibility of a parabolic operator on a sort of variational space involving the half-time derivative. This result has been central to other developments in the field of parabolic problems recently (see *e.g.*, [Nys17, AEN20, AE23]), while it was more like a consequence of the construction of weak solutions in earlier works of the literature, including Kaplan's work [Kap66]. As this result can only be formulated when time describes $\mathbb{R}$, this explains why we proceed by restriction from this case.

2.3. **Fundamental solution.** We come to the notion of fundamental solution and evolution family (or propagators, or Green operators, as suggested by Lions) to represent weak solutions. Although it seems well known that they are the same, we feel it is essential to clarify the two different definitions. This distinction eventually leads to easy arguments, even in this very general context. We assume that $\partial_t + \mathcal{B}$ is a parabolic operator as above for which one can prove existence and uniqueness of weak solutions on I with test functions valued in the core D of the Cauchy problems with the absolute continuity of $t \mapsto \|u(t)\|_H^2$ as in Theorem 2.5, and similarly for its backward adjoint.

Definition 2.6 (Fundamental solution for $\partial_t + \mathcal{B}$ on I)**.** A fundamental solution for $\partial_t + \mathcal{B}$ is a family $\Gamma = (\Gamma(t, s))_{t,s\in I}$ of bounded operators on H such that :

(1) (Uniform boundedness on H) $\sup_{t,s\in I} \|\Gamma(t, s)\|_{\mathcal{L}(H)} < +\infty$.
(2) (Causality) $\Gamma(t, s) = 0$ if $s > t$.
(3) (Measurability) For all $a, \tilde{a} \in \mathrm{D}$, the function $(t, s) \mapsto \langle \Gamma(t, s)a, \tilde{a}\rangle_H$ is Borel measurable on I^2.
(4) (Representation) For all $\phi \in \mathcal{D}(I)$ and $a \in \mathrm{D}$, the weak solution of the equation $\partial_t u + \mathcal{B}u = \phi \otimes a$ in $\mathcal{D}'(I; \mathrm{D})$ satisfies for all $\tilde{a} \in \mathrm{D}$, $\langle u(t), \tilde{a}\rangle_H = \int_{-\infty}^t \phi(s)\langle \Gamma(t, s)a, \tilde{a}\rangle_H \, \mathrm{d}s$, for almost every $t \in I$.

One defines a fundamental solution $\tilde{\Gamma} = (\tilde{\Gamma}(s, t))_{s,t\in I}$ to the backward operator $-\partial_s + \mathcal{B}^\star$ analogously and (2) is replaced by $\tilde{\Gamma}(s, t) = 0$ if $s > t$.

Such an object must be unique (see Lemma 6.24 in the case where $I = \mathbb{R}$, whose proof applies *verbatim*).

Definition 2.7 (Green operators)**.** Let $t, s \in \overline{I}$ and $a, \tilde{a} \in H$.

(1) For $t \geq s$, $G(t, s)a$ is defined as the value at time t of the weak solution to the equation $\partial_t u + \mathcal{B}u = 0$ with initial data a at time s.
(2) For $s \leq t$, $\tilde{G}(s, t)\tilde{a}$ is defined as the value at time s of the weak solution $\tilde{u}$ of the equation $-\partial_s \tilde{u} + \mathcal{B}^\star \tilde{u} = 0$ with final data $\tilde{a}$ at time t.

We set $G(t, s) = 0 = \tilde{G}(s, t)$ if $s > t$. The operators $G(t, s)$ and $\tilde{G}(s, t)$ are called the Green operators for the parabolic operator $\partial_t + \mathcal{B}$ and the backward parabolic operator $-\partial_s + \mathcal{B}^\star$, respectively.

Uniqueness and the integral identities allow to obtain the identification of the two objects as follows, with proof being *verbatim* the ones of Proposition 6.22 and Theorem 6.25.

Theorem 2.8. *The following statements hold.*

(1) *(Adjoint relation) For all $s < t$, $G(t, s)$ and $\tilde{G}(s, t)$ are adjoint operators.*
(2) *(Chapman-Kolmogorov identity) For any $s < r < t$, we have $G(t, s) = G(t, r)G(r, s)$.*
(3) *(Existence) The family of Green operators $(G(t, s))_{s,t\in\overline{I}}$ is a fundamental solution.*

With this in hand, we may first combine the estimates obtained for both families. See Corollary 6.21 for the ones for the Green operators (again, transposed *verbatim*). Next, we obtain full representation for the weak solutions in Theorem 2.5 (again, combining Theorems 7.7 and 8.1).

Theorem 2.9. *Consider the fundamental solution $(\Gamma_{\tilde{\mathcal{B}}}(t,s))_{0\le s\le t\le \mathfrak{T}}$ of $\partial_t+\tilde{\mathcal{B}}$ as in Theorem 2.5. For all $t\in[0,\mathfrak{T}]$, we have the following representation of the weak solution u of (2.4) :*

$$u(t)=\Gamma_{\tilde{\mathcal{B}}}(t,0)a+\int_0^t\Gamma_{\tilde{\mathcal{B}}}(t,s)Sf(s)\mathrm{d}s+\int_0^t\Gamma_{\tilde{\mathcal{B}}}(t,s)S^\beta g(s)\ \mathrm{d}s+\int_0^t\Gamma_{\tilde{\mathcal{B}}}(t,s)h(s)\ \mathrm{d}s,$$

where the two integrals containing f and g are weakly defined in H, while the one involving h converges strongly (i.e., in the Bochner sense). More precisely, for all $\tilde{a}\in H$ and $t\in[0,\mathfrak{T}]$, we have the equality with absolutely converging integrals

$$\begin{aligned}\langle u(t),\tilde{a}\rangle_H=\langle\Gamma_{\tilde{\mathcal{B}}}(t,0)a,\tilde{a}\rangle_H&+\int_0^t\langle f(s),S\tilde{\Gamma}_{\tilde{\mathcal{B}}}(s,t)\tilde{a}\rangle_H\ \mathrm{d}s\\&+\int_0^t\langle g(s),S^\beta\tilde{\Gamma}_{\tilde{\mathcal{B}}}(s,t)\tilde{a}\rangle_H\ \mathrm{d}s+\int_0^t\langle\Gamma_{\tilde{\mathcal{B}}}(t,s)h(s),\tilde{a}\rangle_H\ \mathrm{d}s.\end{aligned}$$

As before, this is obtained from the variants on $(0,\infty)$ and $\mathbb{R}$ which in fact come first. We refer the reader to the text for details.

3. The abstract homogeneous framework

Throughout this article, we are working in a separable complex Hilbert space H whose norm is denoted by $\|\cdot\|_H$ and its inner product by $\langle\cdot,\cdot\rangle_H$, and S is a positive and self-adjoint operator on H. **From Sections 3 to 5, we assume that S is injective and shall not repeat this in statements.** The general case when S might not be injective will be considered in Section 7. We do not assume that $0\in\rho(S)$, that is, S is not necessarily invertible. The spectrum of S is contained in $\mathbb{R}_+=[0,\infty)$. To make our approach accessible, it is useful to present facts from functional calculus and give the construction of spaces of test functions and distributions in an abstract context given that 0 might be in the spectrum of S.

3.1. A review of the Borel functional calculus.

For general background on self-adjoint operators and the spectral theorem, we refer to [RS80] and [Dav95].

By the spectral theorem for self-adjoint operators, there is a unique application $f\mapsto f(S)$ from the space of all locally bounded Borel functions on $(0,\infty)$ that we denote $\mathcal{L}^\infty_{\mathrm{loc}}((0,\infty))$ into the space of closed linear maps on H, which sends 1 to the identity, $(t\mapsto(z-t)^{-1})$ to $(z-S)^{-1}$ for all $z\in\mathbb{C}\setminus\mathbb{R}_+$ and its restriction to the space of all bounded Borel functions on $(0,\infty)$ denoted $\mathcal{L}^\infty((0,\infty))$ is a $\star$-algebra homomorphism into $\mathcal{L}(H)$, the space of bounded linear maps on H. More precisely, we have

$$\forall f\in\mathcal{L}^\infty((0,\infty)):\ \|f(S)\|\le\|f\|_\infty.$$

Moreover, for all $f,g\in\mathcal{L}^\infty_{\mathrm{loc}}((0,\infty))$, we have $f(S)g(S)\subset(fg)(S)$ with equality if $g(S)\in\mathcal{L}(H)$. We also recall that $f(S)^*=f^\star(S)$ with $f^\star=\bar{f}$.

We shall use that if $\varphi:(0,\infty)\to\mathbb{C}$ is a Borel function such that

$$|\varphi(t)|\le C\min(|t|^s,|t|^{-s}),\tag{3.1}$$

for some constants $C,s>0$ and for all $t>0$, then the operators $\int_\varepsilon^{1/\varepsilon}\varphi(aS)\ \frac{\mathrm{d}a}{a}$ are uniformly bounded for $0<\varepsilon<1$ and converge strongly in $\mathcal{L}(H)$, namely for all $v\in H$,

$$\lim_{\varepsilon\to0}\int_\varepsilon^{1/\varepsilon}\varphi(aS)v\ \frac{\mathrm{d}a}{a}=\left(\int_0^{+\infty}\varphi(t)\frac{\mathrm{d}t}{t}\right)v,\tag{3.2}$$

where the limit is in H. This is the so-called Calderón reproducing formula.

In this entire section, we fix a function $\Phi\in\mathcal{D}((0,\infty))$ such that $\int_0^{+\infty}\Phi(t)\frac{\mathrm{d}t}{t}=1$. Remark that for all $\alpha\in\mathbb{R}$, $t\mapsto t^\alpha\Phi(t)\in\mathcal{D}((0,\infty))$ and in particular verifies (3.1) for some constants $\tilde{C},\tilde{s}>0$ and for all $t>0$.

For $\alpha\in\mathbb{R}$, let S^α denote the closed operator $\mathbf{t}^\alpha(S)$, which is also injective, positive and self-adjoint. We recall that for all $\alpha,\beta\in\mathbb{R}$, we have

$$S^{\alpha+\beta}=S^\alpha S^\beta.$$

Denote by $D(S^\alpha)$ the domain of S^α. For any element $u \in D(S^\alpha)$, we set

$$\|u\|_{S,\alpha} := \|S^\alpha u\|_H .$$

We insist on the fact that $\|\cdot\|_{S,\alpha}$ denotes the homogeneous norm on the domain of S^α and the (Hilbertian) graph norm is $(\|\cdot\|_{S,\alpha}^2 + \|\cdot\|_H^2)^{1/2}$. The operator

$$S^\alpha : (D(S^\alpha), \|\cdot\|_{S,\alpha}) \to (H, \|\cdot\|_H) \tag{3.3}$$

is isometric with dense range.

3.2. **An ambient space.** We construct an ambient space in which we can perform all calculations. Consider the vector space

$$E_{-\infty} := \bigcap_{\alpha\in\mathbb{R}} D(S^\alpha),$$

endowed with the topology defined using the norms family $(\|\cdot\|_{S,\alpha})_{\alpha\in\mathbb{R}}$. We recall the following moments inequality

$$\|S^\gamma u\|_H \le \|S^\alpha u\|_H^\theta \|S^\beta u\|_H^{1-\theta} \ (u \in E_{-\infty}),$$

for all $\gamma = \theta\alpha + (1-\theta)\beta$ and $\theta \in [0,1]$ and α and β with same sign [Haa06, Proposition 6.6.4]. Using the moment inequality with the closedness of the powers S^α, one can see that $E_{-\infty}$ endowed with the countable norms family $(\|\cdot\|_{S,\alpha})_{\alpha\in\mathbb{Z}}$ is in fact a Fréchet space. Notice that for all $\alpha \in \mathbb{R}$, $S^\alpha : E_{-\infty} \to E_{-\infty}$ is an isomorphism.

The space $E_{-\infty}$ is to be the test space as evidenced in the following lemma.

Lemma 3.4. *$E_{-\infty}$ is dense in $(D(S^\alpha), \|\cdot\|_{S,\alpha})$ for all $\alpha \in \mathbb{R}$.*

Proof. Let $v \in D(S^\alpha)$. We regularise v by setting for all $\varepsilon \in (0,1)$,

$$v_\varepsilon := \int_\varepsilon^{1/\varepsilon} \Phi(aS)v \ \frac{\mathrm{d}a}{a}.$$

Indeed, we show that $v_\varepsilon \in E_{-\infty}$ and $S^\alpha v_\varepsilon \to S^\alpha v$ in H as $\varepsilon \to 0$. First, for all $\beta \in \mathbb{R}$, since $t \mapsto t^\beta \Phi(at) \in \mathcal{L}^\infty((0,\infty))$, we have $v_\varepsilon \in D(S^\beta)$ with $S^\beta v_\varepsilon = \int_\varepsilon^{1/\varepsilon} S^\beta \Phi(aS)v \ \frac{\mathrm{d}a}{a}$. Hence, $v_\varepsilon \in E_{-\infty}$. Furthermore, as $v \in D(S^\alpha)$, we have $S^\alpha \Phi(aS)v = \Phi(aS)S^\alpha v$, so that $S^\alpha v_\varepsilon$ converges to $S^\alpha v$ by the Calderón reproducing formula. □

Remark 3.5. The approximation is universal, in the sense that if $v \in D(S^\alpha) \cap D(S^\beta)$, then the approximation occurs simultaneously in both semi-norms. In particular, $E_{-\infty}$ is dense in $D(S^\alpha)$ for the graph norm.

Let E_∞ denote the topological anti-dual space of $E_{-\infty}$. The reason for which we are interested in such a space is that it provides an ambient space containing a copy of a completion of all spaces $(D(S^\alpha), \|\cdot\|_{S,\alpha})$. To clarify this claim, we define

$$\|\varphi\|_{E_\alpha} := \sup_{v\in E_{-\infty}\setminus\{0\}} \frac{|\varphi(v)|}{\|v\|_{S,-\alpha}}$$

and the vector space

$$E_\alpha := \{\varphi \in E_\infty : \|\varphi\|_{E_\alpha} < \infty\}.$$

The space $\left(E_\alpha, \|\cdot\|_{E_\alpha}\right)$ is a Banach space. We set

$$j : H \to E_\infty, \quad v \mapsto j(v) := \langle v, \cdot\rangle_H.$$

The application j is injective by the density of $E_{-\infty}$ in H in Lemma 3.4.

Lemma 3.6. *For all $\alpha \in \mathbb{R}$, $j_{|D(S^\alpha)} : (D(S^\alpha), \|\cdot\|_{S,\alpha}) \to \left(E_\alpha, \|\cdot\|_{E_\alpha}\right)$ is isometric with dense range.*

Proof. If $v \in D(S^\alpha)$ then we can write for all $\tilde{v} \in E_{-\infty}$, $j(v)(\tilde{v}) = \langle v, \tilde{v}\rangle_H = \langle S^\alpha v, S^{-\alpha}\tilde{v}\rangle_H$. This implies that $j(v) \in E_\alpha$ and $\|j(v)\|_{E_\alpha} = \|S^\alpha v\|_H = \|v\|_{S,\alpha}$. Now, if $\varphi \in E_\alpha$, then $\varphi \circ S^\alpha$ has a bounded extension on H. Using the Riesz representation theorem, there exists $v \in H$ such that $\varphi \circ S^\alpha = \langle v, \cdot\rangle_H$ or equivalently $\varphi = \langle v, S^{-\alpha}\cdot\rangle_H$. Moreover, we have $\|\varphi\|_{E_\alpha} = \|v\|_H$. Since the range of S^α is dense in H, there exists a sequence $(v_n)_{n\in\mathbb{N}} \in D(S^\alpha)^{\mathbb{N}}$ such that $S^\alpha v_n \to v$ in H. Now, we have for all $\tilde{v} \in E_{-\infty}$,

$$j(v_n)(\tilde{v}) - \varphi(\tilde{v}) = \langle v_n, \tilde{v}\rangle_H - \langle v, S^{-\alpha}\tilde{v}\rangle_H = \langle S^\alpha v_n - v, S^{-\alpha}\tilde{v}\rangle_H.$$

Therefore, $\|j(v_n) - \varphi\|_{E_\alpha} = \|S^\alpha v_n - v\|_H \to 0$. □

To make clear the identification we will adopt in the next paragraph, we temporarily define the operator T on the Hilbert space $j(H) = H^\star$ by setting

$$D(T) := j(D(S)) \ , \quad T := j \circ S \circ j^{-1}.$$

Since $j : H \to j(H)$ is a unitary operator by Lemma 3.6 when $\alpha = 0$, T is unitarily equivalent to S. It follows that T has the same properties as S. More precisely, T is injective, positive and selfadjoint and we have for all $\alpha \in \mathbb{R}$

$$D(T^\alpha) := j(D(S^\alpha)) \ , \quad T^\alpha = j \circ S^\alpha \circ j^{-1}, \quad \|T^\alpha(j(v))\|_{j(H)} = \|S^\alpha v\|_H \, .$$

Using Lemma 3.6, we have for all $\alpha \in \mathbb{R}$, $D(T^\alpha) \subset E_\alpha$ with dense and isometric inclusion for the homogeneous norm of T^α which means that E_α is a completion of $D(T^\alpha)$ for the homogeneous norm. Moreover, it follows from (3.3) that $T^\alpha : D(T^\alpha) \to j(H)$ is isometric with dense range. Notice that by combining Lemma 3.4 and Lemma 3.6, $j(E_{-\infty}) = \bigcap_{\alpha\in\mathbb{R}} D(T^\alpha)$ is dense in all the spaces E_α. Furthermore, we have for all $\alpha \in \mathbb{R}$, $E_\alpha \cap j(H) = D(T^\alpha)$. The advantage is that all the spaces mentioned here are contained in E_∞, an anti-dual space of a Fréchet space, which is in particular a Hausdorff topological vector space.

From now on, by making the identification of H with $j(H)$ and S with $j \circ S \circ j^{-1}$ as above, we assume that H is contained in a Hausdorff topological vector space E_∞ that contains a completion of all the domains of S^α for the homogeneous norms and we denote by $D_{S,\alpha}$ this completion of $D(S^\alpha)$ in E_∞. Moreover, we have that for all $\alpha \in \mathbb{R}$, $D_{S,\alpha} \cap H = D(S^\alpha)$. By Lemma 3.4, the space $E_{-\infty} = \bigcap_{\alpha\in\mathbb{R}} D(S^\alpha)$ is dense in all these completions. Moreover, there is a sesquilinear continuous duality form $\langle w, v\rangle$ on $E_\infty \times E_{-\infty}$ which extends the inner product on H. The functional calculus of S extends to E_∞ by $\langle f(S)w, v\rangle = \langle w, f^*(S)v\rangle$ whenever $f \in \mathcal{L}^\infty_{\rm loc}((0,\infty))$ and $f(t) = O(t^a)$ at 0 and $f(t) = O(t^b)$ at ∞ as $f^*(S) : E_{-\infty} \to E_{-\infty}$ is bounded. In particular, $S^\alpha : E_\infty \to E_\infty$ is an automorphism and we have $D_{S,\alpha} = \{u \in E_\infty : S^\alpha u \in H\}$. The restriction of S^α to $D_{S,\alpha}$ agrees with the unique extension of $S^\alpha : (D(S^\alpha), \|\cdot\|_{S,\alpha}) \to (H, \|\cdot\|_H)$ (see (3.3)). The norm on $D_{S,\alpha}$ is $\|S^\alpha\cdot\|_H$ and we keep denoting it by $\|\cdot\|_{S,\alpha}$ (and it makes it a Hilbert space). We record the following lemma.

Lemma 3.7. *Let $\alpha \in \mathbb{R}$. Then, there are dense inclusions*

$$E_{-\infty} \hookrightarrow D(S^\alpha) \hookrightarrow D_{S,\alpha} \hookrightarrow E_\infty.$$

Moreover, the family $(D_{S,\alpha})_{\alpha\in\mathbb{R}}$ is a complex (and real) interpolation family.

Proof. For the first statement, the first two inclusions are already known to be dense. We show it for the last one. Indeed, if $w \in E_\infty$, then let $\mathcal{F}_w$ be a finite subset of $\mathcal{F}$ such that we have $|\langle w, v\rangle| \le C_w \sup_{\gamma\in\mathcal{F}_w} \|S^\gamma v\|_H$ for all $v \in E_{-\infty}$. The approximation procedure using the duality form and using that $\Phi^\star = \Phi$ shows that w_ε converges to w in E_∞. We claim that $w_\varepsilon \in D_{S,\alpha}$ for all α (hence, $w_\varepsilon \in E_{-\infty}$). Indeed, if we pick $v \in E_{-\infty}$, then $S^\alpha v_\varepsilon \in E_{-\infty}$ and $S^{\alpha+\gamma} v_\varepsilon \in H$ with norm controlled by $C_{\varepsilon,\alpha+\gamma}\|v\|_H$. Thus, we have

$$|\langle S^\alpha w_\varepsilon, v\rangle| = |\langle w, S^\alpha v_\varepsilon\rangle| \le C_w \sup_{\gamma\in\mathcal{F}_w} \|S^{\alpha+\gamma} v_\varepsilon\|_H \le C_w \sup_{\gamma\in\mathcal{F}_w} C_{\varepsilon,\alpha+\gamma}\, \|v\|_H,$$

from which the claim follows. Finally, the fact that family $(D_{S,\alpha})_{\alpha\in\mathbb{R}}$ is a complex (and real) interpolation family is proved in [AMN97]. □

For any real α, the sesquilinear form $(u,v)\mapsto \langle S^\alpha u, S^{-\alpha}v\rangle_H$ defines a canonical duality pairing between $D_{S,\alpha}$ and $D_{S,-\alpha}$ which is simply the inner product $\langle\cdot,\cdot\rangle_H$ extended from $E_{-\infty}\times E_{-\infty}$ to $D_{S,\alpha}\times D_{S,-\alpha}$. In fact, we have for all $(u,v)\in E_{-\infty}\times E_{-\infty}$,

$$\sup_{w\in E_{-\infty}\setminus\{0\}}\frac{|\langle u,w\rangle_H|}{\|w\|_{S,-\alpha}}=\|u\|_{S,\alpha}\quad,\quad \sup_{w\in E_{-\infty}\setminus\{0\}}\frac{|\langle w,v\rangle_H|}{\|w\|_{S,\alpha}}=\|u\|_{S,-\alpha}\,.$$

For $(u,v)\in D_{S,\alpha}\times D_{S,-\alpha}$, we denote $\langle u,v\rangle_{H,\alpha}:=\langle S^\alpha u,S^{-\alpha}v\rangle_H$. It also coincides with the sesquilinear duality $\langle u,v\rangle$ on $E_\infty\times E_{-\infty}$ when $u\in D_{S,\alpha}$ and $v\in E_{-\infty}$. We have the following lemma.

Lemma 3.8. *Let $\alpha,\beta\in\mathbb{R}$. If $u\in D_{S,\alpha}\cap D_{S,\beta}$ and $v\in D_{S,-\alpha}\cap D_{S,-\beta}$, then*

$$\langle u,v\rangle_{H,\alpha}=\langle u,v\rangle_{H,\beta}.$$

Proof. The approximations u_ε and v_ε belong to $E_{-\infty}$ so $\langle u_\varepsilon,v_\varepsilon\rangle_{H,\alpha}=\langle u_\varepsilon,v_\varepsilon\rangle_H=\langle u_\varepsilon,v_\varepsilon\rangle_{H,\beta}$ for all $\varepsilon>0$. The result follows when ε tends to 0 as u_ε converges to u in both spaces $D_{S,\alpha}$ and $D_{S,\beta}$ and v_ε converges to v in both spaces $D_{S,-\alpha}$ and $D_{S,-\beta}$. □

3.3. **Spaces of test functions and distributions.** For I an open interval of $\mathbb{R}$, we denote by $\mathcal{D}(I;E_{-\infty})$ the space of $E_{-\infty}$-valued C^∞ functions on I with compact support. The space $\mathcal{D}(I;E_{-\infty})$ is endowed with the usual inductive limit topology and contains $\mathrm{span}(\mathcal{D}(I)\otimes E_{-\infty})$ as a dense subspace.

We refer to [HvNVW16] for Banach-valued $L^p(I;B)$ spaces. The density lemma below explains why it is relevant to take $\mathcal{D}(I;E_{-\infty})$ as the space of test functions.

Lemma 3.9. *$\mathcal{D}(I;E_{-\infty})$ is dense in $L^p(I;D_{S,\alpha})$ for all $\alpha\in\mathbb{R}$ and $p\in[1,\infty)$.*

Proof. It is enough to consider the case $\alpha=0$, that is to prove the density of $\mathcal{D}(I;E_{-\infty})$ in $L^p(I,H)$ since $S^{-\alpha}:E_{-\infty}\to E_{-\infty}$ is an isomorphism and therefore $S^{-\alpha}:\mathcal{D}(I;E_{-\infty})\to\mathcal{D}(I;E_{-\infty})$, where we set $(S^\alpha f)(t)=S^\alpha(f(t))$ for all $t\in I$ and $f\in\mathcal{D}(I;E_{-\infty})$. It is enough also to prove that $\mathcal{D}(I;E_{-\infty})$ is dense in $\mathcal{D}(I;H)$ for the L^p norm since the latter is dense in $L^p(I;H)$. To do so, we fix $f\in\mathcal{D}(I;H)$ and regularize it by setting for all $\varepsilon>0$, and $t\in I$

$$f_\varepsilon(t):=\int_\varepsilon^{1/\varepsilon}\Phi(aS)f(t)\,\frac{\mathrm{d}a}{a}=\left(\int_\varepsilon^{1/\varepsilon}\Phi(aS)\,\frac{\mathrm{d}a}{a}\right)f(t).$$

It is obvious that $f_\varepsilon\in\mathcal{D}(I;E_{-\infty})$ and by the Calderón reproducing formula $f_\varepsilon(t)\underset{\varepsilon\to0}{\to}f(t)$ pointwisely in H. The uniform boundedness in H of the approximate Calderón operators yields the domination allowing to apply the dominated convergence theorem. □

We can now define a space of distributions in which S plays the role of a differential operator. Specifically, we denote by $\mathcal{D}'(I;E_\infty)$ the space of all bounded anti-linear maps $u:\mathcal{D}(I;E_{-\infty})\to\mathbb{C}$. Bounded means that for any compact set $K\subset I$, there exist two constants $C_K>0$ and $m_K\in\mathbb{N}$, and a finite set $\mathcal{A}_K\subset\mathbb{Z}$ such that

$$\forall\varphi\in\mathcal{D}(I,E_{-\infty}),\ \mathrm{supp}(\varphi)\subset K\Rightarrow\left|\langle\!\langle u,\varphi\rangle\!\rangle_{\mathcal{D}',\mathcal{D}}\right|\le C_K\sup_{|j|\le m_K,\,\alpha\in\mathcal{A}_K}\sup_{t\in K}\|\partial_t^jS^\alpha\varphi(t)\|_H.$$

For convenience, we use the notation with partial derivative in t for the derivative in t thinking of future applications to concrete PDE. We also use a double bracket notation for dualities between vector-valued functions and distributions.

For $\alpha\in\mathbb{R}$, we can embed $L^1_{\mathrm{loc}}(I;D_{S,\alpha})$ in $\mathcal{D}'(I;E_\infty)$ through the classical identification :

$$J_\alpha\colon L^1_{\mathrm{loc}}(I;D_{S,\alpha})\to\mathcal{D}'(I;E_\infty)$$
$$f\mapsto J_\alpha f:\varphi\mapsto\int_I\langle f(t),\varphi(t)\rangle_{H,\alpha}\,\mathrm{d}t=\int_I\langle S^\alpha f(t),S^{-\alpha}\varphi(t)\rangle_H\,\mathrm{d}t.$$

This is well defined by the following lemma.

Lemma 3.10. *For all $\alpha,\beta\in\mathbb{R}$, $J_\alpha=J_\beta$ on $L^1_{\mathrm{loc}}(I;D_{S,\alpha})\cap L^1_{\mathrm{loc}}(I;D_{S,\beta})$.*

Proof. Straightforward corollary of Lemma 3.8. □

Finally, the identification is achieved thanks to the following lemma.

Lemma 3.11. *J_α is injective for all $\alpha \in \mathbb{R}$.*

Proof. Testing with $\varphi = \theta \otimes a$, with $\theta \in \mathcal{D}(\mathbb{R})$ real-valued and $a \in E_{-\infty}$, we easily conclude that $J_\alpha(f) = 0$ implies $\langle S^\alpha f, S^{-\alpha} a\rangle_H = 0$ a.e. on I. This implies that $S^\alpha f = 0$ in H a.e. on I, hence $f = 0$ in $D_{S,\alpha}$ a.e. on I. □

Consequently, the space $L^p(I; D_{S,\alpha})$ can be identified to a sub-space of $\mathcal{D}'(I; E_\infty)$. We would like to apply powers of S to a distribution like we apply them to functions valued in $E_{-\infty}$. The definition below covers this.

Definition 3.12. For $\alpha \in \mathbb{R}$ and $u \in \mathcal{D}'(I; E_\infty)$, we define the distribution $S^\alpha u$ by setting

$$\langle\!\langle S^\alpha u, \varphi\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} := \langle\!\langle u, S^\alpha \varphi\rangle\!\rangle_{\mathcal{D}',\mathcal{D}},\ \forall \varphi \in \mathcal{D}(I; E_{-\infty}).$$

Remark 3.13. For $u \in \mathcal{D}'(I; E_\infty)$, $S^\alpha u \in L^p(I; H)$ is equivalent to $u \in L^p(I; D_{S,\alpha})$. Furthermore, powers of S commute with derivatives in t : for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, $\partial_t^k S^\alpha = S^\alpha \partial_t^k$.

When $I = \mathbb{R}$, we can use the space of tempered distributions adapted to S. Let us start by the Schwartz class $\mathcal{S}(\mathbb{R}; E_{-\infty})$ defined by

$$\mathcal{S}(\mathbb{R}; E_{-\infty}) := \left\{ \varphi \in C^\infty(\mathbb{R}; E_{-\infty}) \ :\ \forall k, \ell \in \mathbb{N},\ t^k \partial_t^\ell \varphi(t) \underset{|t|\to\infty}{\to} 0 \text{ in } E_{-\infty} \right\},$$

which is a Fréchet space for a suitable countable family of norms. Moreover, $\mathcal{D}(\mathbb{R}; E_{-\infty})$ is dense in $\mathcal{S}(\mathbb{R}; E_{-\infty})$ by the same argument as for the usual distributions. We denote by $\mathcal{S}'(\mathbb{R}; E_\infty)$ the topological dual space of $\mathcal{S}(\mathbb{R}; E_{-\infty})$. It is a subspace of $\mathcal{D}'(\mathbb{R}; E_\infty)$ containing $L^p(\mathbb{R}; D_{S,\alpha})$ for all $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$. For the proofs of this and the theorems below, we refer to [Zui02] for the classical distributions and the same proofs work here. It can be proven that $\mathcal{S}(\mathbb{R}; E_{-\infty})$ is dense in $\mathcal{S}'(\mathbb{R}; E_\infty)$, and more generally, that $\mathcal{D}(I; E_{-\infty})$ is dense in $\mathcal{D}'(I; E_\infty)$ for any open set $I \subset \mathbb{R}$. However, this is not important for the discussion that follows, so we leave the verification to the interested reader.

As in the classical case, we will first define $\mathcal{F}$ on $L^1(\mathbb{R}; H)$, then on $\mathcal{S}(\mathbb{R}; E_{-\infty})$ and finally on $\mathcal{S}'(\mathbb{R}; E_\infty)$ by duality.

Definition 3.14. The Fourier transform is defined on $L^1(\mathbb{R}; H)$ by setting for all $f \in L^1(\mathbb{R}; H)$ and $\tau \in \mathbb{R}$

$$\mathcal{F}(f)(\tau) := \hat{f}(\tau) := \int_{\mathbb{R}} e^{-i\tau t} f(t)\ \mathrm{d}t.$$

We define $\overline{\mathcal{F}}$ by changing $-i\tau t$ to $i\tau t$ in the integral.

The Fourier transform on $\mathcal{S}(\mathbb{R}; E_{-\infty})$ enjoys many properties as we recall below.

Proposition 3.15. *The Fourier transform $\mathcal{F}$ enjoys the following properties :*

1. *$\mathcal{F} : \mathcal{S}(\mathbb{R}; E_{-\infty}) \to \mathcal{S}(\mathbb{R}; E_{-\infty})$ is an automorphism verifying for all $\varphi \in \mathcal{S}(\mathbb{R}; E_{-\infty})$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$*
$$\mathcal{F}(S^\alpha \varphi) = S^\alpha \mathcal{F}(\varphi),\ \partial_\tau^k \mathcal{F}(\varphi) = \mathcal{F}((-it)^k \varphi),\ \mathcal{F}(\partial_t^k \varphi) = (i\tau)^k \mathcal{F}(\varphi).$$
2. *For all $\alpha \in \mathbb{R}$, $\mathcal{F}$ extends to an isomorphism on $L^2(\mathbb{R}; D_{S,\alpha})$ which verifies a Plancherel equality.*

We can now transport the Fourier transform $\mathcal{F}$ to $\mathcal{S}'(\mathbb{R}; E_\infty)$ by sesquilinear duality.

Definition 3.16. We define the Fourier transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}; E_\infty)$ by setting

$$\langle\!\langle \mathcal{F}u, \varphi\rangle\!\rangle_{\mathcal{S}',\mathcal{S}} := \langle\!\langle \hat{u}, \varphi\rangle\!\rangle_{\mathcal{S}',\mathcal{S}} := \langle\!\langle u, \overline{\mathcal{F}}(\varphi)\rangle\!\rangle_{\mathcal{S}',\mathcal{S}},\ u \in \mathcal{S}'(\mathbb{R}; E_\infty),\ \varphi \in \mathcal{S}(\mathbb{R}; E_{-\infty}).$$

From Proposition 3.15, we deduce the following proposition regarding the Fourier transform on $\mathcal{S}'(\mathbb{R}; E_\infty)$.

Proposition 3.17. *$\mathcal{F} : \mathcal{S}'(\mathbb{R}; E_\infty) \to \mathcal{S}'(\mathbb{R}; E_\infty)$ is an automorphism and satisfies the property (1) and its restriction to $L^2(\mathbb{R}; D_{S,\alpha})$ agrees with the operator in (2) as in the statement above.*

For $\alpha \in \mathbb{R}$, we denote by D_t^α the time-derivative of order α. More precisely, if $u \in \mathcal{S}'(\mathbb{R}; E_\infty)$ is such that $|\tau|^\alpha \mathcal{F}u \in \mathcal{S}'(\mathbb{R}; E_\infty)$, we set

$$D_t^\alpha u := \mathcal{F}^{-1}\left(|\tau|^\alpha \mathcal{F}u\right).$$

4. Abstract heat equations

In this section, we study well-posedness of the abstract heat equation $\partial_t u + S^2 u = f$ where the role of the Laplacian is played by the square of the self-adjoint operator S. These well-posedness results will imply embeddings and energy inequalities in the spirit of Lions that will be described in the next section. The abstract heat operator in $\mathbb{R}$ is $\partial_t + S^2$. The backward operator $-\partial_t + S^2$ corresponds to reversing time, and the results are exactly the same and are often proved and used simultaneously.

4.1. Solving the abstract heat equation using the Fourier method.

Working on the real line makes the Fourier transform in time available and is a key tool to obtain homogeneous estimates in a simple way.

4.1.1. *Uniqueness in homogeneous energy space.* We begin with a uniqueness result which is key to our discussion.

Proposition 4.1 (Uniqueness in homogeneous energy space)**.** *Let $u \in \mathcal{D}'(\mathbb{R}; E_\infty)$ be a solution of $\partial_t u + S^2 u = 0$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$. If $u \in L^2(\mathbb{R}; D_{S,\alpha})$ for some $\alpha \in \mathbb{R}$, then $u = 0$.*

Proof. As S^α is an isomorphism on E_∞ which commutes with time derivatives, $v = S^\alpha u$ satisfies the same equation, hence we may assume $\alpha = 0$ and $u \in L^2(\mathbb{R}; H)$. As this is a subset of $\mathcal{S}'(\mathbb{R}; E_\infty)$, by applying the Fourier transform to this equation, we have for all $\varphi \in \mathcal{S}(\mathbb{R}; E_{-\infty})$

$$\int_{\mathbb{R}} \langle \hat{u}(\tau), (-i\tau + S^2)\varphi(\tau)\rangle_H \, \mathrm{d}\tau = 0. \tag{4.2}$$

Take a sequence $(\varphi_k)_{k\in\mathbb{N}} \in \mathcal{D}(\mathbb{R}; E_{-\infty})^{\mathbb{N}}$ such that $\varphi_k \to \hat{u}$ in $L^2(\mathbb{R}; H)$ and $0 \notin \operatorname{supp}(\varphi_k)$, for all $k \in \mathbb{N}$. Taking $\tau \mapsto (-i\tau + S^2)^{-1}\varphi_k(\tau)$ as a test function in (4.2) and letting $k \to +\infty$, we have

$$\int_{\mathbb{R}} \|\hat{u}(\tau)\|_H^2 \, \mathrm{d}\tau = 0.$$

By Plancherel, we have then $u = 0$. □

Corollary 4.3 (Invertibility on abstract Schwartz functions and tempered distributions)**.** *The operator $\partial_t + S^2$ is an isomorphism on $\mathcal{S}(\mathbb{R}; E_{-\infty})$ and on $\mathcal{S}'(\mathbb{R}; E_\infty)$.*

Proof. We begin with the result on $\mathcal{S}(\mathbb{R}; E_{-\infty})$. The boundedness is clear. The injectivity follows from the above proposition. The surjectivity is as follows. By Fourier transform, it suffices to show the surjectivity for $i\tau + S^2$. If $\hat{f} \in \mathcal{S}(\mathbb{R}; E_{-\infty})$, then $S^{-2}\hat{f} \in L^2(\mathbb{R}; H)$ and by the uniform boundedness of $(i\tau + S^2)^{-1}S^2$, $g(\tau) = (i\tau + S^2)^{-1}S^2(S^{-2}\hat{f}(\tau)) \in L^2(\mathbb{R}; H)$ with $i\tau g(\tau) + S^2 g(\tau) = \hat{f}(\tau)$. Shifting with powers of S, we have $g \in L^2(\mathbb{R}; E_{-\infty})$. Setting $\hat{u} = g$, we see that $\partial_t u = f - S^2 u$, so $\partial_t u \in L^2(\mathbb{R}; E_{-\infty})$ and by iteration, we have $u \in C^\infty(\mathbb{R}; E_{-\infty})$. The decay is easily checked following the argument and using τ-derivatives of the resolvent $(i\tau + S^2)^{-1}$.

This applies to the backward operator $-\partial_t + S^2$. Hence, by duality, we obtain the result on $\mathcal{S}'(\mathbb{R}; E_\infty)$. □

Remark 4.4. In the sequel, we shall focus on $\alpha = 1$ in Proposition 4.1 to make H the pivotal space, but clearly, one can shift to this case by applying powers of S.

4.1.2. *Solution and source spaces.* We begin with recalling the following result of Lions for the sake of completeness, but we shall not use this result and prove a stronger one.

Proposition 4.5 (Solving the abstract heat equation à la Lions)**.** *If $f \in L^2(\mathbb{R}; D_{S,-1})$, then there exists $u \in L^2(\mathbb{R}; D_{S,1})$ such that $\partial_t u + S^2 u = f$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$.*

Proof. It is straightforward application of the Lions representation theorem [Lio13, Théorème 1.1] in the Hilbert space $L^2(\mathbb{R}; D_{S,1})$. □

As we said, we now argue with in mind that H, or rather $L^2(\mathbb{R};H)$, is the pivotal Hilbert space. Define

$$V_1 := L^2(\mathbb{R};D_{S,1}),$$
$$V_{-1} := \left\{u \in L^2(\mathbb{R};D_{S,1}) \ : \ \partial_t u \in L^2(\mathbb{R};D_{S,-1})\right\}.$$

V_1 is the uniqueness space and V_{-1} is the space to which the solution belongs when the source is taken in $L^2(\mathbb{R};D_{S,-1})$ according to Lions' result. However, for the heat equation, Fourier methods are particularly handy to prove this and also allow more source spaces.

We introduce a hierarchy of intermediate solution and source spaces. For $\alpha \in [-1,1]$, define the following respective solution and source spaces

$$V_\alpha := \left\{u \in L^2(\mathbb{R};D_{S,1}) : D_t^{\frac{1-\alpha}{2}} u \in L^2(\mathbb{R};D_{S,\alpha})\right\},$$
$$W_\alpha := \left\{D_t^{\frac{1+\alpha}{2}} g \ : \ g \in L^2(\mathbb{R};D_{S,\alpha})\right\}.$$

with

$$\|u\|_{V_\alpha} := \left(\|u\|^2_{L^2(\mathbb{R};D_{S,1})} + \|D_t^{\frac{1-\alpha}{2}} u\|^2_{L^2(\mathbb{R};D_{S,\alpha})}\right)^{1/2},$$
$$\|f\|_{W_\alpha} := \|D_t^{-\frac{1+\alpha}{2}} f\|_{L^2(\mathbb{R};D_{S,\alpha})}.$$

We can think of $V_\alpha = L^2(\mathbb{R};D_{S,1}) \cap \dot{H}^{\frac{1-\alpha}{2}}(\mathbb{R};D_{S,\alpha})$ using homogeneous Sobolev spaces on the real line but this presentation avoids having to define these spaces. In the same manner, we think of $W_\alpha = \dot{H}^{-\frac{1+\alpha}{2}}(\mathbb{R};D_{S,\alpha})$. Remark that $W_{-1} = L^2(\mathbb{R};D_{S,-1})$.

The following lemma summarizes some properties of the spaces V_α and W_α and their relation.

Lemma 4.6 (Properties of intermediate spaces)**.** *Fix $-1 \le \alpha \le \alpha' \le 1$. We have the following assertions.*

(1) *V_α is a well-defined subspace of $\mathcal{S}'(\mathbb{R};E_\infty)$, $\left(V_\alpha, \|\cdot\|_{V_\alpha}\right)$ is a Hilbert space, and we have*

$$V_\alpha = \left\{u \in \mathcal{S}'(\mathbb{R};E_\infty) : S^\alpha(S+|\tau|^{1/2})^{1-\alpha}\hat{u} \in L^2(\mathbb{R};H)\right\}, \ \|u\|_{V_\alpha} \sim \|S^\alpha(S+|\tau|^{1/2})^{1-\alpha}\hat{u}\|_{L^2(\mathbb{R};H)}.$$

(2) *We have the following chain of continuous and dense inclusions:*

$$\mathcal{S}(\mathbb{R};E_{-\infty}) \hookrightarrow V_\alpha \hookrightarrow V_{\alpha'} \hookrightarrow \mathcal{S}'(\mathbb{R};E_\infty).$$

(3) *W_α is a subspace of $\mathcal{S}'(\mathbb{R};E_\infty)$, and $\left(W_\alpha, \|\cdot\|_{W_\alpha}\right)$ is a Hilbert space. We have a dense inclusion $\mathcal{S}_0(\mathbb{R};E_{-\infty}) \hookrightarrow W_\alpha$, where $\mathcal{S}_0(\mathbb{R};E_{-\infty}) := \{f \in \mathcal{S}(\mathbb{R};E_{-\infty}) \mid \hat{f}(0) = 0\}$.*

(4) *Let $V_\alpha^\star$ denote the anti-dual space of V_α with respect to $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R};H)}$. It is a subspace of $\mathcal{S}'(\mathbb{R};E_\infty)$ and $V_\alpha^\star = L^2(\mathbb{R};D_{S,-1}) + W_{-\alpha}$ with the following estimate*

$$\|\omega\|_{V_\alpha^\star} \sim \inf\left\{\|f\|_{L^2(\mathbb{R};H)} + \|g\|_{L^2(\mathbb{R};D_{S,-\alpha})} \ : \ \omega = Sf + D_t^{\frac{1-\alpha}{2}} g\right\}.$$

Proof. Let us first prove (1) : V_α is a well-defined subspace of $\mathcal{S}'(\mathbb{R};E_\infty)$. In fact, if $u \in L^2(\mathbb{R};D_{S,1})$, then, by Proposition 3.17, it follows that $|\tau|^{\frac{1-\alpha}{2}} S\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R};H)$. Furthermore, for $\varphi \in \mathcal{S}(\mathbb{R};E_{-\infty})$, we have using Cauchy-Schwarz inequality

$$\int_{\mathbb{R}} |\langle |\tau|^{\frac{1-\alpha}{2}} S\hat{u}, S^{-1}\varphi\rangle_H| \, \mathrm{d}\tau \le \|S\hat{u}\|_{L^2(\mathbb{R};H)} \, \||\tau|^{\frac{(1-\alpha)}{2}} S^{-1}\varphi\|_{L^2(\mathbb{R};H)},$$

and one can define $|\tau|^{\frac{1-\alpha}{2}} \hat{u} \in \mathcal{S}'(\mathbb{R};E_\infty)$ by

$$\langle\langle |\tau|^{\frac{1-\alpha}{2}} \hat{u}, \varphi\rangle\rangle_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}} \langle |\tau|^{\frac{1-\alpha}{2}} S\hat{u}, S^{-1}\varphi\rangle_H \, \mathrm{d}\tau.$$

Finally, $S^\alpha |\tau|^{\frac{1-\alpha}{2}} \hat{u}$ exists in $\mathcal{S}'(\mathbb{R};E_\infty)$ and agrees with $|\tau|^{\frac{1-\alpha}{2}} S^\alpha \hat{u}$. The Hilbert space property (in particular, the completeness) is easy. Next, the proof of the set equality and the norms

equivalence in (1) is easy using the boundedness of the operators $S^{1-\alpha}(S+|\tau|^{1/2})^{-(1-\alpha)}$ and $|\tau|^{\frac{1-\alpha}{2}}(S+|\tau|^{1/2})^{-(1-\alpha)}$ on $L^2(\mathbb{R};H)$.

For the point (2), the inclusion of $\mathcal{S}(\mathbb{R};E_{-\infty})$ in V_α follows easily from (1). To check $V_\alpha \hookrightarrow V_{\alpha'}$ if $\alpha<\alpha'$, write

$$S^{\alpha'}(S+|\tau|^{1/2})^{1-\alpha'} = S^{\alpha'-\alpha}(S+|\tau|^{1/2})^{\alpha-\alpha'}S^{\alpha}(S+|\tau|^{1/2})^{1-\alpha},$$

and use (1) together with the boundedness of $S^{\alpha'-\alpha}(S+|\tau|^{1/2})^{\alpha-\alpha'}$ on $L^2(\mathbb{R};H)$. The density of $\mathcal{S}(\mathbb{R};E_{-\infty})$ into V_α can be deduced using Lemma 3.9 and that of V_α into $V_{\alpha'}$ follows. Finally, although we do not need this later on, the density of V_α in $\mathcal{S}'(\mathbb{R};E_\infty)$ hold as $\mathcal{S}(\mathbb{R};E_{-\infty})$ is dense in $\mathcal{S}'(\mathbb{R};E_\infty)$.

For point (3), since $-1\le\alpha\le 1$, the inclusions are clear together with the Hilbert space property. The density is as follows. Let $f\in W_\alpha$. By definition, let $g\in L^2(\mathbb{R};D_{S,\alpha})$ such that $\hat f = |\tau|^{\frac{1+\alpha}{2}}\hat g$ in $\mathcal{S}'(\mathbb{R};E_\infty)$. Note that the right hand side also belongs to $L^1_{\mathrm{loc}}(\mathbb{R};D_{S,\alpha})$. Take a sequence $(\varphi_k)_{k\in\mathbb{N}}\in\mathcal{S}(\mathbb{R};E_{-\infty})^{\mathbb{N}}$ such that $\hat\varphi_k\to\hat g$ in $L^2(\mathbb{R};D_{S,\alpha})$ and $0\notin\operatorname{supp}(\hat\varphi_k)$. Take $f_k:=\mathcal{F}^{-1}(|\tau|^{(1+\alpha)/2}\hat\varphi_k)$, then $f_k\in\mathcal{S}_0(\mathbb{R};E_{-\infty})$ and $f_k\to f$ in W_α.

The proof of (4) is standard, using Fourier transform and that powers of S commute with multiplication by powers of $|\tau|$, and the calculus occurs using the duality between $\mathcal{S}'(\mathbb{R};E_\infty)$ and $\mathcal{S}(\mathbb{R};E_{-\infty})$. □

4.1.3. *The main Theorem.* Now, we come to the main result of this subsection.

Theorem 4.7 (Invertibility on intermediate spaces)**.** *For all $\alpha\in[-1,1]$, the operator ∂_t+S^2, defined on $\mathcal{S}(\mathbb{R};E_{-\infty})$, extends to a bounded and invertible operator $A_\alpha: V_\alpha\to V_{-\alpha}^\star$, which agrees with the restriction of ∂_t+S^2 acting on $\mathcal{S}'(\mathbb{R};E_\infty)$.*

Proof. From Fourier transform, $u\in V_\alpha$ if and only if $S\hat u\in L^2(\mathbb{R};H)$ and $|\tau|^{\frac{1-\alpha}{2}}S^\alpha\hat u\in L^2(\mathbb{R};H)$. Hence, $f=\partial_t u+S^2u$ is easily seen to belong to $L^2(\mathbb{R};D_{S,-1})+W_\alpha=V_{-\alpha}^\star$. The density of $\mathcal{S}(\mathbb{R};E_{-\infty})$ in V_α yields the bounded extension operator A_α and notice that $A_\alpha=\partial_t+S^2$ on V_α.

To show the invertibility, it suffices to prove that the restriction of the inverse of ∂_t+S^2 on $\mathcal{S}'(\mathbb{R};E_\infty)$ to $V_{-\alpha}^\star$ is bounded into V_α. Let $w\in V_{-\alpha}^\star$ and again, by Fourier transform, write $\hat w = S\hat f+|\tau|^{\frac{1+\alpha}{2}}S^{-\alpha}\hat g$, with $f,g\in L^2(\mathbb{R};H)$. Define $u\in\mathcal{S}'(\mathbb{R};E_\infty)$ by $\hat u(\tau)=\left(i\tau+S^2\right)^{-1}\hat g(\tau)$. That $S\hat u\in L^2(\mathbb{R};H)$ follows from the uniform boundedness (with respect to τ) of $S^2(i\tau+S^2)^{-1}$ and $|\tau|^{\frac{1+\alpha}{2}}S^{1-\alpha}(i\tau+S^2)^{-1}$ and that $|\tau|^{\frac{1-\alpha}{2}}S^\alpha\hat u\in L^2(\mathbb{R};H)$ from that of $|\tau|(i\tau+S^2)^{-1}$ and $|\tau|^{\frac{1-\alpha}{2}}S^{1+\alpha}(i\tau+S^2)^{-1}$. Hence $u\in V_\alpha$ and the estimate follows by taken infimum over all choices of f and g. □

Remark 4.8. When $\alpha=-1$, we recover Proposition 4.5 (Lions' result) : existence in V_{-1}, and uniqueness follows from Proposition 4.1. Note however that uniqueness occurs when $u\in V_1$ which is the largest possible space in that scale.

Remark 4.9. The Fourier method is rather elementary once the setup has been designed, but does not furnish time continuity: we mostly used that ∂_t and S commute. Something specific to time derivatives is the classical embedding theorem of Lions [Lio57] mentioned earlier. This embedding is not true any longer when $V=D(S)$ and $V^\star=D(S^{-1})$ are replaced with their completions $D_{S,1}$ and $D_{S,-1}$ if I is bounded. Indeed, as the embedding $D_{S,1}\hookrightarrow H$ fails, pick $v\in D_{S,1}\setminus H$ and define the function $u(t)=v$, $0\le t\le 1$. We have $u\in L^2((0,1);D_{S,1})$ and $\partial_t u=0$ but $u\notin C([0,1];H)$.

However, this counterexample is ruled out if $I=\mathbb{R}$ or I unbounded and in fact, the continuity holds. This can be obtained when $\alpha=-1$ by approximation from Lions' result but we present a different approach, which has the advantage of allowing $\alpha<0$ to conclude for regularity. Note however, that when $\alpha=0$, continuity cannot hold for all sources in $V_0^\star$ by the isomorphism property. We would have otherwise that any $u\in V_0$ is continuous, valued in H, but this is not the case.

4.2. **Solving the abstract heat equation using the Duhamel method.** Since $-S^2$ generates a C_0 contraction semigroup on H, the Duhamel formula

$$Tf(t):=\int_{-\infty}^t e^{-(t-s)S^2}f(s)\,\mathrm{d}s \tag{4.10}$$

is a way of constructing solutions to $\partial_t u+S^2u=f$ in $\mathcal{D}'(\mathbb{R};E_\infty)$. Remark that the adjoint Duhamel formula

$$\tilde{T}\tilde{f}(s):=\int_s^\infty e^{-(t-s)S^2}\tilde{f}(t)\ \mathrm{d}t \tag{4.11}$$

is a way of constructing solutions to the backward equation $-\partial_s\tilde{u}+S^2\tilde{u}=\tilde{f}$ in $\mathcal{D}'(\mathbb{R};E_\infty)$. All what we shall prove for the (forward) heat equation applies to the backward one. We leave to the reader the care of checking it. For the moment, we assume f to be a test function.

Lemma 4.12 (A priori properties for the Duhamel solution)**.** *If $f\in\mathcal{S}(\mathbb{R};E_{-\infty})$, then Tf defined by* (4.10) *belongs to $\mathcal{S}'(\mathbb{R};E_\infty)$ and is a solution of $\partial_t u+S^2u=f$ in $\mathcal{S}'(\mathbb{R};E_\infty)$.*

Proof. First using the regularity and contractivity of the semigroup,

$$\iint_{\mathbb{R}^2}1_{t>s}|\langle e^{-(t-s)S^2}f(s),\varphi(t)\rangle|\ \ \mathrm{d}s\mathrm{d}t\le\|f\|_{L^1(\mathbb{R};H)}\|\varphi\|_{L^1(\mathbb{R};H)}$$

for any $f,\varphi\in\mathcal{S}(\mathbb{R};E_{-\infty})$. In particular u is defined for all t by a Bochner integral and belongs to $L^\infty(\mathbb{R};H)$. Hence we may apply Fubini's theorem freely, exchanging integrals and inner products in the calculation below:

$$\begin{aligned}
-\int_{\mathbb{R}}\langle u(t),\partial_t\varphi(t)\rangle_H\ \mathrm{d}t&=-\int_{\mathbb{R}}\int_{-\infty}^t\langle f(s),e^{-(t-s)S^2}\partial_t\varphi(t)\rangle_H\ \mathrm{d}s\mathrm{d}t\\
&=-\int_{\mathbb{R}}\int_s^{+\infty}\langle f(s),e^{-(t-s)S^2}\partial_t\varphi(t)\rangle_H\ \mathrm{d}t\mathrm{d}s\\
&=-\int_{\mathbb{R}}\langle f(s),\int_s^{+\infty}e^{-(t-s)S^2}\partial_t\varphi(t)\rangle_H\ \mathrm{d}t\mathrm{d}s\\
&=-\int_{\mathbb{R}}\langle f(s),-\varphi(s)+\int_s^{+\infty}S^2e^{-(t-s)S^2}\varphi(t)\ \mathrm{d}t\rangle_H\ \mathrm{d}s\\
&=\int_{\mathbb{R}}\langle f(s),\varphi(s)\rangle_H\ \mathrm{d}s-\int_{\mathbb{R}}\int_s^{+\infty}\langle e^{-(t-s)S^2}f(s),S^2\varphi(t)\rangle_H\ \mathrm{d}t\mathrm{d}s.
\end{aligned}$$

Using Fubini once more, this shows that $-\langle\!\langle u,\partial_t\varphi\rangle\!\rangle_{\mathcal{S}',\mathcal{S}}=\langle\!\langle f,\varphi\rangle\!\rangle_{\mathcal{S}',\mathcal{S}}-\langle\!\langle u,S^2\varphi\rangle\!\rangle_{\mathcal{S}',\mathcal{S}}$, which means $\partial_t u+S^2u=f$ in $\mathcal{S}'(\mathbb{R};E_\infty)$. □

We now gather a number of a priori estimates which are related to solving the heat equation within $L^2(\mathbb{R};D_{S,1})$.

Lemma 4.13 (A priori estimates for the Duhamel operator)**.** *Let $f\in\mathcal{S}(\mathbb{R};E_{-\infty})$, and define $u=Tf$. For the inequalities involving $\|f\|_{W_\alpha}$, we additionally assume that $f\in\mathcal{S}_0(\mathbb{R};E_{-\infty})$.*

(1) *$u\in C_0(\mathbb{R};H)$ and one has the following uniform bounds*

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\|u(t)\|_H\le\|f\|_{L^1(\mathbb{R};H)}\\
&\sup_{t\in\mathbb{R}}\|u(t)\|_H\le\frac{1}{\sqrt{2}}\|f\|_{L^2(\mathbb{R};D_{S,-1})}\\
&\sup_{t\in\mathbb{R}}\|u(t)\|_H\le C(\alpha)\|f\|_{W_\alpha},\ \alpha\in[-1,0).
\end{aligned}$$

(2) *$u\in L^2(\mathbb{R};D_{S,1})$ and one has the following energy inequalities*

$$\begin{aligned}
&\|u\|_{L^2(\mathbb{R};D_{S,1})}\le\frac{1}{\sqrt{2}}\|f\|_{L^1(\mathbb{R};H)}\\
&\|u\|_{L^2(\mathbb{R};D_{S,1})}\le\|f\|_{L^2(\mathbb{R};D_{S,-1})}\\
&\|u\|_{L^2(\mathbb{R};D_{S,1})}\le C'(\alpha)\|f\|_{W_\alpha},\ \alpha\in[-1,1].
\end{aligned}$$

(3) *$u\in V_\alpha$ for all $\alpha\in[-1,1]$ and one has the following bound*

$$\|D_t^{\frac{1-\alpha}{2}}S^\alpha u\|_{L^2(\mathbb{R};H)}\le\|f\|_{W_\alpha}.$$

Proof. That u belongs to $L^\infty(\mathbb{R};H)$ with $\|u\|_{L^\infty(\mathbb{R};H)} \le \|f\|_{L^1(\mathbb{R};H)}$ has been already observed above. Note that $\partial_t u = f - T(S^2 f) \in L^\infty(\mathbb{R};H)$. Thus u is Lipschitz, hence continuous. The limit 0 at $-\infty$ is clear from the fact that $\|f(s)\|_H$ has rapid decay and the contraction property of the semigroup. As for the limit at $+\infty$, we write for fixed and large A and $t > A$,

$$u(t) = e^{-(t-A)S^2}u(A) + \int_A^t e^{-(t-s)S^2} f(s)\ \mathrm{d}s.$$

The first term tends to 0 in H by properties of the semigroup and, for the second term, one uses again the contraction property and rapid decay of $\|f(s)\|_H$.

We are left with proving the remaining estimates.

Step 1: $\|u(t)\|_H \le \frac{1}{\sqrt{2}} \|f\|_{L^2(\mathbb{R};D_{S,-1})}$ *for all* $t \in \mathbb{R}$. Using Cauchy-Schwarz inequality, we have for all $t \in \mathbb{R}$ and $a \in H$,

$$\int_{-\infty}^t |\langle S^{-1}f(s), Se^{-(t-s)S^2}a\rangle_H|\ \mathrm{d}s \le \frac{1}{\sqrt{2}} \left(\int_{-\infty}^t \|S^{-1}f(s)\|_H^2\ \mathrm{d}s\right)^{1/2} \|a\|_H\,,$$

where we have used the quadratic equality

$$\int_0^\infty \|Se^{-sS^2}a\|_H^2 \mathrm{d}s = \frac{1}{2}\|a\|_H^2\,.$$

As

$$\langle u(t), a\rangle_H = \int_{-\infty}^t \langle S^{-1}f(s), Se^{-(t-s)S^2}a\rangle_H\ \mathrm{d}s,$$

we obtain the desired bound for $\|u(t)\|_H$.

Step 2: $\|u\|_{L^2(\mathbb{R};D_{S,1})} \le \frac{1}{\sqrt{2}}\|f\|_{L^1(\mathbb{R};H)}$. We observe that by Fubini's theorem, we have $\langle Su, \tilde{f}\rangle = \langle u, S\tilde{f}\rangle = \langle f, \tilde{u}\rangle$, where $\tilde{u} = \tilde{T}(S\tilde{f})$. Thus

$$|\langle Su, \tilde{f}\rangle| \le \|f\|_{L^1(\mathbb{R};H)}\, \|\tilde{T}(S\tilde{f})\|_{L^\infty(\mathbb{R};H)} \le \frac{1}{\sqrt{2}}\, \|f\|_{L^1(\mathbb{R};H)}\, \|\tilde{f}\|_{L^2(\mathbb{R};H)}$$

using step 1 for $\tilde{T}$.

Step 3: $\|u\|_{L^2(\mathbb{R};D_{S,1})} \le \|f\|_{L^2(\mathbb{R};D_{S,-1})}$. We already know from step 2 that $\|u\|_{L^2(\mathbb{R};D_{S,1})}$ is finite. To obtain the desired bound, we use again Fubini's theorem several times and obtain

$$\begin{aligned}
\|Su\|_{L^2(\mathbb{R};H)}^2 &= \int_{\mathbb{R}} \langle Su(t), Su(t)\rangle_H\ \mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{-\infty}^t \int_{-\infty}^t \langle S^2 e^{-(t-s)S^2} S^{-1}f(s), S^2 e^{-(t-s')S^2} S^{-1}f(s')\rangle_H\ \mathrm{d}s\mathrm{d}s'\mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\max(s,s')}^{+\infty} \langle S^4 e^{-(2t-(s+s'))S^2} S^{-1}f(s), S^{-1}f(s')\rangle_H\ \mathrm{d}t\mathrm{d}s\mathrm{d}s' \\
&= \frac{1}{2}\int_{\mathbb{R}} \int_{\mathbb{R}} \langle S^2 e^{-(2\max(s,s')-(s+s'))S^2} S^{-1}f(s), S^{-1}f(s')\rangle_H\ \mathrm{d}s\mathrm{d}s' \\
&= \int_{\mathbb{R}} \int_{s\le s'} \langle S^2 e^{-(s'-s)S^2} S^{-1}f(s), S^{-1}f(s')\rangle_H\ \mathrm{d}s\mathrm{d}s' \\
&= \int_{\mathbb{R}} \langle Su(s'), S^{-1}f(s')\rangle_H\ \mathrm{d}s'.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we deduce that

$$\|Su\|_{L^2(\mathbb{R};H)}^2 \le \|Su\|_{L^2(\mathbb{R};H)}\, \|f\|_{L^2(\mathbb{R};D_{S,-1})}\,.$$

Therefore

$$\|u\|_{L^2(\mathbb{R};D_{S,1})} \le \|f\|_{L^2(\mathbb{R};D_{S,-1})}\,.$$

Step 4: $\|u(t)\|_H \leq C(\alpha)\|f\|_{W_\alpha}$, $\alpha \in [-1,0)$. For all $a \in E_{-\infty}$, we define

$$\forall t \in \mathbb{R};\quad \varphi_a(t) := \mathbb{1}_{(-\infty,0]}(t) e^{tS^2} a.$$

Remark that when $a \in H$, $\varphi_a \in L^2(\mathbb{R}, D_{S,1})$, hence $\varphi_a \in L^2(\mathbb{R}, E_{-\infty})$. For $t \in \mathbb{R}$,

$$\begin{aligned}\langle u(t), a\rangle_H &= \int_{-\infty}^0 \langle e^{sS^2} f(s-t), a\rangle_H \, \mathrm{d}s \\ &= \int_{\mathbb{R}} \langle f(s-t), \varphi_a(s)\rangle_H \, \mathrm{d}s \\ &= \langle \tau_t g, D_t^{\frac{1+\alpha}{2}} S^{-\alpha}\varphi_a\rangle_{L^2(\mathbb{R};H),L^2(\mathbb{R};H)}.\end{aligned}$$

In the calculation, we used $\hat{f}(0) = 0$ (see Lemma 4.6, point (3)) and wrote $f = D_t^{\frac{1+\alpha}{2}} S^{-\alpha} g$ with $g \in L^2(\mathbb{R};H)$, defined $\tau_t g(s) = g(s-t)$ and used that translations commute with D_t. If we show that

$$\|D_t^{\frac{1+\alpha}{2}} S^{-\alpha}\varphi_a\|_{L^2(\mathbb{R};H)} = C(\alpha)\,\|a\|_H\,,$$

then

$$|\langle u(t), a\rangle_H| \leq C(\alpha)\,\|g\|_{L^2(\mathbb{R};H)}\,\|a\|_H\,,$$

and we may conclude using the density of $E_{-\infty}$ in H. To see this, applying Fourier transform to φ_a, we get

$$\forall \tau \in \mathbb{R};\quad \hat{\varphi_a}(\tau) = (-i\tau + S^2)^{-1} a,$$

so that

$$|\tau|^{1+\alpha}\left\|S^{-\alpha}(-i\tau+S^2)^{-1}a\right\|_H^2 = |\tau|^{-1}\langle \psi(|\tau|^{-1/2}S)a, a\rangle_H$$

with $\psi(t) = t^{-2\alpha}(1+t^4)^{-1}$. Using simple computations and Calderón's identity

$$\int_{-\infty}^{\infty} \psi(|\tau|^{-1/2}S)a\,\frac{\mathrm{d}\tau}{|\tau|} = \int_0^\infty \frac{t^{-2\alpha}}{1+t^4}\frac{\mathrm{d}t}{t}\; a,$$

we obtain

$$\int_{\mathbb{R}} |\tau|^{1+\alpha}\left\|S^{-\alpha}\hat{\varphi}_a(\tau)\right\|_H^2\,\mathrm{d}\tau = \int_0^\infty \frac{t^{-2\alpha}}{1+t^4}\frac{\mathrm{d}t}{t}\;\|a\|_H^2\,,$$

and conclude using Plancherel identity that $C(\alpha)^2 = \frac{1}{2\pi}\int_0^\infty \frac{t^{-2\alpha}}{1+t^4}\frac{\mathrm{d}t}{t}$.

Step 5: $\|u\|_{L^2(\mathbb{R};D_{S,1})} \leq C'(\alpha)\|f\|_{W_\alpha}$, $\alpha \in [-1,1]$. Since $f \in \mathcal{S}(\mathbb{R};E_{-\infty})$, we know a priori that $u \in L^2(\mathbb{R};D_{S,1})$ from Step 2 and u agrees with the solution given by Fourier transform of Theorem 4.7. Hence, we can use Fourier transform to compute. We have

$$\hat{u}(\tau) = (i\tau + S^2)^{-1}\,|\tau|^{\frac{1+\alpha}{2}}\,S^{-\alpha} g(\tau).$$

where $f = D_t^{\frac{1+\alpha}{2}} S^{-\alpha} g$ with $g \in L^2(\mathbb{R};H)$. Hence

$$\begin{aligned}\|S\hat{u}(\tau)\|_H^2 &= \langle(\tau^2+S^4)^{-1}|\tau|^{1+\alpha}S^{2-2\alpha}\hat{g}(\tau), \hat{g}(\tau)\rangle_H \\ &= \|(|\tau|^{-1/2}S)^{1-\alpha}(1+(|\tau|^{-1/2}S)^4)^{-1/2}\hat{g}(\tau)\|_H^2 \\ &\leq C'(\alpha)^2\|\hat{g}(\tau)\|_H^2\end{aligned}$$

with $C'(\alpha) = \sup_{t>0} t^{1-\alpha}(1+t^4)^{-1/2} < \infty$ when $-1 \leq \alpha \leq 1$.

Step 6: $\|D_t^{\frac{1-\alpha}{2}} S^\alpha u\|_{L^2(\mathbb{R};H)} \le \|f\|_{W_\alpha}$, $\alpha \in [-1,1]$. We proceed as in step 5, and compute

$$\||\tau|^{\frac{1-\alpha}{2}} S^\alpha \hat{u}(\tau)\|_H^2 = \langle (\tau^2 + S^4)^{-1} |\tau|^2 \hat{g}(\tau), \hat{g}(\tau)\rangle_H \le \|\hat{g}(\tau)\|_H^2.$$

The conclusion follows. □

Remark 4.14. As noted in the proof, we can identify the Duhamel solution with the Fourier solution. So this gives an indirect proof that the Duhamel solution belongs to $\mathcal{S}(\mathbb{R}; E_{-\infty})$ for $f \in \mathcal{S}(\mathbb{R}; E_{-\infty})$.

4.3. **Regularity of solutions.** We can now deduce existence and uniqueness results together with regularity. We begin with the simplest case.

Theorem 4.15 (Regularity for source in L^1)**.** *Let $f \in L^1(\mathbb{R}; H)$. Then there exists a unique $u \in L^2(\mathbb{R}; D_{S,1})$ solution of the equation $\partial_t u + S^2 u = f$ in $\mathcal{S}'(\mathbb{R}; E_\infty)$. Moreover $u \in C_0(\mathbb{R}; H)$ with*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H \le \|f\|_{L^1(\mathbb{R};H)} \qquad \text{and} \qquad \|u\|_{L^2(\mathbb{R};D_{S,1})} \le \frac{1}{\sqrt{2}} \|f\|_{L^1(\mathbb{R};H)}.$$

Proof. Uniqueness in $L^2(\mathbb{R}; D_{S,1})$ is provided by Proposition 4.1.

The existence of such a regular solution with the estimates follow from Lemmas 4.12 and 4.13, when $f \in \mathcal{S}(\mathbb{R}; E_{-\infty})$.

Density of $\mathcal{S}(\mathbb{R}; E_{-\infty})$ in $L^1(\mathbb{R}; H)$ allows us to pass to the limit both in the weak formulation of the equation and in the estimates. That the limit stays in $C_0(\mathbb{R}; H)$ follows from the closedness of this space for the sup norm. □

We turn to the second result extending Proposition 4.5 (the case $\beta = -1$).

Theorem 4.16 (Regularity for source in $W_{-\beta}$)**.** *Let $\beta \in (0,1]$ and fix $f \in W_{-\beta}$. Then there exists a unique $u \in L^2(\mathbb{R}; D_{S,1})$ solution of $\partial_t u + S^2 u = f$ in $\mathcal{S}'(\mathbb{R}; E_\infty)$. Moreover $u \in V_{-\beta} \cap C_0(\mathbb{R}; H)$ and there exists a constant $C = C(\beta) > 0$ independent of f such that*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H + \|u\|_{V_{-\beta}} \le C \|f\|_{W_{-\beta}}.$$

Proof. It is a repetition of that of Theorem 4.15, gathering uniqueness of Proposition 4.1, estimates of Lemma 4.13 with $\beta = -\alpha$, and density from Lemma 4.6. □

Remark 4.17. For $\beta \le 0$, there is a solution in $V_{-\beta}$ by Theorem 4.7, but it does not belong to $C_0(\mathbb{R}; H)$.

5. Embeddings and integral identities

The study of the abstract heat equation leads to embeddings for functions spaces in the spirit of Lions and then to integral identities expressing absolute continuity.

5.1. **Embeddings.**

Corollary 5.1 (Extended Lions' embedding)**.** *For $\alpha \in [-1, 0)$, we have $V_\alpha \hookrightarrow C_0(\mathbb{R}; H)$.*

Proof. Fix $\alpha \in [-1, 0)$ and let $u \in V_\alpha$. We have $\partial_t u \in W_\alpha$ and $S^2 u \in L^2(\mathbb{R}; D_{S,-1}) = W_{-1}$, hence $f = \partial_t u + S^2 u \in W_{-1} + W_\alpha = V_{-\alpha}^\star$. As $V_\alpha \subset L^2(\mathbb{R}; D_{S,1})$, by Proposition 4.1, u is the unique solution in $L^2(\mathbb{R}; D_{S,1})$ of the equation

$$\partial_t \tilde{u} + S^2 \tilde{u} = f \ \text{ in } \mathcal{S}'(\mathbb{R}; E_\infty).$$

Using linearity and Theorem 4.16 for $\beta = -\alpha$ and $\beta = 1$, we deduce that $u \in C_0(\mathbb{R}; H)$ and we have

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H \le C(\alpha) \|\partial_t u\|_{W_\alpha} + \left(1 + \frac{1}{\sqrt{2}}\right) \left\|S^2 u\right\|_{L^2(\mathbb{R};D_{S,-1})} \le \tilde{C}(\alpha) \|u\|_{V_\alpha}.$$

□

Remark 5.2. The case $\alpha = -1$ is the homogeneous version of Lions result mentioned before. For $\alpha \in [0,1]$, there is no chance to have an embedding $V_\alpha \hookrightarrow C_0(\mathbb{R}; H)$. In fact, the embedding $\dot{H}^{1/2}(\mathbb{R}; H) \hookrightarrow L^\infty(\mathbb{R}; H)$ fails (case $\alpha = 0$), as the scalar embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for the classical inhomogeneous Sobolev space of order $1/2$ already fails.

We complete the embeddings by exploring further the cases $\alpha \in [0,1]$ although this does not require the heat operator $\partial_t + S^2$.

Lemma 5.3 (Hardy-Littlewood-Sobolev embedding)**.** *Let $\alpha \in (0,1]$ and let $r = 2/\alpha \in [2,\infty)$. Then, we have $V_\alpha \hookrightarrow L^r(\mathbb{R}; D_{S,\alpha})$ and there is a constant $C = C(r) > 0$ such that for all $u \in V_\alpha$,*

$$\|u\|_{L^r(\mathbb{R};D_{S,\alpha})} \leq C(r)\|D_t^{\frac{1-\alpha}{2}} u\|_{L^2(\mathbb{R};D_{S,\alpha})}.$$

Consequently, we have $L^{r'}(\mathbb{R}; D_{S,-\alpha}) \hookrightarrow W_{-\alpha}$, where r' is the Hölder conjugate of r.

Proof. The inequality holds for $u \in \mathcal{S}(\mathbb{R}; E_{-\infty})$ using the Sobolev embedding in $\mathbb{R}$ extended to $D_{S,\alpha}$-valued functions as the inverse of $D_t^{\frac{1-\alpha}{2}}$ is the Riesz potential with exponent $\frac{1-\alpha}{2}$. We conclude by density and a duality argument. □

The next result shows that V_0 and $V_1 \cap L^\infty(\mathbb{R}; H)$ share similar embeddings.

Proposition 5.4 (Mixed norm embeddings)**.** *For $r \in (2,\infty)$ and $\alpha = 2/r$, we have $V_1 \cap L^\infty(\mathbb{R}; H) \hookrightarrow L^r(\mathbb{R}; D_{S,\alpha})$ and $V_0 \hookrightarrow L^r(\mathbb{R}; D_{S,\alpha})$, with*

$$\|u\|_{L^r(\mathbb{R};D_{S,\alpha})} \leq \|u\|^{\alpha}_{L^2(\mathbb{R};D_{S,1})}\|u\|^{1-\alpha}_{L^\infty(\mathbb{R};H)}$$

and

$$\|u\|_{L^r(\mathbb{R};D_{S,\alpha})} \leq \|u\|^{\alpha}_{L^2(\mathbb{R};D_{S,1})}\|D_t^{1/2}u\|^{1-\alpha}_{L^2(\mathbb{R};H)}.$$

Consequently, $L^{r'}(\mathbb{R}; D_{S,-\alpha}) \hookrightarrow L^2(\mathbb{R}; D_{S,-1}) + L^1(\mathbb{R}; H)$ and $L^{r'}(\mathbb{R}; D_{S,-\alpha}) \hookrightarrow L^2(\mathbb{R}; D_{S,-1}) + W_0$.

Proof. For the first inequality, use the moment inequality

$$\|S^\alpha u(t)\|_H \leq \|Su(t)\|^\alpha_H \|u(t)\|^{1-\alpha}_H \leq \|Su(t)\|^\alpha_H \|u\|^{1-\alpha}_{L^\infty(\mathbb{R};H)}$$

and integrate its r-power.

For the second inequality, start with the moment inequality expressed in Fourier side when $u \in \mathcal{S}(\mathbb{R}; E_{-\infty})$, for fixed τ,

$$\||\tau|^{\frac{1-\alpha}{2}} S^\alpha \hat{u}(\tau)\|_H \leq \|S\hat{u}(\tau)\|^\alpha_H \||\tau|^{1/2}\hat{u}(\tau)\|^{1-\alpha}_H.$$

Next, take its square, integrate in τ, use Hölder inequality, Plancherel identity and density to conclude.

The consequences are standard by density and duality and we skip details. □

Remark 5.5. Note that the first inequality and its dual version in the statement hold whenever $\mathbb{R}$ is replaced by any interval. However, the second one and its dual version have a meaning only on $\mathbb{R}$.

Remark 5.6. Let $\alpha \in (0,1]$. Let $S^2 = -\Delta_x$, more precisely $S = (-\Delta_x)^{1/2}$, where Δ_x is the usual Laplace operator defined as a self-adjoint operator on $L^2(\mathbb{R}^n)$. When $2\alpha < n$, Sobolev embedding in $\mathbb{R}^n$ gives us

$$D(S^\alpha) \subset L^q(\mathbb{R}^n), \text{ with } \|v\|_{L^q} \leq C\|S^\alpha v\|_{L^2},\ q = \frac{2n}{n-2\alpha}.$$

This is true for $0 \leq \alpha \leq 1$ if $n \geq 3$, or $\alpha \in (0, \frac{1}{2})$ if $n = 1$ or $\alpha < 1$ if $n = 2$. Thus $D_{S,\alpha} \hookrightarrow L^q(\mathbb{R}^n)$. When $r = 2/\alpha$, we have then $V_\alpha \hookrightarrow L^r(\mathbb{R}; L^q(\mathbb{R}^n))$. The constraints are equivalent to

$$\frac{1}{r} + \frac{n}{2q} = \frac{n}{4} \quad \text{and} \quad 2 \leq r, q < \infty.$$

Thus, we recover the mixed space $L^r_t L^q_x$ that appears in the classical theory [LSU68, chp. 3] and deduce for them the classical embedding $\|u\|_{L^r_t L^q_x} \leq C\|\nabla_x u\|^{2/r}_{L^2_t L^2_x}\|u\|^{1-2/r}_{L^\infty_t L^2_x}$ from the first inequality in Proposition 5.4. This argument is inspired from the one in [AE23].

5.2. **Integral identities.** The Lions' embedding using domains of S and S^{-1} comes with integral identities. We now prove they hold using completions of the domains of S and S^{-1}, and allowing more general right hand sides.

Proposition 5.7 (Integral identities: the real line case)**.** *Let $u \in L^2(\mathbb{R}; D_{S,1})$ and let $\rho \in (2, \infty]$. Assume that $\partial_t u = f + g$ with $f \in L^2(\mathbb{R}; D_{S,-1})$ and $g \in L^{\rho'}(\mathbb{R}; D_{S,-\beta})$, where $\beta = 2/\rho \in [0, 1)$ and ρ' is the Hölder conjugate of ρ. Then $u \in C_0(\mathbb{R}; H)$, $t \mapsto \|u(t)\|_H^2$ is absolutely continuous on $\mathbb{R}$ and for all $\sigma < \tau$,*

$$\|u(\tau)\|_H^2 - \|u(\sigma)\|_H^2 = 2\mathrm{Re} \int_\sigma^\tau \langle f(t), u(t)\rangle_{H,-1} + \langle g(t), u(t)\rangle_{H,-\beta} \, \mathrm{d}t. \tag{5.8}$$

In particular, if $\rho = \infty$, then we infer that

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H \leq \sqrt{2 \|u\|_{L^2(\mathbb{R};D_{S,1})} \|f\|_{L^2(\mathbb{R};D_{S,-1})}} + (1 + \sqrt{2}) \|g\|_{L^1(\mathbb{R};H)}. \tag{5.9}$$

Remark that with our notation, $\mathrm{Re}\langle f(t), u(t)\rangle_{H,-1} = \mathrm{Re}\langle u(t), f(t)\rangle_{H,1}$.

Proof. The assumption $u \in L^2(\mathbb{R}; D_{S,1})$ is equivalent to $S^2 u \in L^2(\mathbb{R}; D_{S,-1})$, hence u verifies the equation

$$\partial_t u + S^2 u = S^2 u + f + g =: h.$$

Using Theorem 4.16 when $\rho < \infty$ and Theorem 4.15 when $\rho = \infty$, we know that $u \in L^2(\mathbb{R}; D_{S,1}) \cap C_0(\mathbb{R}; H)$. It remains to prove the identity.

Let $f_k, g_k \in \mathcal{S}(\mathbb{R}; E_{-\infty})$ with $f_k \to S^2 u + f$ in $L^2(\mathbb{R}; D_{S,-1})$ and $g_k \to g$ in $L^{\rho'}(\mathbb{R}; D_{S,-\beta})$ and set $h_k = f_k + g_k$. Let $u_k \in L^2(\mathbb{R}; D_{S,1})$ be the unique solution of the equation $\partial_t u_k + S^2 u_k = h_k$ given by Corollary 4.3. We have $u_k \in \mathcal{S}(\mathbb{R}; E_{-\infty})$.

The regularity of u_k allows us to write for all $\sigma < \tau$,

$$\|u_k(\tau)\|_H^2 - \|u_k(\sigma)\|_H^2 = 2\mathrm{Re} \int_\sigma^\tau \langle \partial_t u_k(t), u_k(t)\rangle_H \, \mathrm{d}t.$$

Since $\partial_t u_k = -S^2 u_k + h_k$ by the equation, we have for all $\sigma < \tau$,

$$\|u_k(\tau)\|_H^2 - \|u_k(\sigma)\|_H^2 = 2\mathrm{Re} \int_\sigma^\tau \langle f_k(t) - S^2 u_k(t), u_k(t)\rangle_H + \langle g_k(t), u_k(t)\rangle_H \, \mathrm{d}t.$$

To pass to the limit when $k \to \infty$, we observe that $u_k \to u$ in $L^2(\mathbb{R}; D_{S,1})$ and in $C_0(\mathbb{R}; H)$ in all cases, and also in $L^\rho(\mathbb{R}; D_{S,\beta})$ when $\rho < \infty$. In particular $f_k - S^2 u_k \to f$ in $L^2(\mathbb{R}; D_{S,-1})$. We obtain (5.8) at the limit.

In the case $\rho = \infty$, letting $\sigma \to -\infty$ and taking τ at which $\|u(\tau)\|_H = \sup \|u(t)\|_H = X$, we obtain

$$X^2 = \|u(\tau)\|_H^2 \leq 2 \int_{-\infty}^\infty \|S^{-1} f(t)\|_H \|Su(t)\|_H \, \mathrm{d}t + 2X \int_{-\infty}^\infty \|g(t)\|_H \, \mathrm{d}t.$$

Solving the inequality for X, we obtain the conclusion. □

We stress that the above result is false on bounded intervals as evidenced by the counter-example in Remark 4.9. But it remains valid on half-lines. On $(0, \infty)$ say, it can be shown either using the backward heat equation or an extension method. We describe the second method below.

Corollary 5.10 (Integral identities: the half-line case)**.** *Let I be an open half-line of $\mathbb{R}$. Let $u \in L^2(I; D_{S,1})$ and let $\rho \in (2, \infty]$. Assume that $\partial_t u = f + g$ with $f \in L^2(I; D_{S,-1})$ and $g \in L^{\rho'}(I; D_{S,-\beta})$, where $\beta = 2/\rho \in [0, 1)$. Then $u \in C_0(\bar{I}, H)$, $t \mapsto \|u(t)\|_H^2$ is absolutely continuous on $\bar{I}$ and (5.8) holds for all $\sigma, \tau \in \bar{I}$ such that $\sigma < \tau$.*

Proof. We assume that $I = (0, \infty)$ because it is always possible to go back to this case. We will construct an even extension u_e of u and odd extensions g_o, f_o of g, f to $\mathbb{R}$. These extensions belong to the same spaces as u, f, g but in $\mathbb{R}$ and $\partial_t u_e = f_o + g_o$. Thus, Proposition 5.7 applies to u_e. We obtain the conclusion by restricting to $\bar{I}$.

We start by defining for all $a \in E_{-\infty}$ the distribution $\langle u, a\rangle$ on $(0,\infty)$ by setting

$$\forall \phi \in \mathcal{D}((0,\infty);\mathbb{C}), \langle\langle u, a\rangle, \phi\rangle_{\mathcal{D}',\mathcal{D}} := \langle\!\langle u, \phi \otimes a\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} = \int_0^\infty \langle Su(t), S^{-1}a\rangle_H \bar\phi(t)\ \mathrm{d}t.$$

Hence $\langle u, a\rangle$ is locally integrable and agrees with $\langle u, a\rangle(t) = \langle Su(t), S^{-1}a\rangle_H$ almost everywhere. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, a\rangle = \langle g, a\rangle_{H,-\alpha} + \langle f, a\rangle_{H,-1} \quad in\ \mathcal{D}'((0,\infty);\mathbb{C}).$$

The assumptions on u, f, g imply that $\langle u, a\rangle \in W^{1,1}(0,T)$ for any $T > 0$. It follows that $\langle u, a\rangle$ can be identified with a absolutely continuous function on $[0,\infty)$. We define $u_e \in \mathcal{D}'(\mathbb{R}; E_\infty)$ by

$$\langle\!\langle u_e, \phi \otimes a\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} := \int_0^\infty \langle u, a\rangle(t)(\bar\phi(t) + \bar\phi(-t))\ \mathrm{d}t.$$

using that distributions $\mathcal{D}'(\mathbb{R}; E_\infty)$ are uniquely determined on tensor products $\phi\otimes a$ with $\phi \in \mathcal{D}(\mathbb{R})$ and $a \in E_{-\infty}$. We have $u_e = u$ in $\mathcal{D}'((0,\infty); E_\infty)$ by taking ϕ supported in $(0,\infty)$. Next, integration by parts shows that

$$\begin{aligned}\langle\!\langle u_e, \frac{\mathrm{d}}{\mathrm{d}t}(\phi \otimes a)\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} &= -\int_0^\infty (\langle S^{-\beta}g(t), S^\beta a\rangle_H + \langle S^{-1}f(t), Sa\rangle_H)(\bar\phi(t) + \bar\phi(-t))\ \mathrm{d}t \\ &= -\int_{\mathbb{R}} (\langle g_o(t), a\rangle_{H,-\beta} + \langle f_o(t), a\rangle_{H,-1})\bar\phi(t)\ \mathrm{d}t,\end{aligned}$$

where g_o and f_o are the odd extensions of g and f, respectively. Hence $\partial_t u_e = g_o + f_o$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$. Lastly,

$$\begin{aligned}\langle\!\langle Su_e, \phi \otimes a\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} = \langle\!\langle u_e, S(\phi \otimes a)\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} &= \int_0^\infty \langle Su(t), a\rangle_H(\bar\phi(t) + \bar\phi(-t))\ \mathrm{d}t \\ &= \int_{\mathbb{R}} \langle (Su)_e(t), a\rangle_H \bar\phi(t)\ \mathrm{d}t,\end{aligned}$$

where $(Su)_e$ is the even extension of Su, so that $Su_e = (Su)_e$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$. □

The conclusion of Corollary 5.10 can be polarized, given two functions u, $\tilde u$ that verify the assumptions of Corollary 5.10 with the same exponent $\rho \in (2,\infty]$ and $\beta = 2/\rho$. Thanks to the extendability seen in the previous proof, the same also works with open, half-infinite intervals and the conclusion is as follows.

Corollary 5.11 (Polarized integral identities)**.** *Assume that u, $\tilde u$ satisfy the same assumptions as in Corollary 5.10 on two open infinite intervals I and J with non empty intersection. Then $t \mapsto \langle u(t), \tilde u(t)\rangle_H$ is absolutely continuous on $\bar I \cap \bar J$ and we have for all $\sigma, \tau \in \bar I \cap \bar J$ such that $\sigma < \tau$*

$$\begin{aligned}\langle u(\tau), \tilde u(\tau)\rangle_H - \langle u(\sigma), \tilde u(\sigma)\rangle_H = &\int_\sigma^\tau \langle f(t), \tilde u(t)\rangle_{H,-1} + \langle g(t), \tilde u(t)\rangle_{H,-\beta}\ \mathrm{d}t \\ &+ \int_\sigma^\tau \langle u(t), \tilde f(t)\rangle_{H,1} + \langle u(t), \tilde g(t)\rangle_{H,\beta}\ \mathrm{d}t.\end{aligned}$$

Remark 5.12. We note that by linearity, the above identities hold with g replaced by a sum of several terms in $L^{\rho'}(\mathbb{R}; D_{S,-\beta})$ for different pairs (ρ,β) and similarly for the polarized version. However, the inequality (5.9) should be modified accordingly.

On a bounded interval there is a statement with an extra $L^1(I;H)$ hypothesis on u.

Corollary 5.13 (Integral identities: the bounded case)**.** *Let I be a bounded, open interval of $\mathbb{R}$. Let $u \in L^2(I; D_{S,1}) \cap L^1(I;H)$ and let $\rho \in (2,\infty]$. Assume that $\partial_t u = f + g$ with $f \in L^2(I; D_{S,-1})$ and $g \in L^{\rho'}(I; D_{S,-\beta})$, where $\beta = 2/\rho \in [0,1)$. Then $u \in C(\bar I, H)$, $t \mapsto \|u(t)\|_H^2$ is absolutely continuous on $\bar I$ and* (5.8) *holds for all $\sigma, \tau \in \bar I$ such that $\sigma < \tau$.*

Proof. Assume that $I = (0, \mathfrak{T})$. If we take χ a smooth real-valued function that is equal to 1 near 0 and 0 near $\mathfrak{T}$ and set $v = \chi u$ on $(0, \infty)$ then we can see that $v \in L^2((0,\infty); D_{S,1})$ and $\partial_t v \in L^2((0,\infty); D_{S,-1}) + L^{\rho'}((0,\infty); D_{S,-\beta}) + L^1((0,\infty); H)$. We may apply Corollary 5.10 with the above remark, and by restriction we have the conclusion on any subinterval $[0, \mathfrak{T}']$ with $\mathfrak{T}' < \mathfrak{T}$. If we now do this with a smooth real-valued function that is equal to 0 near 0 and 1 near $\mathfrak{T}$, and apply Corollary 5.10 on $(-\infty, \mathfrak{T})$ we have by restriction the conclusion for u on any subinterval $[\mathfrak{T}', \mathfrak{T}]$ with $0 < \mathfrak{T}'$. We conclude on $[0, \mathfrak{T}]$ by gluing. □

6. Abstract parabolic equations

In this section, we study parabolic equations of type

$$\partial_t u + \mathcal{B} u = f,$$

where $\partial_t + \mathcal{B}$ is a parabolic operator with a time-dependent elliptic part $\mathcal{B}$ under "divergence structure". Here, we do not assume any time-regularity on $\mathcal{B}$ apart its weak measurability. We provide a complete framework to prove well-posedness and to construct propagators and fundamental solution operators avoiding density arguments from parabolic operators with time regular elliptic part. We also avoid time regularization like Steklov approximations. Uniqueness implies that our construction agrees with others under common hypotheses.

6.1. **Setup.** Throughout this section, we fix an operator

$$T : D(T) \subset H \to K$$

which is injective, closed and densely defined from $D(T) \subset H$ to another complex separable Hilbert space K. The operator $T^\star T$ is an injective, positive self-adjoint operator on H, so is $S := (T^\star T)^{1/2}$. Moreover, by the Kato's second representation theorem [Kat13], we have

$$D(S) = D(T) \quad \text{and} \quad \forall u, v \in D(T),\ \langle Su, Sv\rangle_H = \langle T^\star T u, v\rangle_H = \langle Tu, Tv\rangle_K.$$

As a result, $D_{S,1}$ is a completion of $D(T)$ for the norm $\|T\cdot\|_K$.

Next, $(B_t)_{t\in\mathbb{R}}$ is a fixed family of bounded and coercive sesquilinear forms on $D(T) \times D(T)$ with respect to the homogeneous norm on $D(T)$ and with uniform bounds (independent of t). To be precise, $B_t : D(T) \times D(T) \to \mathbb{C}$ is a sesquilinear form verifying

$$|B_t(u,v)| \le M \|u\|_{S,1} \|v\|_{S,1} \ , \quad \nu \|u\|_{S,1}^2 \le \mathrm{Re}(B_t(u,u)), \tag{6.1}$$

for some $M, \nu > 0$ and for all $t \in \mathbb{R}$ and $u, v \in D(S)$. This is the equivalent to saying that for all $t \in \mathbb{R}$, there exists a bounded and strictly accretive linear map $A(t)$ on $\overline{\mathrm{ran}(T)}$ such that

$$\forall u, v \in D(T),\ B_t(u,v) = \langle A(t) Tu, Tv\rangle_K. \tag{6.2}$$

We assume in addition that the family $(B_t)_{t\in\mathbb{R}}$ is weakly measurable, *i.e.*, $t \mapsto B_t(u,v)$ is a measurable function on $\mathbb{R}$, for all $u, v \in D(T)$.

We keep denoting by B_t the unique extension of B_t to $D_{S,1} \times D_{S,1}$. Remark that the family $(B_t)_{t\in\mathbb{R}}$ is automatically weakly measurable for the reason that for all $u, v \in D_{S,1}$ the function $t \mapsto B_t(u,v)$ is a pointwise limit of a sequence of measurable functions.

Note that the adjoint forms $B_t^\star$ defined by $B_t^\star(u,v) = \overline{B_t(v,u)}$ have the same properties and are associated to $A(t)^*$.

As

$$\int_{\mathbb{R}} |B_t(u(t), v(t))| \,\mathrm{d}t \le M \|u\|_{L^2(\mathbb{R}; D_{S,1})} \|v\|_{L^2(\mathbb{R}; D_{S,1})},$$

the operator $\mathcal{B}$ defined by

$$\langle\!\langle \mathcal{B}u, v\rangle\!\rangle = \int_{\mathbb{R}} B_t(u(t), v(t)) \,\mathrm{d}t \quad \text{when } u, v \in L^2(\mathbb{R}; D_{S,1})$$

is a bounded operator from $L^2(\mathbb{R}; D_{S,1})$ to $L^2(\mathbb{R}; D_{S,-1})$ with

$$\|\mathcal{B}u\|_{L^2(\mathbb{R}; D_{S,-1})} \le M \|u\|_{L^2(\mathbb{R}; D_{S,1})}.$$

Next, the partial derivative is a well-defined tempered distribution given by

$$\langle\!\langle \partial_t u, \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} = -\langle\!\langle u, \partial_t \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}}.$$

When $u \in L^2(\mathbb{R}; D_{S,1})$, one can compute the right hand side as

$$\langle\!\langle u, \partial_t \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} = \langle\!\langle Su, S^{-1}\partial_t \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}} \langle Su(t), S^{-1}\partial_t \varphi(t) \rangle_H \, \mathrm{d}t.$$

Definition 6.3 (The forward parabolic operator associated to the family $(B_t)_{t\in\mathbb{R}}$)**.** The operator

$$\partial_t + \mathcal{B} : L^2(\mathbb{R}; D_{S,1}) \to \mathcal{S}'(\mathbb{R}; E_\infty)$$

defined using the weak formulation

$$\forall u \in L^2(\mathbb{R}; D_{S,1}), \forall \varphi \in \mathcal{S}(\mathbb{R}; E_{-\infty}), \ \langle\!\langle \partial_t u + \mathcal{B}u, \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} := -\langle\!\langle u, \partial_t \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} + \int_{\mathbb{R}} B_t(u(t), \varphi(t)) \, \mathrm{d}t$$

is called the parabolic operator associated to the family $(B_t)_{t\in\mathbb{R}}$. The definition is the same as above when $\mathbb{R}$ is substituted by an open interval $I \subset \mathbb{R}$, replacing $\mathcal{S}'(\mathbb{R}; E_\infty)$ by $\mathcal{D}'(I; E_\infty)$ and $\mathcal{S}(\mathbb{R}; E_{-\infty})$ by $\mathcal{D}(I; E_{-\infty})$. In both cases, we formally write $\partial_t + \mathcal{B} = \partial_t + T^\star A(t) T$.

Remark that this definition needs no assumption $u \in L^1_{\mathrm{loc}}(\mathbb{R}; H)$ but if it is the case one can use also $\langle\!\langle u, \partial_t \varphi \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}} \langle u(t), \partial_t \varphi(t) \rangle_H \, \mathrm{d}t$ (see Section 8). For $u, v \in \mathcal{S}(\mathbb{R}; E_{-\infty})$, an integration by parts then yields

$$\langle\!\langle \partial_t u + \mathcal{B}u, v \rangle\!\rangle_{\mathcal{S}',\mathcal{S}} = \overline{\langle\!\langle -\partial_t v + \mathcal{B}^\star v, u \rangle\!\rangle}_{\mathcal{S}',\mathcal{S}},$$

where $-\partial_t + \mathcal{B}^\star$ is the backward parabolic operator associated to the adjoint family of forms $(B_t^\star)_{t\in\mathbb{R}}$ defined similarly.

We wish to find (weak) solutions $u \in L^2(\mathbb{R}; D_{S,1})$ to $\partial_t u + \mathcal{B}u = f$ for appropriate source terms. The challenge here is that we cannot use Fourier Transform anymore, nor a semi-group. We could start with Lions representation theorem but we choose a different route, introducing a variational parabolic operator.

We denote by H_t the Hilbert transform with symbol $i\tau/|\tau|$. More precisely, if $u \in \mathcal{S}'(\mathbb{R}; E_\infty)$ is such that we have $i\tau/|\tau| \, \mathcal{F}u \in \mathcal{S}'(\mathbb{R}; E_\infty)$, then we set

$$H_t u := \mathcal{F}^{-1}\left(i\frac{\tau}{|\tau|} \mathcal{F}u \right).$$

We define a bounded sesquilinear form $B_{V_0} : V_0 \times V_0 \to \mathbb{C}$ by

$$\forall u, v \in V_0, \ B_{V_0}(u, v) := \int_{\mathbb{R}} \langle H_t D_t^{1/2} u(t), D_t^{1/2} v(t) \rangle_H + B_t(u(t), v(t)) \, \mathrm{d}t.$$

By the Riesz representation theorem, there exists a unique $\mathcal{H} \in \mathcal{L}(V_0, V_0^\star)$ such that

$$\langle\!\langle \mathcal{H}u, v \rangle\!\rangle_{V_0^\star, V_0} := B_{V_0}(u, v), \ u, v \in V_0.$$

We have

$$(\partial_t + \mathcal{B})_{|V_0} = \mathcal{H} \ , \quad (-\partial_t + \mathcal{B}^\star)_{|V_0} = \mathcal{H}^\star,$$

where $\mathcal{H}^\star : V_0 \to V_0^\star$ is the adjoint of $\mathcal{H}$. Indeed, we have the almost everywhere equality

$$\langle H_t D_t^{1/2} u(t), D_t^{1/2} v(t) \rangle_H = -\langle u(t), \partial_t v(t) \rangle_H = -\langle Su(t), S^{-1}\partial_t v(t) \rangle_H$$

when $u, v \in \mathcal{S}(\mathbb{R}; E_{-\infty})$ so that $\mathcal{H}$ and $\partial_t + \mathcal{B}$ agree on $\mathcal{S}(\mathbb{R}; E_{-\infty})$ and we conclude by density in V_0. Thus, we may call $\mathcal{H}$ the variational parabolic operator associated to $\mathcal{B}$ as it comes from the sesquilinear form B_{V_0} and V_0 plays the role of a variational space.

6.2. **Existence and uniqueness results.** We now prove our main results.

6.2.1. *Source term in $V_0^\star$: Kaplan's method.* The following lemma is essentially due to Kaplan [Kap66]. It expresses hidden coercivity of the variational parabolic operator $\mathcal{H}$. We reproduce the argument for completeness.

Lemma 6.4 (Kaplan's lemma: invertibility on the pivotal variational space)**.** *For each $f \in V_0^\star$, there exists a unique $u \in V_0$ such that $\mathcal{H}u = f$. Moreover,*

$$\|u\|_{V_0} \leq C(M,\nu)\,\|f\|_{V_0^\star}\,.$$

Proof. By the Plancherel theorem and the fact that the Hilbert transform H_t commutes with $D_t^{1/2}$ and S, it is a bijective isometry on V_0. As it is skew-adjoint, for all $\delta \in \mathbb{R}$, $1+\delta H_t$ is an isomorphism on V_0 and $\|(1+\delta H_t)u\|_{V_0}^2 = (1+\delta^2)\|u\|_{V_0}^2$. The same equality holds on $V_0^\star$.

Let $\delta > 0$ to be chosen later. The modified sesquilinear form $B_{V_0}(\cdot,(1+\delta H_t)\cdot)$ is bounded on $V_0 \times V_0$ and for all $u \in V_0$

$$\begin{aligned}\mathrm{Re}\, B_{V_0}(u,(1+\delta H_t)u) &= \mathrm{Re}\int_{\mathbb{R}} \langle H_t D_t^{1/2}u, D_t^{1/2}(1+\delta H_t)u\rangle_H + B_t(u(t),(1+\delta H_t)u(t))\,\mathrm{d}t\\ &= \mathrm{Re}\int_{\mathbb{R}} \delta\langle H_t D_t^{1/2}u(t), H_t D_t^{1/2}u(t)\rangle_H + B_t(u(t),u(t)) + \delta B_t(u(t),H_t u(t))\,\mathrm{d}t.\end{aligned}$$

where we have used that H_t is skew-adjoint, hence

$$\mathrm{Re}\int_{\mathbb{R}} \langle H_t D_t^{1/2}u(t), D_t^{1/2}u(t)\rangle_H\,\mathrm{d}t = 0.$$

We obtain

$$\mathrm{Re}(B_{V_0}(u,(1+\delta H_t)u)) \geq \delta\|D_t^{1/2}u\|^2_{L^2(\mathbb{R};H)} + (\nu-\delta M)\,\|u\|^2_{L^2(\mathbb{R};D_{S,1})}\,.$$

Choosing $\delta = \frac{\nu}{1+M}$, it becomes

$$\mathrm{Re}(B_{V_0}(u,(1+\delta H_t)u)) \geq \frac{\nu}{1+M}\,\|u\|^2_{V_0}\,,\ \forall u \in V_0.$$

Fix $f \in V_0^\star$. The Lax-Milgram lemma implies that there exists a unique $u \in V_0$ such that

$$B_{V_0}(u,(1+\delta H_t)\cdot) = (1+\delta H_t)^\star \circ f.$$

Furthermore, we have the estimate

$$\|u\|_{V_0} \leq \frac{1+M}{\nu}\,\|(1+\delta H_t)^\star \circ f\|_{V_0^\star}\,.$$

Using the fact that $(1+\delta H_t)^\star$ is an isomorphism on $V_0^\star$ with operator norm equal to $\sqrt{1+\delta^2}$, we have that for each $f \in V_0^\star$ there exists a unique $u \in V_0$ such that $B_{V_0}(u,\cdot) = f$ with

$$\|u\|_{V_0} \leq \frac{1+M}{\nu} \times \sqrt{1+\left(\frac{\nu}{1+M}\right)^2}\,\|f\|_{V_0^\star}\,.$$

□

Now, we come to the uniqueness result below.

Proposition 6.5 (Uniqueness in energy space)**.** *Let I be an interval which is a neighbourhood of $-\infty$. If $u \in L^2(I;D_{S,1})$ is a solution of $\partial_t u + \mathcal{B}u = 0$ in $\mathcal{D}'(I;E_\infty)$, then $u = 0$.*

Proof. We have $u \in L^2(I;D_{S,1})$ and $\partial_t u = -\mathcal{B}u \in L^2(I;D_{S,-1})$. Using Corollary 5.10, we have $u \in C_0(\overline{I};H)$ and verifies for $\sigma,\tau \in \bar{I}$ such that $\sigma < \tau$,

$$\|u(\tau)\|_H^2 - \|u(\sigma)\|_H^2 = -2\mathrm{Re}\int_\sigma^\tau B_t(u(t),u(t))\,\mathrm{d}t \leq 0\,.$$

When $\sigma \to -\infty$, we deduce that $u(\tau) = 0$, for all $\tau \in \overline{I}$. □

Remark 6.6. When $I = \mathbb{R}$, one can directly prove Proposition 6.5 using uniqueness in Lemma 6.4. In fact, if $u \in L^2(\mathbb{R};D_{S,1})$ is a solution of $\partial_t u + \mathcal{B}u = 0$ in $\mathcal{D}'(\mathbb{R};E_\infty)$ then $\partial_t u = -\mathcal{B}u \in L^2(\mathbb{R};D_{S,-1})$, so $u \in V_{-1} \subset V_0$ and $\mathcal{H}u = 0$, therefore $u = 0$ by Lemma 6.4.

6.2.2. *Source term in $W_{-\beta}$, $\beta \in (0,1]$.* Let us start with the following theorem.

Proposition 6.7 (Existence and uniqueness for $W_{-\beta}$ source)**.** *Let $\beta \in (0,1]$ and let $f \in W_{-\beta}$. Then, there exists a unique $u \in L^2(\mathbb{R}; D_{S,1})$ solution to $\partial_t u + \mathcal{B}u = f$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$. Moreover, $u \in C_0(\mathbb{R}; H) \cap V_{-\beta}$ and there exists $C = C(M, \nu, \beta) > 0$ such that*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H + \|u\|_{V_{-\beta}} \leq C \|f\|_{W_{-\beta}}.$$

Proof. Since $W_{-\beta} \hookrightarrow V_\beta^\star \hookrightarrow V_0^\star$, Lemma 6.4 provides us with a solution $u = \mathcal{H}^{-1} f \in V_0$ with the estimate $\|u\|_{V_0} \leq C(M,\nu) \|f\|_{V_0^\star}$, so in particular, $\|u\|_{L^2(\mathbb{R};D_{S,1})} \leq C(M,\nu,\beta) \|f\|_{W_{-\beta}}$. Uniqueness in $L^2(\mathbb{R}; D_{S,1})$ is provided by Proposition 6.5. Writing the equation as

$$\partial_t u + S^2 u = S^2 u - \mathcal{B}u + f \quad \text{in } \mathcal{D}'(\mathbb{R}; E_\infty),$$

we may combine Theorems 4.15, 4.16 together with $\mathcal{B}u \in L^2(\mathbb{R}; D_{S,-1})$, to see that $u \in C_0(\mathbb{R}; H) \cap V_{-\beta}$ with

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H + \|u\|_{V_{-\beta}} \leq C(M,\nu,\beta) \left(\|u\|_{L^2(\mathbb{R};D_{S,1})} + \|f\|_{W_{-\beta}} \right).$$

Therefore,

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H + \|u\|_{V_{-\beta}} \leq C(M,\nu,\beta) \|f\|_{W_{-\beta}}.$$

□

Corollary 6.8 (Boundedness properties of $\mathcal{H}^{-1}$)**.** *Fix $\rho \in [2,\infty)$ and set $\beta = 2/\rho \in (0,1]$. Then,*

$$\mathcal{H}^{-1} : L^{\rho'}(\mathbb{R}; D_{S,-\beta}) \to V_{-\beta} \cap C_0(\mathbb{R}; H) \text{ is bounded.}$$

The same holds for $(\mathcal{H}^\star)^{-1}$.

Proof. Combine Proposition 6.7, Lemma 5.3 and Proposition 5.4. □

Remark 6.9. For fixed $f \in L^{\rho'}(\mathbb{R}; D_{S,-\beta})$, we have $\mathcal{H}^{-1} f \in V_{-\beta} \subset V_0$. In particular, using Proposition 5.4, we have $\mathcal{H}^{-1} f \in L^r(\mathbb{R}; D_{S,\alpha})$ for any $r \in (2,\infty)$ where $\alpha = 2/r$ and there exists a constant $C = C(M,\nu,\beta) > 0$ such that

$$\left\|\mathcal{H}^{-1} f\right\|_{L^r(\mathbb{R};D_{S,\alpha})} \leq C \|f\|_{L^{\rho'}(\mathbb{R};D_{S,-\beta})}.$$

The same is true for $(\mathcal{H}^\star)^{-1}$.

6.2.3. *Source term in $L^1(\mathbb{R}; H)$.* The previous theorems rely on Lemma 6.4 to prove the existence, so they do not apply anymore when $f \in L^1(\mathbb{R}; H)$ since $L^1(\mathbb{R}; H) \not\subseteq V_0^\star$. Yet, we can solve with such source terms using a duality scheme.

Proposition 6.10 (Existence and uniqueness for source in L^1)**.** *Let $f \in L^1(\mathbb{R}; H)$. Then there exists a unique $u \in L^2(\mathbb{R}, D_{S,1})$ solution to $\partial_t u + \mathcal{B}u = f$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$. Moreover, $u \in C_0(\mathbb{R}; H) \cap L^2(\mathbb{R}; D_{S,1})$ and there exists a constant $C = C(M,\nu) > 0$ such that*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_H + \|u\|_{L^2(\mathbb{R};D_{S,1})} \leq C \|f\|_{L^1(\mathbb{R};H)}. \tag{6.11}$$

Proof. Uniqueness is provided by Proposition 6.5. To prove the existence, we remark that Corollary 6.8 for the backward operator $\mathcal{H}^\star$ in the case $\rho = 2$ implies that $(\mathcal{H}^\star)^{-1}$ is bounded from $L^2(\mathbb{R}; D_{S,-1})$ into $C_0(\mathbb{R}; H)$. We define

$$\mathcal{T} : L^1(\mathbb{R}; H) \to \mathcal{D}'(\mathbb{R}; E_\infty), \ \langle\!\langle \mathcal{T} f, \varphi \rangle\!\rangle_{\mathcal{D}',\mathcal{D}} := \langle\!\langle f, (\mathcal{H}^\star)^{-1} \varphi \rangle\!\rangle_{L^1(\mathbb{R};H),L^\infty(\mathbb{R};H)},$$

and we have

$$\|\mathcal{T} f\|_{L^2(\mathbb{R};D_{S,1})} \leq C(M,\nu) \|f\|_{L^1(\mathbb{R};H)}. \tag{6.12}$$

Next, let f in $\mathcal{D}(\mathbb{R}; E_{-\infty})$. We write for all $\varphi \in \mathcal{D}(\mathbb{R}; E_{-\infty})$, observing that $\mathcal{D}(\mathbb{R}; E_{-\infty}) \subset V_0^\star$,

$$\langle\!\langle \mathcal{T} f, \varphi \rangle\!\rangle_{\mathcal{D}',\mathcal{D}} = \langle\!\langle \mathcal{H}\mathcal{H}^{-1} f, (\mathcal{H}^\star)^{-1} \varphi \rangle\!\rangle_{V_0^\star,V_0} = \langle\!\langle \mathcal{H}^{-1} f, \varphi \rangle\!\rangle_{\mathcal{D}',\mathcal{D}}.$$

Hence, $\mathcal{T}f = \mathcal{H}^{-1}f \in C_0(\mathbb{R};H)$ and

$$\forall \varphi \in \mathcal{D}(\mathbb{R};E_{-\infty}),\quad -\langle\!\langle \mathcal{T}f, \partial_t\varphi\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} + \int_{\mathbb{R}} B_t(\mathcal{T}f(t),\varphi(t))\,\mathrm{d}t = \int_{\mathbb{R}} \langle f(t),\varphi(t)\rangle_H\,\mathrm{d}t. \tag{6.13}$$

Using (5.9), we obtain

$$\sup_{t\in\mathbb{R}} \|\mathcal{T}f(t)\|_H \leq \sqrt{2M}\,\|\mathcal{T}f\|_{L^2(\mathbb{R};D_{S,1})} + (1+\sqrt{2})\,\|f\|_{L^1(\mathbb{R};H)} \leq C(M,\nu)\,\|f\|_{L^1(\mathbb{R};H)}\,. \tag{6.14}$$

Now, let us pick $f \in L^1(\mathbb{R};H)$. Let $(f_k)_{k\in\mathbb{N}} \in \mathcal{D}(\mathbb{R};E_{-\infty})^{\mathbb{N}}$ such that $f_k \to f$ in $L^1(\mathbb{R};H)$. By (6.12) and (6.14), we have $\mathcal{T}f_k \to \mathcal{T}f$ in $L^2(\mathbb{R};D_{S,1}) \cap C_0(\mathbb{R};H)$. Using (6.13) with $\mathcal{T}f_k$ for a fixed $\varphi \in \mathcal{D}(\mathbb{R};E_{-\infty})$ and letting $k\to\infty$ imply that $\partial_t(\mathcal{T}f) + \mathcal{B}(\mathcal{T}f) = f$ in $\mathcal{D}'(\mathbb{R};E_\infty)$. □

6.2.4. *Source term is a bounded measure on H.* First, we define the space of bounded H-valued measures on $\mathbb{R}$, denoted $\mathcal{M}(\mathbb{R};H)$, as the topological anti-dual space of $C_0(\mathbb{R};H)$ with respect to the sup-norm. We denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{M},C_0}$ the anti-duality bracket. We equip the space $\mathcal{M}(\mathbb{R};H)$ with the operator norm, that is

$$\|\mu\|_{\mathcal{M}} := \sup_{\varphi\in C_0(\mathbb{R};H)\setminus\{0\}} \frac{|\langle\!\langle\mu,\varphi\rangle\!\rangle_{\mathcal{M},C_0}|}{\|\varphi\|_{L^\infty(\mathbb{R};H)}}.$$

It is a Banach space containing a subspace isometric to $L^1(\mathbb{R};H)$.

For $\mu\in\mathcal{M}(\mathbb{R};H)$, $\mathrm{supp}(\mu)$ is the complement of the largest open set of $\mathbb{R}$ on which μ is equal to 0. More precisely, we say that μ equals 0 on an open set $\Omega\subset\mathbb{R}$ if for all $\phi\in C_0(\mathbb{R},H)$ with support contained in Ω, $\langle\!\langle\mu,\phi\rangle\!\rangle_{\mathcal{M},C_0} = 0$. Let $\mathcal{N}$ denotes the set of all such open sets. We have

$$\mathrm{supp}(\mu) := \mathbb{R}\setminus\bigcup_{\Omega\in\mathcal{N}}\Omega.$$

An important example are Dirac measures. For any $s\in\mathbb{R}$ and $a\in H$, we denote by $\delta_s\otimes a$ the Dirac measure on s carried by a which is defined by

$$\langle\!\langle\delta_s\otimes a,\phi\rangle\!\rangle_{\mathcal{M},C_0} = \langle a,\varphi(s)\rangle_H,\ \varphi\in C_0(\mathbb{R};H).$$

We first state the classical lemma below for later use.

Lemma 6.15. *Let $\mu\in\mathcal{M}(\mathbb{R};H)$. Then there exists a sequence $(f_\varepsilon)_{\varepsilon>0}$ in $L^1(\mathbb{R};H)$ such that $f_\varepsilon \rightharpoonup \mu$ (weak-$\star$ convergence) and $\sup_{\varepsilon>0}\|f_\varepsilon\|_{L^1(\mathbb{R};H)} \leq \|\mu\|_{\mathcal{M}}$.*

Proof. We obtain the sequence $(f_\varepsilon)_{\varepsilon>0}$ by convoluting μ with a scalar mollifying sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ and we easily check that we have all the required properties. □

Proposition 6.16 (Existence and uniqueness for bounded measure source)**.** *Let $\mu\in\mathcal{M}(\mathbb{R};H)$. Then there exists a unique $u\in L^2(\mathbb{R};D_{S,1})$ solution to $\partial_t u + \mathcal{B}u = \mu$ in $\mathcal{D}'(\mathbb{R};E_\infty)$. Moreover, $u\in L^\infty(\mathbb{R};H)$ f and there is a constant $C = C(M,\nu) > 0$ such that*

$$\|u\|_{L^2(\mathbb{R};D_{S,1})} + \|u\|_{L^\infty(\mathbb{R};H)} \leq C(M,\nu)\,\|\mu\|_{\mathcal{M}}\,. \tag{6.17}$$

If $I\subset\mathbb{R}\setminus\mathrm{supp}(\mu)$ an unbounded open interval, then $u\in C_0(\overline{I},H)$ and $t\mapsto\|u(t)\|_H^2$ is absolutely continuous on $\overline{I}$. Moreover, if I is a neighbourhood of $-\infty$, then $u=0$ on $\overline{I}$.

Proof. Uniqueness is provided by Proposition 6.5. To prove the existence, we use lemma 6.15 to pick $(f_n)_{n\in\mathbb{N}}\in L^1(\mathbb{R};H)^{\mathbb{N}}$ such that $f_n\rightharpoonup\mu$ and $\sup_{n\in\mathbb{N}}\|f_n\|_{L^1(\mathbb{R};H)} \leq \|\mu\|_{\mathcal{M}}$. By Proposition 6.10, for all $n\in\mathbb{N}$, there is a unique $u_n\in L^2(\mathbb{R};D_{S,1})$ solution of the equation $\partial_t u_n + \mathcal{B}u_n = f_n$ in $\mathcal{D}'(\mathbb{R};E_\infty)$ and we have (6.11), implying

$$\|u_n\|_{L^2(\mathbb{R};D_{S,1})} + \|u_n\|_{L^\infty(\mathbb{R};H)} \leq C(M,\nu)\,\|\mu\|_{\mathcal{M}}\,. \tag{6.18}$$

Using the Banach-Alaoglu theorem, there exists $u\in L^2(\mathbb{R};D_{S,1})\cap L^\infty(\mathbb{R};H)$ such that, up to extracting a sub-sequence, $u_n\rightharpoonup u$ weakly in $L^2(\mathbb{R};D_{S,1})$ and weakly-$\star$ in $L^\infty(\mathbb{R};H)$ for the duality pairing L^∞-L^1. We have (6.17) and easily pass to the limit in the equation to obtain the desired solution.

Now, if $I \subset \mathbb{R}$ is an unbounded open interval such that $I \cap \operatorname{supp}(\mu) = \varnothing$, then by Corollary 5.10, $u \in C_0(\overline{I}, H)$ and $t \mapsto \|u(t)\|_H^2$ is absolutely continuous on $\overline{I}$. If I is a neighbourhood of $-\infty$, then $u = 0$ by Proposition 6.5 on I. □

The next corollary is crucial to construct fundamental solution and Green operators.

Corollary 6.19 (Existence and uniqueness for Dirac measure source)**.** *Let $s \in \mathbb{R}$ and $a \in H$. Then there exists a unique $u \in L^2(\mathbb{R}; D_{S,1})$ solution to $\partial_t u + \mathcal{B}u = \delta_s \otimes a$ in $\mathcal{D}'(\mathbb{R}; E_\infty)$. Moreover, $u \in C_0(\mathbb{R} \setminus \{s\}; H)$, equals 0 on $(-\infty, s)$ and $\lim_{t \to s^+} u(t) = a$ in H, and there is a constant $C = C(M, \nu) > 0$ such that*

$$\sup_{t \in \mathbb{R} \setminus \{s\}} \|u(t)\|_H + \|u\|_{L^2(\mathbb{R}; D_{S,1})} \leq C \|a\|_H .$$

Furthermore, $u_{|(s,\infty)}$ is the restriction of an element in V_{-1}.

Proof. Applying the previous result Proposition 6.16, we have existence and uniqueness of u with the estimates and $u \in C_0(\mathbb{R} \setminus \{s\}; H)$, equals 0 on $(-\infty, s)$ and has a limit $u(s^+)$ when $t \to s^+$.

That $u_{|(s,\infty)}$ is a restriction to (s, ∞) of an element in V_{-1} follows from the fact that $\partial_t u = f$ in $\mathcal{D}'((s, \infty); E_\infty)$ with $f = -\mathcal{B}u$ on (s, ∞) and the method of Corollary 5.10, by taking the even extension of u and the odd extension of f with respect to s.

It remains to show $u(s^+) = a$. Let $\tilde{a} \in E_{-\infty}$ and $\theta \in \mathcal{D}(\mathbb{R})$ with $\theta(s) = 1$ and set $\varphi = \theta \otimes \tilde{a}$. Using the absolute continuity of $t \mapsto \langle u(t), \varphi(t) \rangle_H$ on both $[s, \infty)$ and $(-\infty, s]$ using Corollary 5.11, we have

$$-\langle u(s^+), \tilde{a} \rangle_H = \int_s^\infty -B_t(u(t), \varphi(t)) + \langle u(t), \partial_t \varphi(t) \rangle_H \, \mathrm{d}t$$
$$+\langle u(s^-), \tilde{a} \rangle_H = \int_{-\infty}^s -B_t(u(t), \varphi(t)) + \langle u(t), \partial_t \varphi(t) \rangle_H \, \mathrm{d}t$$

By the equation for u on $\mathbb{R}$, and since $u \in L^1_{\mathrm{loc}}(\mathbb{R}; H)$, we obtain

$$\int_{-\infty}^\infty B_t(u(t), \varphi(t)) - \langle u(t), \partial_t \varphi(t) \rangle_H \, \mathrm{d}t = \langle\!\langle \delta_s \otimes a, \varphi \rangle\!\rangle_{\mathcal{M}, C_0} = \langle a, \tilde{a} \rangle_H .$$

Summing up, $-\langle u(s^+), \tilde{a} \rangle_H + \langle u(s^-), \tilde{a} \rangle_H = -\langle a, \tilde{a} \rangle_H$. As $u(s^-) = 0$, this yields $\langle u(s^+), \tilde{a} \rangle_H = \langle a, \tilde{a} \rangle_H$ and we conclude by density of $E_{-\infty}$ in H. □

6.3. **Green operators.** The notion below of Green operators was first introduced by J.-L. Lions [Lio13].

Definition 6.20 (Green operators)**.** Let $t, s \in \mathbb{R}$ and $a, \tilde{a} \in H$.

(1) For $t \neq s$, $G(t, s)a$ is defined as the value at time t of the solution $u \in L^2(\mathbb{R}; D_{S,1})$ of the equation $\partial_t u + \mathcal{B}u = \delta_s \otimes a$ in Corollary 6.19.
(2) For $s \neq t$, $\tilde{G}(s, t)\tilde{a}$ is defined as the value at time s of the solution $u \in L^2(\mathbb{R}; D_{S,1})$ of the equation $-\partial_s u + \mathcal{B}^\star u = \delta_t \otimes \tilde{a}$ in Corollary 6.19.

The operators $G(t, s)$ and $\tilde{G}(s, t)$ are called the Green operators for the parabolic operator $\partial_t + \mathcal{B}$ and the backward parabolic operator $-\partial_t + \mathcal{B}^\star$, respectively.

The properties discussed in the last section can be summarized in the following corollary.

Corollary 6.21 (Estimates for Green operators)**.** *There is a constant $C = C(M, \nu) > 0$ such that one has the following statements.*

(1) *For all $t < s \in \mathbb{R}$, $G(t, s) = 0$ and for all $a \in H$, $t \mapsto G(t, s)a \in C_0([s, \infty); H)$ with $G(s, s)a = a$ and it is a restriction to (s, ∞) of an element in V_{-1}, and for any $r \in [2, \infty)$, $a \in H$ and $s \in \mathbb{R}$, we have $G(\cdot, s)a \in L^r((s, \infty); D_{S,\alpha})$ where $\alpha = 2/r$ with*

$$\sup_{t \geq s} \|G(t, s)\|_{\mathcal{L}(H)} \leq C \quad \text{and} \quad \int_s^{+\infty} \|G(t, s)a\|_{S,\alpha}^r \, \mathrm{d}t \leq C^r \|a\|_H^r .$$

(2) *For all* $s > t \in \mathbb{R}$, $\tilde{G}(s,t) = 0$ *and for all* $\tilde{a} \in H$, $s \mapsto \tilde{G}(s,t)\tilde{a} \in C_0((-\infty,t];H)$ *with* $\tilde{G}(t,t)\tilde{a} = \tilde{a}$ *and it is a restriction to* $(-\infty,t)$ *of an element in* V_{-1}, *and for any* $r \in [2,\infty)$, $t \in \mathbb{R}$, *we have* $\tilde{G}(s,\cdot)a \in L^r((-\infty,t);D_{S,\alpha})$ *where* $\alpha = 2/r$ *with*

$$\sup_{t\geq s} \|\tilde{G}(s,t)\|_{\mathcal{L}(H)} \leq C \quad \text{and} \quad \int_{-\infty}^{t} \|\tilde{G}(s,t)\tilde{a}\|_{S,\alpha}^r \mathrm{d}s \leq C^r \|\tilde{a}\|_H^r.$$

Proof. The properties follows from construction in Corollary 6.19 and the interpolation inequalities in Proposition 5.4 for $2 < r < \infty$. □

Moreover, expected adjointness and Chapman-Kolmogorov relations hold.

Proposition 6.22 (Adjointess and Chapman-Kolmogorov identities)**.** *The following statements hold.*

(1) *For all* $s < t$, $G(t,s)$ *and* $\tilde{G}(s,t)$ *are adjoint operators.*
(2) *For any* $s < r < t$, *we have* $G(t,s) = G(t,r)G(r,s)$.

Proof. We first prove point (1). We fix $t, s \in \mathbb{R}$ such that $s < t$. For $a, \tilde{a} \in H$, we can apply the integral identity of Corollary 5.11 to $u := G(\cdot,s)a$ and $v = \tilde{G}(\cdot,t)\tilde{a}$ between s and t. Note that by duality the integrand vanishes almost everywhere, hence

$$\langle G(t,s)a, \tilde{a}\rangle_H = \langle G(t,s)a, \tilde{G}(t,t)\tilde{a}\rangle_H = \langle G(s,s)a, \tilde{G}(s,t)\tilde{a}\rangle_H = \langle a, \tilde{G}(s,t)\tilde{a}\rangle_H.$$

The adjunction property follows. For point (2), we apply the same equality between r and t and use that the adjoint of $G(t,r)$ is $\tilde{G}(r,t)$ from point (1), to obtain

$$\langle G(t,s)a, \tilde{a}\rangle_H = \langle G(t,s)a, \tilde{G}(t,t)\tilde{a}\rangle_H = \langle G(r,s)a, \tilde{G}(r,t)\tilde{a}\rangle_H = \langle G(t,r)G(r,s)a, \tilde{a}\rangle_H.$$

□

6.4. **Fundamental solution.** We define the fundamental solution as representing the inverse of $\partial_t + \mathcal{B}$.

Definition 6.23 (Fundamental solution for $\partial_t + \mathcal{B}$)**.** A fundamental solution for $\partial_t + \mathcal{B}$ is a family $\Gamma = (\Gamma(t,s))_{t,s\in\mathbb{R}}$ such that :

(1) $\sup_{t,s\in\mathbb{R}} \|\Gamma(t,s)\|_{\mathcal{L}(H)} < +\infty$.
(2) $\Gamma(t,s) = 0$ if $s > t$.
(3) For all $a, \tilde{a} \in E_{-\infty}$, the function $(t,s) \mapsto \langle \Gamma(t,s)a, \tilde{a}\rangle_H$ is Borel measurable on $\mathbb{R}^2$.
(4) For all $\phi \in \mathcal{D}(\mathbb{R})$ and $a \in E_{-\infty}$, the solution $u \in L^2(\mathbb{R};D_{S,1})$ of the equation $\partial_t u + \mathcal{B}u = \phi \otimes a$ in $\mathcal{D}'(\mathbb{R};E_\infty)$ satisfies for all $\tilde{a} \in E_{-\infty}$, $\langle u(t), \tilde{a}\rangle_H = \int_{-\infty}^t \phi(s)\langle \Gamma(t,s)a, \tilde{a}\rangle_H \, \mathrm{d}s$, for almost every $t \in \mathbb{R}$.

One defines a fundamental solution $(\tilde{\Gamma}(s,t))_{s,t\in\mathbb{R}}$ to the backward operator $-\partial_s + \mathcal{B}^\star$ analogously and (ii) is replaced by $\tilde{\Gamma}(s,t) = 0$ if $s > t$.

Lemma 6.24 (Uniqueness of fundamental solutions)**.** *There is at most one fundamental solution to* $\partial_t + \mathcal{B}$ *in the sense of Definition 6.23.*

Proof. Assume Γ_1, Γ_2 are two fundamental solutions to $\partial_t + \mathcal{B}$. Fix a and $\tilde{a}$ in $E_{-\infty}$. The function $(t,s) \mapsto \langle \Gamma_k(t,s)a, \tilde{a}\rangle_H$ is bounded and measurable for $k \in \{1,2\}$ by (1) and (3), hence Fubini's Theorem with (2) and (4) yield for all $\phi, \tilde{\phi} \in \mathcal{D}(\mathbb{R})$,

$$\iint_{\mathbb{R}^2} \tilde{\phi}(s)\phi(t)\langle \Gamma_1(t,s)a, \tilde{a}\rangle_H \, \mathrm{d}s\mathrm{d}t = \iint_{\mathbb{R}^2} \tilde{\phi}(s)\phi(t)\langle \Gamma_2(t,s)a, \tilde{a}\rangle_H \, \mathrm{d}s\mathrm{d}t.$$

Therefore, we obtain $\langle \Gamma_1(t,s)a, \tilde{a}\rangle_H = \langle \Gamma_2(t,s)a, \tilde{a}\rangle_H$ for almost every $(t,s) \in \mathbb{R}^2$. At this stage, the almost everywhere equality can depend on a and $\tilde{a}$. Applying this for test elements $a, \tilde{a} \in E_{-\infty}$ describing a countable set in $E_{-\infty}$ that is dense in H and using that the operators $\Gamma_1(t,s)$, $\Gamma_2(t,s)$ are bounded on H by (1), one deduces that Γ_1 and Γ_2 agree almost everywhere. □

6.5. **The Green operators are the fundamental solution operators.** The two notions are well defined and we show that they lead to the same families. We borrow partially ideas from [AE23].

Theorem 6.25 (Green operators and fundamental solution operators agree)**.** *The family of Green operators is the fundamental solution (up to almost everywhere equality) and (4) holds for all $t \in \mathbb{R}$.*

Proof. As there is at most one fundamental solution, it suffices to show that the family of Green operators satisfies the requirements (1)-(4) in Definition 6.23.

The Green operators verify (1) and (2) of the Definition 6.23 by Corollary 6.21. For the measurability issue (3), remark that for all $a, \tilde{a} \in E_{-\infty}$, we have $(t,s) \mapsto \langle G(t,s)a, \tilde{a}\rangle_H$ is separately continuous on $\mathbb{R}^2 \setminus \{(t,t), t \in \mathbb{R}\}$, so Borel measurable on $\mathbb{R}^2$. We only have to prove (4), namely that for any $\phi \in \mathcal{D}(\mathbb{R})$ and $a, \tilde{a} \in E_{-\infty}$, if u is the weak solution for the source term $\phi \otimes a$, we have for all (not just almost all) $t \in \mathbb{R}$,

$$\langle u(t), \tilde{a}\rangle_H = \int_{-\infty}^{t} \phi(s) \, \langle G(t,s)a, \tilde{a}\rangle_H \ \mathrm{d}s. \tag{6.26}$$

Fix $t \in \mathbb{R}$. Introduce $\tilde{u} = \tilde{G}(\cdot, t)\tilde{a}$. Using the absolute continuity of $s \mapsto \langle u(s), \tilde{u}(s)\rangle_H$ on $(-\infty, t]$ with its zero limit at $-\infty$, and seing $\phi \otimes a \in L^1(\mathbb{R}; H)$, we have by Corollary 5.11

$$\langle u(t), \tilde{a}\rangle_H = \langle u(t), \tilde{u}(t)\rangle_H = \int_{-\infty}^{t} \langle \phi(s)a, \tilde{u}(s)\rangle_H \ \mathrm{d}s.$$

For $s \leq t$, using Proposition 6.22 in the last equality below

$$\langle \phi(s)a, \tilde{u}(s)\rangle_H = \phi(s)\langle a, \tilde{G}(s,t)\tilde{a}\rangle_H = \phi(s)\langle G(t,s)a, \tilde{a}\rangle_H$$

and we are done. □

6.6. **Representation with the fundamental solution operators.** Having identified Green operators to fundamental solution operators, the latter inherits the properties of the former. From now on, we use the more traditional notation $\Gamma(t,s)$. We can now state a complete representation theorem for all the distributional solutions seen in the last subsections, with specified convergence issues.

Theorem 6.27. *Let $s \in \mathbb{R}$, $a \in H$, $g \in L^{\rho'}(\mathbb{R}; H)$, where $\rho \in [2, \infty]$ and $\beta = 2/\rho$. Then the unique solution $u \in L^2(\mathbb{R}; D_{S,1})$ of the equation*

$$\partial_t u + \mathcal{B}u = \delta_s \otimes a + S^\beta g \ \text{ in } \mathcal{D}'(\mathbb{R}; E_\infty)$$

obtained by combining Propositions 6.7, 6.10 and Corollary 6.19 can be represented pointwisely by the equation

$$u(t) = \Gamma(t,s)a + \int_{-\infty}^{t} \Gamma(t,\tau) S^\beta g(\tau) \ \mathrm{d}\tau,$$

where the integral is weakly defined in H when $\rho < \infty$ and strongly defined when $\rho = \infty$ (i.e., in the Bochner sense). More precisely, for all $\tilde{a} \in H$, we have the equality with absolutely converging integral

$$\langle u(t), \tilde{a}\rangle_H = \langle \Gamma(t,s)a, \tilde{a}\rangle_H + \int_{-\infty}^{t} \langle g(\tau), S^\beta \tilde{\Gamma}(\tau, t)\tilde{a}\rangle_H \ \mathrm{d}\tau. \tag{6.28}$$

Remark 6.29. Remark that by Proposition 5.4, one could even reduce to proving the result when $\rho = 2$ and $\rho = \infty$.

Proof. It is enough to prove (6.28). By uniqueness and linearity, we start by writing $u = u_1 + u_2$ where u_k is the solution of the equation considering only the k^{th} term in the right-hand side of the equation. Recall that we have identification of G and Γ. Fix $t \in \mathbb{R}$.

The first term involving a is $u_1(t) = \Gamma(t,s)a$ by construction and identification.

The argument for u_2 is as follows. According to the proof of Theorem 6.24 the weak solution v obtained from source $\phi \otimes c$, where $\phi \in \mathcal{D}(\mathbb{R})$ and $c \in E_{-\infty}$, satisfies for any $\tilde{a} \in E_{-\infty}$ and $t \in \mathbb{R}$,

$$\begin{aligned}\langle v(t), \tilde{a}\rangle_H &= \int_{-\infty}^{t} \phi(\tau)\langle \Gamma(t,\tau)c, \tilde{a}\rangle_H \, \mathrm{d}\tau \\ &= \int_{-\infty}^{t} \langle (\phi \otimes c)(\tau), \tilde{\Gamma}(\tau,t)\tilde{a}\rangle_H \, \mathrm{d}\tau.\end{aligned}$$

For any $\rho \in [2,\infty]$, we have

$$\int_{-\infty}^{t} |\langle h(\tau), \tilde{\Gamma}(\tau,t)\tilde{a}\rangle_{H,-\beta}| \, \mathrm{d}\tau \leq C\|h\|_{L^{\rho'}(\mathbb{R};D_{S,-\beta})}\|\tilde{a}\|_H$$

by using Cauchy-Schwarz inequality in H and Hölder inequality invoking estimates for $\tilde{G}$ in Proposition 6.22. Writing $\langle (\phi \otimes c)(\tau), \tilde{\Gamma}(\tau,t)\tilde{a}\rangle_H$ as $\langle (\phi \otimes c)(\tau), \tilde{\Gamma}(\tau,t)\tilde{a}\rangle_{H,-\beta}$, we can conclude for u_2 by density of the span of tensor products $\phi \otimes c$ in $L^{\rho'}(\mathbb{R}; D_{S,-\beta})$, and density of $E_{-\infty}$ in H. For $\rho = \infty$, we may also verify the strong convergence. □

We record the following operator-valued Schwartz kernel result.

Proposition 6.30. *Let $\phi, \tilde{\phi} \in \mathcal{D}(\mathbb{R})$ and $a, \tilde{a} \in H$. Then,*

$$\langle\!\langle \mathcal{H}^{-1}(\phi \otimes a), \tilde{\phi} \otimes \tilde{a}\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} = \iint \phi(s)\langle \Gamma(t,s)a, \tilde{a}\rangle_H \overline{\tilde{\phi}}(t) \, \mathrm{d}s\mathrm{d}t.$$

In other words, one can see $\langle \Gamma(t,s)a, \tilde{a}\rangle_H$ as the Schwartz kernel of the sesquilinear map $(\phi, \tilde{\phi}) \mapsto \langle \mathcal{H}^{-1}(\phi \otimes a), \tilde{\phi} \otimes \tilde{a}\rangle_{\mathcal{D}',\mathcal{D}}$ on $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R})$.

Proof. By density of $E_{-\infty}$ in H and boundedness of the Green operators, we may use (6.26) for $a, \tilde{a} \in H$ and we obtain,

$$\begin{aligned}\langle\!\langle \mathcal{H}^{-1}(\phi \otimes a), \tilde{\phi} \otimes \tilde{a}\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} &= \int_{\mathbb{R}} \langle \mathcal{H}^{-1}(\phi \otimes a)(t), \tilde{a}\rangle_H \overline{\tilde{\phi}}(t) \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \left(\int_{-\infty}^{t} \phi(s)\langle \Gamma(t,s)a, \tilde{a}\rangle_H \, \mathrm{d}s \right) \overline{\tilde{\phi}}(t) \, \mathrm{d}t \\ &= \iint \phi(s)\langle \Gamma(t,s)a, \tilde{a}\rangle_H \overline{\tilde{\phi}}(t) \, \mathrm{d}s\mathrm{d}t,\end{aligned}$$

where we have used Fubini's theorem and $\Gamma(s,t) = 0$ for $s > t$ in the last line. □

6.7. **The Cauchy problem and the fundamental solution.** In this section, we consider the Cauchy problem on the interval $(0,\infty)$. The coefficients B_t are defined on $(0,\infty)$ and satisfy (6.1) there. We fix $\rho \in [2,\infty]$ and set $\beta = 2/\rho$. The Cauchy problem with initial condition $a \in H$ and $g \in L^{\rho'}((0,\infty); H)$ consists in finding a weak solution to

$$\begin{cases} \partial_t u + \mathcal{B}u = S^\beta g \text{ in } \mathcal{D}'((0,\infty); E_\infty), \\ u(0) = a \text{ in } E_\infty. \end{cases} \tag{6.31}$$

Remark 6.32. Note that when $f \in L^2((0,\infty); K)$, there exists $g \in L^2((0,\infty); H)$ such that $T^* f = Sg$, hence the $\beta = 1$ case covers the classical Lions equation.

Definition 6.33. A weak solution to (6.31) is a function $u \in L^2((0,\infty); D_{S,1})$, with

(i) u solves the the first equation in $\mathcal{D}'((0,\infty); E_\infty)$, that is, for all $\varphi \in \mathcal{D}((0,\infty); E_{-\infty})$

$$\int_0^\infty -\langle u(t), \partial_t\varphi(t)\rangle_{H,1} + B_t(u(t), \varphi(t)) \, \mathrm{d}t = \int_0^\infty \langle g(t), S^\beta\varphi(t)\rangle_H \, \mathrm{d}t.$$

(ii) $\forall \tilde{a} \in E_{-\infty}$, $\langle u(t), \tilde{a}\rangle_H \to \langle a, \tilde{a}\rangle_H$, along a sequence tending to 0.

A weaker formulation testing against functions $\varphi \in \mathcal{D}([0,\infty); E_{-\infty})$ with right hand side containing the additional term $\langle a, \varphi(0)\rangle_H$ is often encountered. In the end it amounts to the same solutions thanks to *a priori* continuity in H, which only uses the upper bound on B_t.

Proposition 6.34. *Any weak solution to (i) belongs to $C_0([0,\infty);H)$ and $t\mapsto \|u(t)\|_H^2$ satisfies the energy equality for any $\sigma,\tau\in[0,\infty]$ such that $\sigma<\tau$,*

$$\|u(\tau)\|_H^2+2\mathrm{Re}\int_\sigma^\tau B_t(u(t),u(t))\ \mathrm{d}t=\|u(\sigma)\|_H^2+2\mathrm{Re}\int_\sigma^\tau \langle g(t),S^\beta u(t)\rangle_H\ \mathrm{d}t.$$

Proof. We assume $u\in L^2((0,\infty);D_{S,1})$ and the equation implies that $\partial_t u\in L^2((0,\infty);D_{S,-1})+L^{\rho'}((0,\infty);D_{S,-\beta})$. We may apply Corollary 5.10. □

The main result of this section is the following theorem which puts together all the theory developed so far.

Theorem 6.35. *Consider the above assumptions on B_t, ρ,β and g,a.*

(1) *There exists a unique weak solution u to the problem (6.31). Moreover, $u\in C_0([0,\infty);H)\cap L^r((0,\infty);H)$ for any $r\in(2,\infty)$ with $\alpha=2/r$ (if $\rho<\infty$, then u is also the restriction to $(0,\infty)$ of an element in $V_{-\beta}$) and*

$$\sup_{t\in[0,\infty)}\|u(t)\|_H+\|u\|_{L^2((0,\infty);D_{S,1})}+\|u\|_{L^r((0,\infty);D_{S,\alpha})}\le C\left(\|g\|_{L^{\rho'}(\mathbb{R};H)}+\|a\|_H\right),$$

where $C=C(M,\nu,\rho)>0$ is a constant independent of g,a.

(2) *There exists a unique fundamental solution $\Gamma=\{\Gamma(t,s)\}_{0\le s\le t<\infty}$ for $\partial_t+\mathcal{B}$ in the sense of Definition 6.23 in $(0,\infty)$. In particular, for all $t\in[0,\infty)$, we have the following representation of u :*

$$u(t)=\Gamma(t,0)a+\int_0^t\Gamma(t,s)S^\beta g(s)\ \mathrm{d}s, \tag{6.36}$$

where the integral is weakly defined in H when $\rho<\infty$ and strongly defined in H when $\rho=\infty$ (i.e., in the Bochner sense). More precisely, for all $\tilde a\in H$ and $t\in[0,\infty)$, we have equality with absolutely converging integral

$$\langle u(t),\tilde a\rangle_H=\langle\Gamma(t,0)a,\tilde a\rangle_H+\int_0^t\langle g(s),S^\beta\tilde\Gamma(s,t)\tilde a\rangle_H\ \mathrm{d}s. \tag{6.37}$$

Proof. We start with the existence of such a solution. We extend g by 0 on $(-\infty,0]$ and keep the same notation for the extensions. We also extend the family $(B_t)_t$ to $\mathbb{R}$ by setting $B_t=\nu\langle S\cdot,S\cdot\rangle_H$ on $\mathbb{R}\setminus(0,\infty)$ and we keep calling $\mathcal{B}$ the operator associated to this family.

Using Proposition 6.7 when $\rho<\infty$ or Proposition 6.10 when $\rho=\infty$ and Corollary 6.19, there exists a unique $\tilde u\in L^2(\mathbb{R};D_{S,1})$ solution of the equation

$$\partial_t\tilde u+\mathcal{B}\tilde u=\delta_0\otimes a+S^\beta g\ \text{ in }\mathcal{D}'(\mathbb{R};E_\infty).$$

Moreover, $\tilde u\in C_0(\mathbb{R}\setminus\{0\};H)$, $\tilde u=0$ on $(-\infty,0)$ with $\lim_{t\to0^+}\tilde u(t)=a$ in H and if $\rho<\infty$ then the restriction to $(0,\infty)$ of $\tilde u$ is an element in $V_{-\beta}$. Furthermore, we have $\tilde u\in L^r(\mathbb{R};D_{S,\alpha})$ for any $r\in(2,\infty)$ with $\alpha=2/r$ by Proposition 5.4 and we have the estimate

$$\|\tilde u\|_{L^\infty(\mathbb{R};H)}+\|\tilde u\|_{L^2(\mathbb{R};D_{S,1})}+\|\tilde u\|_{L^r(\mathbb{R};D_{S,\alpha})}\le C(M,\nu,\rho)\left(\|S^\beta g\|_{L^{\rho'}(\mathbb{R};H)}+\|a\|_H\right).$$

In addition, (6.28) in Theorem 6.27 implies (6.37) and (6.36) for $\tilde u(t)$ for all $t\in\mathbb{R}$ with the fundamental solution defined on $\mathbb{R}$. The candidate $u:=\tilde u_{|(0,\infty)}$ satisfies all the required properties of the theorem, proving existence and representation.

Next, we check uniqueness in the space $L^2((0,\infty);D_{S,1})$. We assume that u is a solution to (6.31) with $a=0$, $f=0$ and $g=0$. By Proposition 6.34,

$$2\mathrm{Re}\int_0^\infty B_t(u(t),u(t))\ \mathrm{d}t=0.$$

Using the coercivity of B_t, we deduce that $u=0$ on $(0,\infty)$. Definition, existence and uniqueness of the fundamental solution in $(0,\infty)$ is *verbatim* as in Section 6.4. □

Remark 6.38. Uniqueness in the previous proof does not work if we are working on a bounded interval $(0,\mathfrak{T})$ because Corollary 5.10 fails in this case.

Remark 6.39. Of course, by linearity, we can replace $S^\beta g$ by a linear combination of terms in $L^{\rho'}((0,\infty);D_{S,-\beta})$ for different ρ.

7. Inhomogeneous version

One would like to treat parabolic operators with elliptic part being $\mathcal{B}$ plus lower order terms allowing T to be not injective (e.g., differential operators with Neumann boundary conditions). Here is a setup for doing this effortlessly given the earlier developments.

7.0.1. *Setup for the inhomogeneous theory.* As before, consider T and $S=(T^*T)^{1/2}$ without assuming that T is injective. One can still define a Borel functional calculus associated to S as in Subsection 3.1 by replacing $(0,\infty)$ by $[0,\infty)$. In the right hand side of the Calderón reproducing formula (3.2), v is replaced by its orthogonal projection onto $\overline{\mathrm{ran}(S)}$. The most important fact is that for any $\alpha\geq 0$, we can still define S^α as the closed operator $\mathbf{t}^\alpha(S)$, which is also positive and self-adjoint but not necessarily injective, having the same null space as S.

Let $\tilde{T}:D(\tilde{T})=D(T)\to H\oplus K$ the operator defined by $\tilde{T}u:=(\lambda u,Tu)$ where $\lambda\in\mathbb{R}_+$. Assume $\lambda>0$. Then $\tilde{T}$ is injective and $\tilde{S}_\lambda=(\tilde{T}^*\tilde{T})^{1/2}=(\lambda^2+S^2)^{1/2}$ is a self-adjoint, positive and invertible operator on H, with domain $D(\tilde{S}_\lambda)=D(\tilde{T})=D(T)=D(S)$.

Using that $\tilde{S}_\lambda=(\lambda^2+S^2)^{1/2}$, we have that for $\lambda>0$ and $\alpha\geq 0$,

$$D_{\tilde{S}_\lambda,\alpha}=D(S^\alpha)\ ,\quad \|\cdot\|_{\tilde{S}_\lambda,\alpha}\simeq\|\cdot\|_{S,\alpha}+\|\cdot\|_H.$$

For $\alpha<0$, we know that the sesquilinear form $(u,v)\mapsto\langle\tilde{S}_\lambda^\alpha u,\tilde{S}_\lambda^{-\alpha}v\rangle_H$ defines a canonical duality pairing between $D_{\tilde{S}_\lambda,\alpha}$ and $D_{\tilde{S}_\lambda,-\alpha}$. Therefore, for any $u\in D_{\tilde{S}_\lambda,\alpha}$, there exists $(w,\tilde{w})\in H^2$ such that

$$\langle\tilde{S}_\lambda^\alpha u,\tilde{S}_\lambda^{-\alpha}v\rangle_H=\langle w,S^{-\alpha}v\rangle_H+\langle\tilde{w},v\rangle_H,$$

for all $v\in D(S^{-\alpha})$. In this sense, we write $D_{\tilde{S}_\lambda,\alpha}=S^{-\alpha}H+H$ with norm equivalent to the quotient norm

$$\|u\|_{\tilde{S}_\lambda,\alpha}\simeq\inf_{u=S^{-\alpha}w+\tilde{w}}(\|w\|_H+\|\tilde{w}\|_H).$$

From now on and as before, we set $\tilde{S}:=\tilde{S}_1=(1+S^2)^{1/2}$. In conclusion, the "inhomogeneous" fractional spaces for S become the "homogeneous" fractional spaces for $\tilde{S}$, so that applying the above theory with $\tilde{S}$ leads to the inhomogeneous theory for S (even if S is non injective).

Finally, we set

$$\tilde{E}_{-\infty}:=\bigcap_{\alpha\in\mathbb{R}}D(\tilde{S}^\alpha).$$

7.0.2. *Embeddings and Integral identities.* We begin by noting that Proposition 5.4 holds *verbatim* with the same proof, even if S is not necessarily injective. As for continuity and integral identities, we have to modify the statement as follows.

Proposition 7.1. *Let $\mathfrak{T}\in(0,\infty]$. Let $u\in L^1((0,\mathfrak{T});D(S))$ with $\int_0^{\mathfrak{T}}\|Su(t)\|_H^2\,\mathrm{d}t<\infty$ if $\mathfrak{T}<\infty$, or $u\in L^1((0,\mathfrak{T}');D(S))$ for all $\mathfrak{T}'<\infty$, with $\int_0^{\infty}\|Su(t)\|_H^2\,\mathrm{d}t<\infty$ if $\mathfrak{T}=\infty$. Assume that $\partial_t u=Sf+S^\beta g$ in $\mathcal{D}'((0,\mathfrak{T});\tilde{E}_\infty)$, where $f\in L^2((0,\mathfrak{T});H)$ and $g\in L^{\rho'}((0,\mathfrak{T});H)$, with $\beta=2/\rho\in[0,1)$. When $\mathfrak{T}<\infty$, then $u\in C([0,\mathfrak{T}];H)$, and the function $t\mapsto\|u(t)\|_H^2$ is absolutely continuous on $[0,\mathfrak{T}]$. For all $\sigma,\tau\in[0,\mathfrak{T}]$ such that $\sigma<\tau$, the following integral identity holds:*

$$\|u(\tau)\|_H^2-\|u(\sigma)\|_H^2=2\,\mathrm{Re}\int_\sigma^\tau\langle f(t),Su(t)\rangle_H\,\mathrm{d}t+\int_\sigma^\tau\langle g(t),S^\beta u(t)\rangle_H\,\mathrm{d}t.$$

If $\mathfrak{T}=\infty$, then the same conclusion holds on any bounded interval, and u is bounded in H.

Proof. Using Proposition 5.4, we can express $Sf+S^\beta g=S\tilde{f}+h$, with $\tilde{f}\in L^2((0,\mathfrak{T});H)$ and $h\in L^1((0,\mathfrak{T});H)$.

We start with the case $\mathfrak{T}<\infty$. Consider the orthogonal decomposition $H=\overline{\mathrm{ran}(S)}\oplus\mathrm{nul}(S)$, and write $u=u_1+u_2$, where $u_1\in L^1((0,\mathfrak{T});\overline{\mathrm{ran}(S)})$ satisfies $\int_0^{\mathfrak{T}}\|Su_1(t)\|_H^2\,\mathrm{d}t<\infty$, and $u_2\in L^1((0,\mathfrak{T});\mathrm{nul}(S))$. Similarly, we decompose $\tilde{f}=\tilde{f}_1+\tilde{f}_2$ and $h=h_1+h_2$. We have $\partial_t u_1=S\tilde{f}_1+h_1$ and $\partial_t u_2=h_2$ where both equalities hold in $\mathcal{D}'((0,\mathfrak{T});\tilde{E}_\infty)$. We obtain that $\partial_t u_2\in$

$L^1((0,\mathfrak{T});\mathrm{nul}(S))$, hence $u_2 \in W^{1,1}((0,\mathfrak{T});\mathrm{nul}(S)) \hookrightarrow C([0,\mathfrak{T}];\mathrm{nul}(S))$. Using Corollary 5.13 with $S_{|\overline{\mathrm{ran}(S)}}$ which is injective, we conclude that $u_1 \in C([0,\mathfrak{T}];\overline{\mathrm{ran}(S)})$. Finally, we obtain the energy equality using orthogonality.

When $\mathfrak{T}=\infty$, the conclusion is already valid on $[0,\infty)$. To see the behavior at ∞, we can use the same decomposition and $u_1 \in C_0([0,\mathfrak{T}];\overline{\mathrm{ran}(S)})$ from Corollary 5.10. As for u_2 we have by direct integration, that for all $t\geq 0$, $\|u_2(t)\|_H \leq \inf_{\tau\geq 0}\|u_2(\tau)\|_H + \|h_2\|_{L^1((0,\infty);\mathrm{nul}(S))}$. □

7.0.3. *The Cauchy problem.* In this section, we are interested in the Cauchy problem on segments and half-lines, in a non-homogeneous manner. Recall that $\tilde{S}_\lambda = (\lambda^2+S^2)^{1/2}$ with $D(\tilde{S}_\lambda)=D(T)$ and we assume $\lambda\geq 0$ for the moment. Let $0<\mathfrak{T}\leq\infty$. First, let us consider $(\tilde{B}_t)_{t\in(0,\mathfrak{T})}$ a weakly measurable family of bounded sesquilinear forms on $D(T)\times D(T)$. More precisely, we assume that

$$|\tilde{B}_t(u,v)| \leq M\|\tilde{S}_\lambda u\|_H \|\tilde{S}_\lambda v\|_H, \tag{7.2}$$

for some $M>0$ and for all $u,v\in D(T)$ and $t\in(0,\mathfrak{T})$. In addition, we assume that the family $(\tilde{B}_t+\kappa)_{t\in(0,\mathfrak{T})}$ is uniformly coercive for some $\kappa\in\mathbb{R}_+$, *i.e.*,

$$\mathrm{Re}(\tilde{B}_t(v,v)) + \kappa\,\|v\|_H^2 \geq \nu\|\tilde{S}_\lambda v\|_H^2, \tag{7.3}$$

for some $\nu>0$ and for all $v\in D(T)$ and $t\in(0,\mathfrak{T})$. Notice that $\tilde{B}_t+\kappa$ satisfies the lower bound in (6.1) with $\tilde{S}$ replacing S and the upper bound with $M+\kappa$ on $(0,\mathfrak{T})$. We denote by $\tilde{\mathcal{B}}$ the operator associated to the family $(\tilde{B}_t)_{t\in(0,\mathfrak{T})}$. One may represent $\tilde{B}_t$ as $\tilde{T}^*\tilde{A}(t)\tilde{T}$, where $\tilde{A}(t)$ is bounded on $H\oplus\overline{\mathrm{ran}(\mathrm{T})}$. If we decide to represent $\tilde{A}(t)$ in 2×2 matrix form, then $\tilde{\mathcal{B}}$ writes as $\mathcal{B}$ plus lower order terms with bounded operator-valued coefficients.

On segments, say $(0,\mathfrak{T})$, we can consider the Cauchy problem for all possible values of $\lambda\geq 0$ and $\kappa\geq 0$. On half-lines, say $(0,\infty)$, we restrict the range of the parameters. This leads to the following cases (we will not attempt to track λ and κ quantitatively).

(a) $\mathfrak{T}<\infty$.
(b) $\mathfrak{T}=\infty$, $\lambda>0$ and $\kappa=0$.

See Remark 7.9 and Remark 7.8 for more when $\mathfrak{T}=\infty$.

We fix $\rho\in[2,\infty]$ and set $\beta=2/\rho$. Given an initial condition $a\in H$ and $g\in L^{\rho'}((0,\mathfrak{T});H)$, we wish to solve the Cauchy problem

$$\begin{cases} \partial_t u + \tilde{\mathcal{B}}u = \tilde{S}^\beta g \ \text{ in } \mathcal{D}'((0,\mathfrak{T});\tilde{E}_\infty), \\ u(0)=a \ \text{ in } \tilde{E}_\infty. \end{cases} \tag{7.4}$$

Recall that $\tilde{S}^\beta g$ can be written as $S^\beta g_1 + g_2$, with $g_i\in L^{\rho'}((0,\mathfrak{T});H)$, $i=1,2$.

Definition 7.5. A weak solution to (7.4) is a function $u\in L^1((0,\mathfrak{T});D(S))$ with $\int_0^{\mathfrak{T}}\|Su(t)\|_H^2\,\mathrm{d}t<\infty$ if $\mathfrak{T}<\infty$ or $u\in L^1((0,\mathfrak{T}');D(S))$ for all $\mathfrak{T}'<\infty$ with $\int_0^\infty\|Su(t)\|_H^2\ \mathrm{d}t<\infty$ if $\mathfrak{T}=\infty$ and such that

(i) u solves the first equation in $\mathcal{D}'((0,\mathfrak{T});\tilde{E}_\infty)$, that is, for all $\varphi\in\mathcal{D}((0,\mathfrak{T});\tilde{E}_{-\infty})$

$$\int_0^{\mathfrak{T}} -\langle u(t),\partial_t\varphi(t)\rangle_H + \tilde{B}_t(u(t),\varphi(t))\ \mathrm{d}t = \int_0^{\mathfrak{T}}\langle g(t),\tilde{S}^\beta\varphi(t)\rangle_H\ \mathrm{d}t.$$

(ii) $\forall\tilde{a}\in\tilde{E}_{-\infty}$, $\langle u(t),\tilde{a}\rangle_H\to\langle a,\tilde{a}\rangle_H$ along a sequence tending to 0.

The difference with the homogeneous situation is the global or local $L^1(H)$ condition.

Again, a weaker formulation testing against functions $\varphi\in\mathcal{D}([0,\mathfrak{T});\tilde{E}_{-\infty})$ with right hand side containing the additional term $\langle a,\varphi(0)\rangle_H$ can be considered. In the end it amounts to the same solutions thanks to *a priori* continuity in H, which only uses the upper bound on $\tilde{B}_t$.

Lemma 7.6. *In case (a), any weak solution to (i) belongs to $C([0,\mathfrak{T}];H)$, and $t\mapsto\|u(t)\|_H^2$ satisfies the energy equality for any $\sigma,\tau\in[0,\mathfrak{T}]$ such that $\sigma<\tau$,*

$$\|u(\tau)\|_H^2 + 2\mathrm{Re}\int_\sigma^\tau \tilde{B}_t(u(t),u(t))\ \mathrm{d}t = \|u(\sigma)\|_H^2 + 2\mathrm{Re}\int_\sigma^\tau\langle g(t),\tilde{S}^\beta u(t)\rangle_H\ \mathrm{d}t.$$

In case (b), we have the same conclusion on any bounded interval.

Proof. Case (b) follows directly from case (a). To prove the latter, using Proposition 5.4 with $\tilde{S}$ replacing S, we can write $\tilde{S}^\beta g = \tilde{S}\tilde{f} + \tilde{h}$ with $\tilde{f} \in L^2((0,\mathfrak{T});H)$ and $\tilde{h} \in L^1((0,\mathfrak{T});H)$. This can be expressed as $Sf + h$ with $\tilde{f} \in L^2((0,\mathfrak{T});H)$ and $\tilde{h} \in L^1((0,\mathfrak{T});H)$. We conclude on using Proposition 7.1. □

The main result of this section is the following theorem which puts together the inhomogeneous version of all the theory developed so far.

Theorem 7.7. *Consider the above assumptions on $\tilde{B}_t$, λ, κ, f, g and a.*

(1) *There exists a unique weak solution u to the problem (7.4). Moreover, $u \in C([0,\mathfrak{T}];H) \cap L^r((0,\mathfrak{T});D(\tilde{S}^\alpha))$ for all $r \in [2,\infty]$ with $\alpha = 2/r$, with $u(\infty) = 0$ in case (b) where $\mathfrak{T} = \infty$, and we have the estimate*

$$\sup_{t\in[0,\mathfrak{T}]} \|u(t)\|_H + \|\tilde{S}^\alpha u\|_{L^r((0,\mathfrak{T});H)} \leq C(\|g\|_{L^{\rho'}((0,\mathfrak{T});H)} + \|a\|_H),$$

where $C = C(M,\nu,\rho,\kappa,\mathfrak{T}) > 0$ is a constant independent of g and a.

(2) *There exists a unique fundamental solution $\Gamma_{\tilde{\mathcal{B}}} = (\Gamma_{\tilde{\mathcal{B}}}(t,s))_{0\leq s\leq t\leq \mathfrak{T}}$ for $\partial_t + \tilde{\mathcal{B}}$ in the sense of Definition 6.23 in $(0,\mathfrak{T})$ (by convention, set $\Gamma(\infty,s) = 0$ if $\mathfrak{T} = \infty$). In particular, for all $t \in [0,\mathfrak{T}]$, we have the following representation of u :*

$$u(t) = \Gamma_{\tilde{\mathcal{B}}}(t,0)a + \int_0^t \Gamma_{\tilde{\mathcal{B}}}(t,s)\tilde{S}^\beta g(s)\ \mathrm{d}s,$$

where the integral is weakly defined in H when $\rho < \infty$ and strongly defined when $\rho = \infty$ (i.e., in the Bochner sense). For all $\tilde{a} \in H$ and $t \in [0,\mathfrak{T}]$,

$$\langle u(t),\tilde{a}\rangle_H = \langle \Gamma_{\tilde{\mathcal{B}}}(t,0)a,\tilde{a}\rangle_H + \int_0^t \langle g(s), \tilde{S}^\beta \tilde{\Gamma}_{\tilde{\mathcal{B}}}(s,t)\tilde{a}\rangle_H\ \mathrm{d}s.$$

Proof. We begin with existence.
Existence in case (b). Apply Theorem 6.35 with $\tilde{S}$ replacing S and as the right hand side belongs to $L^{\rho'}((0,\infty);D_{\tilde{S},-\beta})$. This shows the existence of a weak solution v in $L^2((0,\infty);D(\tilde{S}))$, which also belongs to $C_0([0,\infty);H)$ and $L^r((0,\infty);D(\tilde{S}^\alpha))$.
Existence in case (a). Extend g by 0 and $\tilde{B}_t$ by $\nu\langle\tilde{S}\cdot,\tilde{S}\cdot\rangle_H$ on $(\mathfrak{T},\infty)$ if $\mathfrak{T} < \infty$ and use the same notation. Let $\kappa' > \kappa$. Apply Theorem 6.35 with $\tilde{S}$ replacing S and with right hand side in $L^{\rho'}((0,\infty);D_{\tilde{S},-\beta})$ to the auxiliary Cauchy problem

$$\begin{cases} \partial_t v + (\tilde{\mathcal{B}} + \kappa')v = \tilde{S}^\beta(e^{-\kappa' t}g) \ \text{ in } \mathcal{D}'((0,\infty);\tilde{E}_\infty), \\ v(0) = a \text{ in } \tilde{E}_\infty, \end{cases}$$

and obtain a weak solution v in $L^2((0,\infty);D(\tilde{S}))$. The function $u := e^{\kappa' t}v$ restricted to $[0,\mathfrak{T}]$ gives us a weak solution with the desired properties.

Next, we prove uniqueness. Assume u is a weak solution to (7.4) with $a = 0$ and $g = 0$.
Uniqueness in case (b). We have $u \in L^1((0,\mathfrak{T}');D(S))$ for all $\mathfrak{T}' < \infty$ and $\int_0^\infty \|Su(t)\|_H^2\ \mathrm{d}t < \infty$. Applying Lemma 7.6, we have $u \in C([0,\mathfrak{T}'];H)$ for all $\mathfrak{T}' > 0$, $u(0) = 0$ and

$$\left\|u(\mathfrak{T}')\right\|_H^2 + 2\mathrm{Re}\int_0^{\mathfrak{T}'} \tilde{B}_t(u(t),u(t))\ \mathrm{d}t = 0.$$

Using the coercivity of $\tilde{B}_t$, we deduce that $u = 0$ on $(0,\mathfrak{T}')$ and therefore, $u = 0$ on $[0,\infty)$.
Uniqueness in case (a). We have $u \in L^1((0,\mathfrak{T});D(S))$ with $\int_0^{\mathfrak{T}} \|Su(t)\|_H^2\,\mathrm{d}t < \infty$. Let $\kappa' > \kappa$. Set $v = e^{-\kappa' t}u$ on $(0,\mathfrak{T})$ so that $v \in L^1((0,\mathfrak{T});D(S))$ with $\int_0^{\mathfrak{T}} \|Sv(t)\|_H^2\,\mathrm{d}t < \infty$ and $\partial_t v + (\tilde{\mathcal{B}} + \kappa')v = 0$ in $\mathcal{D}'((0,\mathfrak{T});\tilde{E}_\infty)$. Applying Lemma 7.6, we have $v \in C([0,\mathfrak{T}];H)$, $v(0) = 0$ and

$$\|v(\mathfrak{T})\|_H^2 + 2\mathrm{Re}\int_0^{\mathfrak{T}} \tilde{B}_t(v(t),v(t)) + \kappa'\|v(t)\|_H^2\ \mathrm{d}t = 0.$$

Using the coercivity of $\tilde{B}_t + \kappa'$ resulting from (7.3), we deduce that $v = 0$ and therefore, $u = 0$ on $[0, \mathfrak{T}]$.

Finally, definition, existence and uniqueness of the fundamental solution $\Gamma_{\tilde{\mathcal{B}}}$ can be obtained easily by proceeding as in Section 6.4. □

Remark 7.8. If $\mathfrak{T} = \infty$ with $\kappa > 0$, then we can construct a weak solution but it does not satisfy $\int_0^\infty \|Su(t)\|_H^2 \, \mathrm{d}t < \infty$.

Remark 7.9. For $\mathfrak{T} = \infty$, $\lambda = 0$, $\kappa = 0$ and $\tilde{S}^\beta g$ replaced by $S^\beta g$, we can apply Theorem 6.35 provided that T is injective. However, when T (hence S) is not injective then the proof of Theorem 7.7 provides us with a global solution but not with limit 0 at ∞. In fact, the zero limit at ∞ fails. Take $u_0 \in \mathrm{nul}(S) \setminus \{0\}$ and set $u(t) = u_0$ for all $t \geq 0$. We have $u \in L^2((0, \mathfrak{T}'); H)$ for all $\mathfrak{T}' < \infty$ with $\int_0^\infty \|Su(t)\|_H^2 \, \mathrm{d}t = 0 < \infty$. Moreover, u is a weak solution to the abstract heat equation

$$\begin{cases} \partial_t u + S^2 u = 0 \ \text{ in } \mathcal{D}'((0,\infty); \tilde{E}_\infty), \\ u(0) = u_0 \text{ in } \tilde{E}_\infty, \end{cases}$$

with $\lim_{t\to\infty} u(t) = u_0$.

Remark 7.10. Consider the special case $\tilde{\mathcal{B}} = \mathcal{B} + \omega$, $\omega > 0$, on $(0, \infty)$, keeping the condition (6.1) for $\mathcal{B}$ with T (and S) injective. We have (b) with $\lambda = 1$, constant $\sup(M, \omega)$ in (7.2) and constant $\inf(\nu, \omega)$ in (7.3). The theorem above applies and gives us fundamental solution operators $\Gamma_{\mathcal{B}+\omega}(t, s)$, defined for $0 \leq s \leq t < \infty$. Call $\Gamma_{\mathcal{B}}(t, s)$ the one obtained in the previous section. Uniqueness for the Cauchy problem for $\partial_t + \mathcal{B} + \omega$ holds in $L^2((0, \mathfrak{T}), D(S))$ for all $\mathfrak{T} < \infty$ and this shows that $\Gamma_{\mathcal{B}+\omega}(t, s) = e^{-\omega(t-s)}\Gamma_{\mathcal{B}}(t, s)$. Working on $\mathbb{R}$, then we obtain the equality for $-\infty < s \leq t < \infty$.

8. The final step towards concrete situations

The reader might wonder how to apply our theory in concrete situations, where the abstract spaces of test functions $\mathcal{D}(I; E_{-\infty})$ or $\mathcal{D}(I; \tilde{E}_{-\infty})$ might not be related to usual spaces of test functions. The following result gives us a sufficient condition showing that one can replace $E_{-\infty}$ or $\tilde{E}_{-\infty}$ by an arbitrary dense set in the domain of S, sometimes called a core of $D(S) = D(T)$.

Theorem 8.1. *Let* D *be a Hausdorff topological vector space with continuous and dense inclusion* $\mathrm{D} \hookrightarrow D(S)$*, where* $D(S)$ *is equipped with the graph norm. Assume a priori that weak solutions belong to* $L^1_{\mathrm{loc}}(I; H)$*, and replace the test function space by* $\mathcal{D}(I; \mathrm{D})$ *in their definition, with in the latter case,* $\partial_t u$ *computed via :*

$$\forall \varphi \in \mathcal{D}(I; \mathrm{D}), \ \langle\langle \partial_t u, \varphi \rangle\rangle = -\int_I \langle u(t), \partial_t \varphi(t) \rangle_H \, \mathrm{d}t.$$

Then our well-posedness results are the same: this means that existence with estimates, uniqueness (requiring additionally $u \in L^1_{\mathrm{loc}}(I; H)$ *in the uniqueness class), and energy equalities hold.*

The proof relies on the following density lemma. Denote by $\|u\|_{D(S)} = (\|u\|_H^2 + \|Su\|_H^2)^{1/2}$ the Hilbertian graph norm and $\langle \cdot, \cdot \rangle_{D(S)}$ the corresponding inner product.

Lemma 8.2. *Let* D *be as in the above theorem. For all open interval* $I \subset \mathbb{R}$*,* $\mathcal{D}(I; \mathrm{D})$ *is a dense subspace of* $\mathcal{D}(I; D(S))$ *in the following sense : for all* $\varphi \in \mathcal{D}(I; D(S))$*, there exists a sequence* $(\varphi_k)_{k\geq 0} \in \mathrm{span}(\mathcal{D}(I) \otimes \mathrm{D})^{\mathbb{N}}$ *such that*

1. *For all* $k \geq 0$, $\mathrm{supp}(\varphi_k) \subset \mathrm{supp}(\varphi)$.
2. *For all* $k \geq 0$ *and* $t \in I$,
$$\|\varphi_k(t)\|_{D(S)} \leq 3 \|\varphi(t)\|_{D(S)}, \quad \|\partial_t \varphi_k(t)\|_{D(S)} \leq 3 \|\partial_t \varphi(t)\|_{D(S)}.$$
3. *For all* $t \in I$, $\|\varphi_k(t) - \varphi(t)\|_{D(S)} + \|\partial_t \varphi_k(t) - \partial_t \varphi(t)\|_{D(S)} \to 0$ *as* $k \to \infty$.
4. *For all* $\beta \in [0, 1]$, $t \in I$ *and* $k \geq 0$, $\|S^\beta \varphi_k(t)\|_H \leq 3 \|\varphi(t)\|_{D(S)}$ *and* $\|S^\beta(\varphi_k(t) - \varphi(t))\|_H \to 0$ *as* $k \to \infty$.

Proof. The space $(D(S), \|\cdot\|_{D(S)})$ is separable as it is isometric to a subspace of $H \times H$ which is separable. Let $(w_j)_{j\in\mathbb{N}} \in D(S)^{\mathbb{N}}$ be a Hilbertian basis of $\Big(D(S), \|\cdot\|_{D(S)}\Big)$. As D is dense in $D(S)$ then for all $j \geq 0$, one can find a sequence $(v_j^k)_{k\in\mathbb{N}} \in \mathrm{D}^{\mathbb{N}}$ such that for all $k \geq 0$, $\|w_j - v_j^k\|_{D(S)} \leq \frac{1}{2^{j+k}}$. For $h = \sum_{j\geq 0} \alpha_j w_j \in D(S)$, one can see using Cauchy-Schwarz inequality and Plancherel that $\| \sum_{j=0}^k \alpha_j (v_j^k - w_j)\|_{D(S)} \leq \sum_{j=0}^k \frac{|\alpha_j|}{2^{j+k}} \leq \frac{4}{3\cdot 2^k}\|h\|_{D(S)}$, so that $h_k = \sum_{j=0}^k \alpha_j v_j^k$ satisfies $\|h_k - h\|_{D(S)} \leq \frac{4}{3\cdot 2^k}\|h\|_{D(S)} + \| \sum_{j\geq k+1} \alpha_j w_j\|_{D(S)}$ and $\|h_k\|_{D(S)} \leq (\frac{4}{3}+1)\|h\|_{D(S)}$. Now, fix $\varphi \in \mathcal{D}(I; D(S))$ and set for all $k \geq 0$ and $t \in I$,

$$\varphi_k(t) := \sum_{j=0}^k \langle \varphi(t), w_j\rangle_{D(S)} v_j^k \ .$$

Clearly, the sequence $(\varphi_k)_{k\in\mathbb{N}} \in \mathrm{span}(\mathcal{D}(I)\otimes \mathrm{D})^{\mathbb{N}}$ with (1), and (2) and (3) follow from the above estimates. Finally, (4) follows from the moments inequality combined with (2) and (3). □

Proof of Theorem 8.1. The case using $\mathcal{D}(I; \tilde{E}_{-\infty})$ being similar, it only suffices to show that with the a priori requirement that weak solutions also belong to $L^1_{\mathrm{loc}}(I; H)$, the formulations of the equations against test functions in $\mathcal{D}(I; E_{-\infty})$ and in $\mathcal{D}(I; \mathrm{D})$ are equivalent, because then they have the same solutions. In fact, they are equivalent to a formulation against test functions in $\mathcal{D}(I; D(S))$. Indeed, if $u \in L^2(I; D_{S,1}) \cap L^1_{\mathrm{loc}}(I; H)$ then by Lemma 3.10, we have for all $\varphi \in \mathcal{D}(I; E_{-\infty})$,

$$\langle\!\langle u, \partial_t\varphi\rangle\!\rangle_{\mathcal{D}',\mathcal{D}} = \int_I \langle u(t), \partial_t\varphi(t)\rangle_{H,1}\ \mathrm{d}t = \int_I \langle Su(t), S^{-1}\partial_t\varphi(t)\rangle_H\ \mathrm{d}t = \int_I \langle u(t), \partial_t\varphi(t)\rangle_H\ \mathrm{d}t.$$

Applying that $\mathcal{D}(I; E_{-\infty})$ is dense $\mathcal{D}(I; D(S))$ as in Lemma 8.2 and dominated convergence, we can see that the weak formulation for all $\varphi \in \mathcal{D}(I; D(S))$ holds. Of course we can conversely restrict to test functions $\mathcal{D}(I; E_{-\infty})$, showing that the formulations testing with $\varphi \in \mathcal{D}(I; E_{-\infty})$ or $\varphi \in \mathcal{D}(I; D(S))$ are equivalent. This would be the same starting from another dense set D. Finally, the initial data property in the Cauchy problems testing against elements in $E_{-\infty}$ is equivalent to testing against arbitrary elements in H by density as $u(t)$ belongs almost everywhere to H. This would be the same replacing $E_{-\infty}$ by another dense set D in $D(S)$ as it would also be dense in H. □

9. Three applications

9.1. **Parabolic Cauchy problems on domains with Dirichlet boundary condition.** Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ an open set. We denote by $L^2(\Omega)$ the Hilbert space of square integrable functions on Ω with respect to the Lebesgue measure $\mathrm{d}x$ with norm denoted by $\|\cdot\|_2$ and its inner product by $\langle\cdot,\cdot\rangle$. As usual, we denote by $\mathcal{D}(\Omega)$ the class of smooth and compactly supported functions on Ω. We set $H^1(\Omega) := \big\{u \in L^2(\Omega) : \nabla_x u \in L^2(\Omega)\big\}$ and it is a Hilbert space for the norm $\|u\|_{H^1(\Omega)} := (\|u\|_2^2 + \|\nabla_x u\|_2^2)^{1/2}$. Finally, $H^1_0(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ in $(H^1(\Omega), \|\cdot\|_{H^1(\Omega)})$.

We denote by $-\Delta_D$ the unbounded operator on $L^2(\Omega)$ associated to the positive symmetric sesquilinear form on $H^1_0(\Omega)\times H^1_0(\Omega)$ defined by

$$(u,v) \mapsto \int_\Omega \nabla_x u(x)\cdot\overline{\nabla_x v(x)}\ \mathrm{d}x.$$

Let $A : I\times\Omega \to M_n(\mathbb{C})$ be a matrix-valued function with complex measurable entries and such that

$$|A(t,x)\xi\cdot\zeta| \leq M\,|\xi|\,|\zeta|\,, \qquad \nu\,|\xi|^2 \leq \mathrm{Re}(A(t,x)\xi\cdot\overline{\xi}) \tag{9.1}$$

for some $M,\nu > 0$ and for all $\zeta,\xi \in \mathbb{C}^n$ and $(t,x) \in I\times\Omega$. We let $-\mathrm{div}_x$ be the adjoint of $\nabla_x : H^1_0(\Omega) \to L^2(\Omega)^n$ and use the customary notation $\mathcal{B} = -\mathrm{div}_x A(t,\cdot)\nabla_x$.

Fix $\mathfrak{T} > 0$, $\rho \in (2,\infty)$, set $\beta = 2/\rho \in (0,1)$ and let ρ' be the Hölder conjugate of ρ. For $f \in L^2((0,\mathfrak{T}); L^2(\Omega)^n)$, $g \in L^{\rho'}((0,\mathfrak{T}); L^2(\Omega))$, $h \in L^1((0,\mathfrak{T}); L^2(\Omega))$ and $\psi \in L^2(\Omega)$, consider the

following Cauchy problem

$$\begin{cases} \partial_t u - \mathrm{div}_x(A(t,\cdot)\nabla_x u) = -\mathrm{div}_x f + (-\Delta_D)^{\beta/2} g + h \text{ in } \mathcal{D}'((0,\mathfrak{T})\times\Omega), \\ u(t) \to \psi \text{ in } \mathcal{D}'(\Omega) \text{ as } t \to 0^+. \end{cases} \tag{9.2}$$

The first equation is interpreted in the weak sense according to the following definition.

Definition 9.3. A weak solution to the first equation in (9.2) is a (complex-valued) function $u \in L^1((0,\mathfrak{T}); H^1_0(\Omega))$ with $\int_0^{\mathfrak{T}} \|\nabla_x u(t)\|_2^2 \,\mathrm{d}t < \infty$ such that for all $\varphi \in \mathcal{D}((0,\mathfrak{T})\times\Omega)$,

$$\int_0^{\mathfrak{T}} \int_\Omega - u(t,x)\partial_t\varphi(t,x) + A(t,x)\nabla_x u(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^{\mathfrak{T}} \int_\Omega f(t,x)\cdot\nabla_x\varphi(t,x) + g(t,x)(-\Delta_D)^{\beta/2}\varphi(t,x)\,\mathrm{d}x + h(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t.$$

The consequence of our theory is

Theorem 9.4 (Cauchy problem on $(0,\mathfrak{T})$)**.** *Let f, g, h, ψ be as above.*

(1) *There exists a unique weak solution to the Cauchy problem (9.2) as defined above. Moreover, $u \in C([0,\mathfrak{T}]; L^2(\Omega))$ with $u(0) = \psi$, the application $t \mapsto \|u(t)\|_2^2$ is absolutely continuous on $[0,\mathfrak{T}]$ and we can write the energy equalities. Furthermore, $u \in L^r((0,\mathfrak{T}); D((-\Delta_D)^{\alpha/2}))$ for any $\alpha \in (0,1]$ with $r = 2/\alpha \in [2,\infty)$ and we have*

$$\sup_{t\in[0,\mathfrak{T}]} \|u(t)\|_2 + \|(-\Delta_D)^{\alpha/2}u\|_{L^r((0,\mathfrak{T});H)}$$
$$\le C(\|f\|_{L^2((0,\mathfrak{T});L^2(\Omega)^n)} + \|g\|_{L^{\rho'}((0,\mathfrak{T});L^2(\Omega))} + \|h\|_{L^1((0,\mathfrak{T});L^2(\Omega))} + \|\psi\|_2),$$

where $C = C(M,\nu,\rho,\mathfrak{T}) > 0$ is a constant independent of the data f, g, h and ψ.

(2) *There exists a unique fundamental solution $\Gamma = (\Gamma(t,s))_{0\le s\le t\le\mathfrak{T}}$ for $\partial_t - \mathrm{div}_x A(t,\cdot)\nabla_x$. In particular, for all $t \in [0,\mathfrak{T}]$, we have the following representation of $u(t)$:*

$$u(t) = \Gamma(t,0)\psi + \int_0^t \Gamma(t,\tau)(-\mathrm{div}_x f)(\tau)\,\mathrm{d}\tau + \int_0^t \Gamma(t,\tau)(-\Delta_D)^{\beta/2} g(\tau)\,\mathrm{d}\tau + \int_0^t \Gamma(t,\tau)h(\tau)\,\mathrm{d}\tau,$$

where the two integrals with f and g are weakly defined in $L^2(\Omega)$ while the other one converges strongly (i.e., in the Bochner sense). More precisely, we have for all $\tilde\psi \in L^2(\Omega)$ and $t \in [0,\mathfrak{T}]$,

$$\langle u(t), \tilde\psi\rangle = \langle \Gamma(t,0)\psi, \tilde\psi\rangle + \int_0^t \langle f(s), \nabla_x \tilde\Gamma(s,t)\tilde\psi\rangle\,\mathrm{d}s$$
$$+ \int_0^t \langle g(s), (-\Delta_D)^{\beta/2}\tilde\Gamma(s,t)\tilde\psi\rangle\,\mathrm{d}s + \int_0^t \langle \Gamma(t,s)h(s), \tilde\psi\rangle\,\mathrm{d}s.$$

Proof. As $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$ with respect to the graph norm of the injective self-adjoint operator $S = (-\Delta_D)^{1/2}$ by definition, we are in the context of Theorem 8.1 in Section 8, which corresponds to Theorem 7.7 for each f, g, h by linearity and using that $-\mathrm{div}_x f = (-\Delta_D)^{1/2}\tilde f$ with $\tilde f \in L^2((0,\mathfrak{T}); L^2(\Omega))$, with $B_t : H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{C}$ being the sesquilinear form defined via

$$\forall u, v \in H^1_0(\Omega) :\ B_t(u,v) := \int_\Omega A(t,x)\nabla_x u(x)\cdot\overline{\nabla_x v(x)}\,\mathrm{d}x.$$

□

Remarks 9.5. (1) With the modification in the definition of weak solutions, the statement applies for the Cauchy problem on $(0,\infty)$ and u has limit 0 at ∞. (Use Theorems 8.1 and 6.35).

(2) Remark that the theory applies for complex coefficients. In particular, we do not assume any local regularity for weak solutions and fundamental solutions are merely bounded operators. Bounds on their kernels need additional assumptions.

(3) If Ω is bounded (or only bounded in one direction), then Poincaré inequality holds on $H^1_0(\Omega)$ [EHMT24, Proposition 3.25], and it follows that $D(S) = D_{S,1} = H^1_0(\Omega)$ with equivalent norms. In particular, the inhomogeneous and homogeneous theories developed in Section 6 are the same for this concrete case.
(4) We may want to replace the spaces $L^r((0,\mathfrak{T}); D((-\Delta_D)^{\alpha/2}))$ by mixed Lebesgue spaces $L^r((0,\mathfrak{T}); L^q(\Omega))$. The embeddings of the domains of the fractional powers $(-\Delta_D)^{\alpha/2}$ into Lebesgue spaces $L^q(\Omega)$ depend on the geometry of the domain. See the discussion in [AE23].

9.2. **Parabolic integro-differential operators.** The second application is for integro-differential parabolic operators $\partial_t + \mathcal{B}$ where $\mathcal{B}$ is associated with a sesquilinear form B_t satisfying (6.1) for $t \in I$ (I open interval) with $T = S = (-\Delta)^{\gamma/2}$ for some $\gamma > 0$. The most notable example from the references mentioned in the introduction is that of $\mathcal{B}$ arising from the family of forms

$$B_t(u,v) := \iint_{\mathbb{R}^n\times\mathbb{R}^n} K(t,x,y)\frac{(u(x)-u(y))\overline{(v(x)-v(y))}}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y,$$

for some $\gamma \in (0,1)$ and $u, v \in W^{\gamma,2}(\mathbb{R}^n)$. We assume here $K : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ to be a measurable kernel that satisfies the accretivity condition for some $\lambda > 0$,

$$0 < \lambda \le \operatorname{Re} K(t,x,y) \le |K(t,x,y)| \le \lambda^{-1} \qquad (\text{a.e. } (t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n). \tag{9.6}$$

The Sobolev space $W^{\gamma,2}(\mathbb{R}^n)$ is the space of measurable functions u on $\mathbb{R}^n$ with norm $\|u\|_\gamma$ given by

$$\|u\|^2_\gamma = \int_{\mathbb{R}^n} |u(x)|^2\,\mathrm{d}x + \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y,$$

and it is well known that $W^{\gamma,2}(\mathbb{R}^n)$ agrees with the domain of $(-\Delta)^{\gamma/2}$ and that the last term in the expression above is comparable to $\|(-\Delta)^{\gamma/2}u\|_2^2$. Using this observation, (9.6) and Cauchy-Scwharz inequality, we can check (6.1) with $T = S = (-\Delta)^{\gamma/2}$.

From now on, we can apply the theory developed so far and obtain well-posedness results on $I \times \mathbb{R}^n$ but we shall not repeat the statements and leave that to the reader. May be the most notable outcome is that there always exists a unique fundamental solution, and this seems new at this level of generality.

Theorem 9.7. *Let $\gamma > 0$. The integro-differential parabolic operator $\partial_t + \mathcal{B}$ on $I \times \mathbb{R}^n$ has a unique fundamental solution.*

9.3. **Degenerate parabolic operators.** The third application concerns degenerate parabolic operators on $\mathbb{R}^n$. We fix a weight ω in the Muckenhoupt class $A_2(\mathbb{R}^n, \mathrm{d}x)$, meaning that $\omega : \mathbb{R}^n \to \mathbb{R}$ is a measurable and positive function satisfying

$$[\omega]_{A_2} := \sup_{Q\subset\mathbb{R}^n} \left(\fint_Q \omega(x)\,\mathrm{d}x\right)\left(\fint_Q \omega^{-1}(x)\,\mathrm{d}x\right) < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For background on Muckenhoupt weights and related results, we refer to [Ste93, Ch. V].

We denote by $L^2_\omega(\mathbb{R}^n) := L^2(\mathbb{R}^n, \mathrm{d}\omega)$ the Hilbert space of square-integrable functions with respect to $\mathrm{d}\omega$, with norm denoted by $\|\cdot\|_{2,\omega}$ and inner product $\langle\cdot,\cdot\rangle_{2,\omega}$. It is known that

$$\mathcal{D}(\mathbb{R}^n) \subset L^2_\omega(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}x) \subset \mathcal{D}'(\mathbb{R}^n)$$

and the first inclusion is dense.

We define the weighted Sobolev space $H^1_\omega(\mathbb{R}^n)$ (or $W^{1,2}_\omega(\mathbb{R}^n)$) as the space of functions $f \in L^2_\omega(\mathbb{R}^n)$ for which the distributional gradient $\nabla_x f$ belongs to $L^2_\omega(\mathbb{R}^n)^n$, and equip this space with the norm $\|f\|_{H^1_\omega} := (\|f\|^2_{2,\omega} + \|\nabla_x f\|^2_{2,\omega})^{1/2}$ making it a Hilbert space. It is also known that $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^1_\omega(\mathbb{R}^n)$ (see [Kil94, Thm. 2.5]).

Let $I \subset \mathbb{R}$ be an open interval. Let $A : I \times \mathbb{R}^n \to M_n(\mathbb{C})$ be a matrix-valued function with complex measurable coefficients such that

$$|A(t,x)\xi\cdot\zeta| \le M\omega(x)\,|\xi|\,|\zeta|\,, \quad \nu\,|\xi|^2\,\omega(x) \le \operatorname{Re}(A(t,x)\xi\cdot\overline{\xi}),$$

for some constants $M, \nu > 0$ and for all $\xi, \zeta \in \mathbb{C}^n$ and $(t, x) \in I \times \mathbb{R}^n$.

For each $t \in I$, we define the sesquilinear form $B_t : H^1_\omega(\mathbb{R}^n) \times H^1_\omega(\mathbb{R}^n) \to \mathbb{C}$ by

$$B_t(u, v) := \int_{\mathbb{R}^n} A(t, x)\nabla_x u(x) \cdot \overline{\nabla_x v(x)}\, \mathrm{d}x,$$

for all $u, v \in H^1_\omega(\mathbb{R}^n)$. The assumptions on A yield

$$|B_t(u, v)| \leq M \|\nabla_x u\|_{2,\omega} \|\nabla_x v\|_{2,\omega}, \quad \nu \|\nabla_x u\|^2_{2,\omega} \leq \mathrm{Re}(B_t(u, u)).$$

This is (6.1) with $T = \nabla_x : H^1_\omega(\mathbb{R}^n) \to L^2_\omega(\mathbb{R}^n)^n$. We note that T is injective since $d\omega$ has infinite mass as a doubling measure on $\mathbb{R}^n$. We denote by $\partial_t - \omega^{-1}(x)\mathrm{div}_x A(t, x)\nabla_x$ the degenerate parabolic operator associated with the family $(B_t)_{t\in I}$. At this point, we can apply the theory developed above to obtain well-posedness results on $I \times \mathbb{R}^n$, for the Cauchy problems with test functions in $I \times \mathbb{R}^n$ using Theorem 8.1, assuming weak solutions to *a priori* be in $L^1_{\mathrm{loc}}(I; L^2(\mathbb{R}^n))$ if I is unbounded.

Theorem 9.8. *The operator $\partial_t - \omega^{-1}(x)\mathrm{div}_x A(t, x)\nabla_x$ on $I \times \mathbb{R}^n$ has a unique fundamental solution.*

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Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405 Orsay, France
Email address: pascal.auscher@universite-paris-saclay.fr

Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405 Orsay, France
Email address: khalid.baadi@universite-paris-saclay.fr