Justifying assumptions in mathematical descriptions: a modeling practice methodized by biological theory

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Abstract

A mathematical description relies on assumptions about some aspects of things which are not under focus in the description. In models, they are manifested by symmetries, initial and boundary conditions, parameters, etc. Those things are externally determining in the sense that they contribute to the determination of the mathematically formalized unknowns (state), but cannot be affected by them. Now, most assumptions rely on pieces of knowledge regarding the *relative invariance* of those determining aspects of things. Yet, those invariances are, in fact, bounded in time. Thus, assumptions and therefore mathematical descriptions relying on them cease to be valid at some point.

Based on this epistemological difficulty, we introduce a modeling framework to justify the undetermined knowns (assumptions) of a mathematical description by the determined unknowns of *other* mathematical descriptions. Notably, we introduce objects defined by mathematical descriptions, oriented relations of determination between them, and discuss some elementary diagrams with examples. We distinguish synchronic from diachronic relations of determinations, which allows us to consider the beginning, change, and end of lasting relations of determinations.

Apart from shedding new lights on the object of physics, this work formalizes a modeling practice that is theoretically relevant in other sciences resorting in part to the same epistemology. To emphasize this aspect, we show how it integrates a theory in biology via closure of constraints. By positing as fundamentally invariant the *closure* of a self-determining *set* of objects describing a biological system, the possible variations of the latter are unbounded as long as it remains organizationally closed. This work therefore paves the way for modeling the open-ended evolution of an organized entity using *organized mathematical descriptions*.

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Contents

Introduction

What justifies the validity of mathematical descriptions in sciences? In physics, the answer is at least partly provided by theories (Weyl, 1952; Feynman, 1967; Van Fraassen, 1989; Bailly and Longo, 2011; Longo and Montévil, 2014). They posit principles of conservation, fundamental symmetries, and involve fundamental constants whose measurements are invariant—although historically refined by varying instruments. One formalizes those theoretical elements using mathematical writing, thus introducing mathematical objects and enabling the construction of a mathematical structure. The latter's validity is *in fine* at least in part justified by the validity of the theories involved.

In general, in addition to theoretically grounded mathematical objects, other objects are necessary to construct physically descriptive mathematical structures. They formalize the assumptions of the modeler about a situation of interest. Therefore, the validity of a description also depends on the validity of those assumptions. From a mathematical perspective, elements stemming from assumptions and theories have the same status: their validity is *a priori* taken for granted because they enable the determination of some scientifically relevant unknown that motivates the description. Moreover, their validity is not determined by the mathematical description. By analogy with mathematical logic, they are both similar to axioms.

Nevertheless, mathematical elements stemming from assumptions and theories are distinct in natural sciences because assumptions may cease to be valid at some point in time. Of course, the validity of theories may also end due to empirical or theoretical discoveries. Yet, theoretical principles are posited precisely to be invariant in time. In contrast, the modeler usually makes assumptions that she knows to be bounded in time— otherwise, she would suggest a new theoretical principle. In sum, on the one hand, we can consider that theoretical elements are independent of time, so formalizing them using mathematical objects is adequate. On the other hand, this is not the case of assumptions, hence an irreducible tension when using abstract objects to formalize them. Our initial question can, therefore, be rephrased: What justifies the use of mathematical objects formalizing assumptions that are valid only for some time in mathematical descriptions?

Generally, they are justified by things exhibiting some invariance, during an interval of time, and relative to what the mathematical structure describes. For example, boundary conditions for particles in a volume may be justified by a metallic box enclosing them. This box may vary if it corrodes, but its relevant aspects —as far as the particles are concerned—, are invariant during an interval of time. At some point in time, the box no longer justifies the boundary conditions because it breaks. Also, before the box was fabricated, it did not justify the boundary conditions. Numerous physicists make explicit the consequences of this epistemological remark in different contexts (see e.g., Reiner (1964); Feynman (1972); Palmer (1982); Beekman et al. (2019)). For instance, Richard Feynman famously begins his set of lectures on statistical mechanics defining thermal equilibrium as follows: it is when "all the 'fast' things have happened and all the 'slow' things not".

In physics, the consequences of the bounded in time validity of mathematical descriptions only affect the modeler's practice. For his description to be steadily grounded for some interval of time, he resorts to assumptions that he can support, be it by involving the teleological actions of human beings (e.g., experimenters, engineers). He can also rely on elementary and simple entities whose properties are perennial or measure the relevant aspects of its object of study. In sum, the end of validity of a mathematical description does not play any fundamental role in current theories, and is usually avoided. From a mathematical perspective, chemistry follows the same epistemology. However, raising this question is certainly relevant to articulate experimental and modeling practices in physics and for a better epistemological understanding of the objects of physics. In that regard, it may shed new light on foundational questions in this field.

Now, physicists have actively developed interfaces with other fields, and scientists have taken (classical) physics as the model of sciences, leading to the presence of mathematical descriptions in disciplines where the question of the bounded validity of mathematical description can matter theoretically. It is singularly the case in biology.

There is no physical or chemical theory defining what are biological functions, organizations, and individuals. However, those concepts are ubiquitous in biology, which studies objects made of physicochemical entities. One may expect of a biological theory to formalize those concepts while being compatible if not articulated with physical and chemical ones (Longo and Soto, 2016). Yet, although the latter posit fundamental invariants or symmetries, the most notable and unanimous achievement of theoretical biology so far has been to posit a principle based on variation, which, for instance, comes before natural selection in the theory of evolution (Darwin, 1859). This theoretical move has far-reaching consequences, notably regarding the role of mathematical descriptions in

biology (Longo and Montévil, 2014). Yet, only a few scientists made explicit the fundamental difficulty of using a descriptive approach that understands changes based on invariances in biology. René Thom realized this incompatibility without fear of drawing the consequences of his rigorous reasoning: "It is the lack of the definition [of the virtual possible] that affects—very seriously— the scientific nature of Darwin's Theory of Evolution" (Pomian and Amsterdamski (1990), p. 271, our translation). A change of a mathematical structure in time (here, a space of possibilities, a state space) is indeed incompatible with the common use of mathematical descriptions in physics. For R. Thom, not complying with this aspect of the epistemology of physics makes a theory unscientific.

In contrast, a recent theoretical framework embraces Darwin's ontological reversal and draws its consequences regarding mathematics (Soto et al., 2016). It posits a principle of variation, which is equivalent to stating that the validity of a mathematical description of a biological phenomenon is not stable in time (Montévil et al., 2016). Consequently, the justification of the validity of mathematical descriptions has a theoretical role, at least in biology.

To understand why from an informal perspective, consider a biological system constituted of n parts respectively described by n mathematical descriptions. On a coarse-graining level relevant to biology, there are no entities with intrinsic perennial properties. Physically isolating one part usually entails its fast degradation and possibly the death of the organism. Continuous measurements *in vivo* are not possible in general, and many parameters are usually fitted in models (Lillacci and Khammash, 2010). Therefore, it is the n described parts of the biological system that justify at least in part the assumptions of the n mathematical descriptions. Explaining the origin of those assumptions unveils the biological organization of the system by exhibiting how parts are related to one another. It ultimately explains the stability of the biological system, by identifying the role of its parts in maintaining one another. The functioning of the biological system being explained, the respective assumptions in the n mathematical descriptions of its parts—usually assuming the functioning of the overall system— are justified.

These anti-reductionist and circular trademarks are the signatures of a long organicist tradition in philosophical and theoretical biology (Kant, 1790; Bernard, 1865; Cannon, 1929; Schrödinger, 1944; Wiener, 1948; Von Bertalanffy, 1952; Piaget, 1967; Waddington, 1968; Maturana and Varela, 1973; Rosen, 1991; Kauffman, 2000; Montévil and Mossio, 2015). The stance of philosophers and scientists who recognize themselves in this tradition is that organisms are capable of *self-determination*. A major epistemological obstacle is that self-determination implies a circularity between causes and effects which, without a precise characterization, is not compatible with the stance of modern sciences —effects cannot determine their causes. However, given a precise definition of *biological organization*, ubiquitous but problematic concepts of biology can be naturalized, such as biological functions, norms, teleology, and individuals (see Mossio (2024) and references therein for a recent review).

Biological organization constitutes a second principle in the above-mentioned theoretical framework (Mossio et al., 2016). For the latter to be tested against scientific practices, efforts need to be dedicated to articulating the theory with modeling and empirical practices (Montévil and Mossio, 2020). "Whether you can observe a thing or not depends on the theory which you use. It is the theory which decides what can be observed" (A. Einstein, cited in Heisenberg (1971)). From an experimental point of view, the theory points to the critical role of past observables (Montévil, 2019a). Systematic work is lacking as far as mathematical descriptions are concerned. One should expect that the theory also points to what deserves to be determined by a mathematical structure, i.e., what requires explanation in biology. So far, the very first models, be them mathematized or not, have not been constructed using a systematic method (Montévil et al., 2016; Montévil, 2018; Bich et al., 2020; Montévil, 2022; El-Hani et al., 2024).

Mathematical descriptions of parts of biological systems are abundant in the literature (see e.g., Murray (2002); Britton (2003)) and are the backbone of what the US National Research Council of the National Academies calls the *new biology* (National Research Council of the National Academies, 2009). Considered as models, they can be seen as biophysical models in the sense that they use mathematical descriptions like physicists do: they describe changes determined by invariants. Our stance is that those models lack a theoretical framework to be *biologically* meaningful (O'Malley and Dupré, 2005). As mentioned, they are indifferent to key biological concepts, such as function —which is absent from the theories of physics for good reasons. This goes hand in hand with the fact that what requires a description is not defined *a priori*. Moreover, they cannot constitute biological models according to the above-mentioned theory without contradicting the principle of variation and (therefore) face epistemological difficulties (Montévil, 2020). A *single* mathematical description cannot be a model of a biological system that undergoes open-ended biological evolution (Longo, 2018; Montévil, 2019b; Kauffman, 2019). This is because it cannot but assume the invariant effect of determining entities external to the description, i.e., the relative

invariance of the causal context of what is described.

The gist of the problem is therefore the place of mathematics in this theoretical framework. Different approaches may be proposed. The first one consists in elaborating new mathematics (Sarti et al., 2022). Similar to the co-evolution of physical theories and mathematics throughout the history of science, one could argue that biology deserves its own mathematics. Second, one may argue that by resorting to computational tools of growing power, the conceptual problems at stake could effectively be overcome in a single large enough mathematical description in the future (e.g., see (Soros and Stanley, 2014) in the field of artificial life). Although those two approaches aim to address the principle of variation, they do not tackle explicitly the principle of organization.

The approach we submit in this article is compatible with both principles and usual modeling practices in biology, i.e., biophysics. It builds on the following reasoning. The fruitful reductionist approach of biophysical models is undeniably valuable in describing parts of biological systems. Yet, by hypothesis, biological systems exhibit a circular regime of causation. Therefore, a mathematical description of a part of a biological system relies on at least one assumption whose validity *ultimately* depends on what this very mathematical description determines. Now, those assumptions involve the effects of other parts of the biological system, so they can be justified by what mathematical descriptions of other parts of the system determine. Thus, our approach builds on the following idea: we aim to justify an assumption in one description by what *another* mathematical description determines. Using this oriented relation, circularity will then take place at the level of mathematical descriptions.

This article thus introduces a modeling practice methodized by biological theory. However, this framework may be valuable in other sciences that resort to mathematical descriptions too—for epistemological or theoretical reasons. After introducing objects defined by mathematical descriptions (Part. 1), we define relations of determinations between such objects (Part. 2), and combinations of those relations (Part. 3). Up to this point, there is no biological specificity to our framework. To emphasize this point, we provide physical and chemical examples all along. The last part of this paper (Part. 4) finally expresses, in this framework, the circularity of a biological organization *sensu* closure of constraints (Montévil and Mossio, 2015)—this principle will be briefly summarized. We finally conclude that by positing as a fundamental invariant of biological systems a circular relation of determination between biophysical models, biological functions can be defined in modeling practices and they can diachronically change while leaving the biological system organized.

1 An object defined by a mathematical description

In this first part, we introduce several concepts to identify elements of a mathematical description. We use them in the next parts, in which we introduce relations of determinations between two objects defined by such descriptions. To visualize how we articulate those concepts from a global perspective, one may refer to Fig. 1, which summarizes this section. We also provide two detailed examples to illustrate the practicality of those concepts.

1.1 Things and objects

1.1.1 Things and their aspects

We first identify what a mathematical description describes — some aspects of a thing —, and what kind of objects those descriptions define. We introduce this distinction because different objects may describe different aspects of the same thing. For example, we may describe an iron box with solid mechanics to understand its mechanical properties, condensed matter physics to understand its electrical properties, or chemistry to understand its corrosion.

We use the term "thing" to refer to *what* can be designated, observed and possibly manipulated in empirical works. In the context of this paper, things are or can be objectivized by a scientific description; however, a thing is always partially described because a description focuses on some of its aspects only. Moreover, things may be involved in different causal contexts in which they may do very different things (Mohan and Dwivedi, 2018)—accordingly, our use of "thing" should not be considered with a rigid metaphysics.

We adopt the crucial following stance: *the aspects of a thing that can change are not indefinitely invariant*. This statement is neither a physical principle, nor a wild ontological commitment. First, it is grounded by the second law of thermodynamics, considered in a large enough isolated system. Correlatively, it is the mere consequence of the most plausible hypotheses regarding the ultimate fate of the universe—roughly, an energetically homogeneous soup of elementary particles: any

situation differing from this state is ultimately transient. Second, from another perspective, the equivalence we adopt between aspects that "can change" and aspects that "will change" can be understood by an inevitable interaction with another thing that ultimately impacts those aspects. Nevertheless, note that that this statement is not formal, and does not always have practical value: for example, the lifetime of protons is such that it is reasonable to say that some of their aspects are infinitely invariant in most situations. In sum, we adopt this stance to *take seriously* the far-reaching consequences of the problem of time scales in the overwhelming majority of scientific descriptions.

1.1.2 Studied aspects of the studied thing (SAST)

In the first part of this paper, we need to highlight the particular thing that the mathematical description at hand describes. We call it the *studied thing*. Similarly, its scrutinized aspects are the *studied aspects of the studied thing* (SAST), and they hopefully describe the phenomenon of interest. As a running example throughout this part, we use a particle in a squared box as a studied thing. The SAST are its position and velocity. Other aspects of it are unstudied (e.g., its mass, charge, color, etc.).

1.1.3 Object

Scientific objects describe some aspects of things. In this paper, we exclusively refer to objects defined by mathematical descriptions. One can create such an object *in abstracto* on a piece of paper and it is assumed to capture the causation of the intended phenomenon (in contrast, other epistemologies base the definition of an object on a particular thing—see e.g., International Commission on Zoological Nomenclature (1999)). This abstract definition, detached from things, implies that an object can actually describe a class of things that share some aspects. In other words, an object defined by a mathematical description is a *generic* object (Bailly and Longo, 2011; Longo and Montévil, 2014; Zalamea, 2015). This has consequences for scientific practices: experimenters can work on the same object using different things belonging to the same class. In the following, we refer to an object describing the SAST as *the* object, i.e., the one that scientists (traditionally physicists) effectively construct in their study.

1.2 Mathematical structure (MS)

We now turn to the analysis of a mathematical description. The term of "mathematical model" is ambiguous, since it may refer to the product of a modelization process, i.e. a mathematical structure, or to the rationale leading to this product, i.e. the construction of the mathematical structure. These two meanings structure this part, and we contend that the scientific meaning of a mathematical structure requires the supplement provided by the arguments in its construction. We analyze mathematical structures in this section, and their construction in the next. Here, when we speak of a mathematical description, it includes those two parts.

The MS is the part of the object that, once constructed, can be, at least in principle, scrutinized without knowing the physical meaning of the mathematical signs (say, by a fictitious mathematician without any physical knowledge). It is composed of mathematical spaces and equations that relate mathematical objects living in these spaces (quantities or more elaborate constructs).

The introduction or the use of the MS is motivated by the studied thing (SAST). In order for the model to provide an understanding of the SAST, the MS is polarized by the modeler into two parts: a part that is determined by the rest of the mathematical structure, the determined unknown (DU), and the determining mathematical structure (determining MS).

1.2.1 State of the object or determined unknown (DU)

In the MS, the phenomenon of interest (SAST) is formalized by the determined unknown (DU), typically the state of the object. The state of the object is unknown, but it is determined by the mathematical description: it is the DU. Determining the unknown is what motivates the use of the mathematical description. It can be of any appropriate mathematical nature (e.g., point, vector, function, etc.).

For instance, for the particle in the box, a value of the vectors describing the position $\mathbf{q}$, and momentum $\mathbf{p}$ of the particle defines a state of the object $(\mathbf{q}, \mathbf{p})$. In another context in which a spatially resolved density of charge is a DU, a function ρ may define a state of the object, where $\rho(\mathbf{x})$ is the charge density at a position defined by a three-dimensional vector $\mathbf{x}$.

1.2.2 Determining mathematical structure (DMS)

What determines the DU? It is the *determining mathematical structure* (DMS). The DMS may take various forms, but always includes two kinds of elements: a *state space* and *equations*.

The state space contains the possible states of the object, i.e., the possible DUs.

The equation is what determines the state of the object or determines how the state changes in the state space over time (dynamical systems). Equations typically involve variables, parameters, initial conditions, etc. In a vast majority of cases, they are shaped by an optimization principle (e.g., least action, maximal entropy, etc.). There is a crucial hierarchy between the DMS and the DU: the former determines the latter, and the state or a change of state of the object cannot affect the DMS.

For instance, the state space of the particle may be $[0, L]^3 \times \mathbb{R}^3$ where L is the length of the box. If boundary conditions are used, a state space such as $\mathbb{R}^6$ could lead to a physically identical situation. The equation determining how the state changes could be Newton's equation from classical mechanics. In another context in which the unknown pressure P of a perfect gas is determined for a given volume V, number of particles N, and temperature T, the equation of equilibrium thermodynamics reads $P = Nk_BT/V$ where k_B is the Boltzmann constant[1].

Together, the DU and the DMS constitute the *mathematical structure* (MS) of the object. It is purely mathematical, and is expressed in arbitrarily sophisticated mathematics.

The MS depends on the employed theory. In physics, there are attempts to posit a standard MS for a given physical theory (Schweigert, 2023). Yet, it is important to note that an adequate MS is not univocal (Zalamea, 2015). Different MS may lead to the same physical result. Defining classes of MSs is of no use from a practical point of view—one must choose one to operate. On the other hand, the same MS may describe different aspects of different things, which is typically the case of analogical models (Achinstein, 1964). In sum, the MS of the object and the SAST are far from a one-to-one relationship. Mathematics is highly valuable in the construction of objectivity, but in an instrumental sense, not in an ontological sense (Knuuttila, 2011). For an MS to make scientific sense, it requires supplements; therefore, we need to consider the construction of the MS of the object (see below) in detail.

1.2.3 Explicit property of the determined unknowns (EPDU)

The MS enables one to exhibit an *explicit mathematical property of the DUs* (EPDU). It is a relevant property that may involve multiple states of the object (hence the plural for DU*s*), and which is meaningful from a scientific point of view. While the DMS mechanically determines the DU, this is not the case of the EPDU, because it requires the additional choice of the property to exhibit, the identification of the *explanans*. A special case is when the EPDU of choice is the solution to the DMS, for example a trajectory as a function of time. One derives an EPDU from a mathematical description in a concise manner to ease its interpretation.

For instance, the time-averaged position of the particle may be an EPDU. In other contexts, asymptotic properties, such as attractors, limit cases, steady states as well as transient ones, such as relaxation times, may be EPDUs.

1.3 Construction of the mathematical structure (CMS)

We now focus of the rationale leading to the MS of the object. We are not interested in the research process from which the MS results, and that involves the know-how of modelers, tinkering and explorations. Rather, our focus is on the argumentative steps that modelers must report to account for the MS they submit, i.e., the *scientific construction of the mathematical structure* (CMS). It defines the meaning of the mathematical signs, and decomposes analytically into positing undetermined knowns (Sect. 1.3.1) and formalizing them mathematically by external determinants (Sect. 1.3.3).

1.3.1 Undetermined known (UK)

An *undetermined known* (UK) is a piece of knowledge that *enables* scientists to determine the state (DU). We call it undetermined because, in contrast to the DU, the DMS does not determine it. Because it is a piece of knowledge, it requires an argument. The latter may involve a theoretical principle, an empirical regularity, a measurement, a heuristic[2], an explicit restriction of

[1] In this case, the state of the object cannot change without a change of the DMS.

[2] The argument is that it facilitates the mathematical description without qualitatively changing the EPDU *a priori*.

the description, etc. In general, UKs enable the introduction of symmetries, conserved quantities, fundamental constants, parameters, initial/boundary conditions, approximations, etc..

For instance, we may posit that the box constantly isolates the particle from the rest of the universe (UK), because of some properties of the box (see below). We may then invoke the principle of energy conservation to posit the energy conservation of the particle (UK). We may consider that the mass of the particle and the length of the box do not vary (UKs).

1.3.2 Determining aspects of a determining thing (DADT)

In this paper, we focus on two classes of UKs that involve aspects of things which are not the SAST. The first class contains UKs that refer to initial conditions[3]. The second class contains UKs that involve things for some duration. Not all UKs fall into those two classes. As mentioned, some may stem from theoretical principles, heuristics, refer to fundamental constants, etc.

If a UK involves a thing, we say that the latter is a *determining thing* (DT) of the object. In general, only some of its aspects are involved in a UK and at least some of those aspects may change—otherwise they would not be proper to this thing and be theoretical principles. Therefore, the concept of *determining aspects of a DT* (DADT) is more precise. The UKs of the object that involves DADTs for some duration form this second class. The UK in this class cannot be indefinitely valid over time, as discussed above. Accordingly, if an object relies on a UK of this class, its MS is no longer justified at some point in time.

Note that the studied thing can itself be a DT, because some of its unstudied aspects may be involved in a UK. Moreover, note that UKs of the first class may involve DADTs—but their validity cannot change once they are known.

For instance, the particle is a DT because the invariance of its mass (DADT) is a UK of the second class. The invariant force field of the box on the particle (UK) involves the positions of the atoms the box (DADT), or simply the mass and the planarity of its faces (DADT). The constant isolation of the particle (UK) involves the mass of the box, its rigidity, its impermeability, and its specific heat capacity (DADT).

1.3.3 External determinant (ED)

External determinants (EDs) are mathematical objects formalizing knowledge (UKs). Notably, they have the above-mentioned hierarchical relation with the DU: a state or a change of state of the object cannot affect one of its EDs. EDs ground the MS from a mathematical perspective. Note that the term "external" is to be understood from a causal perspective, not a spatial one.

In the ideal case, when publishing a model, the CMS requires positing UKs, formalizing them as EDs, and synthesizing the latter to build the MS. Reciprocally, for example, if the CMS is incomplete, finding the construction of the object from its MS goes from identifying the EDs, then the UKs that may involve DADTs.

For instance, the parameter m is an ED formalizing the invariant mass of the particle (UK). Depending on the state space (see Sect. 1.2.2), either L is an ED formalizing the invariant length of the box (UK), or the equation of the boundary condition is an ED formalizing the invariant force field that the box applies on the particle (UK). The energy conservation of the particle (UK) is formalized by the following ED: $\mathrm{d}E/\mathrm{d}t = 0$, where E is the energy of the particle and t refers to time.

1.4 Summary

We summarize this section with the help of Fig. 1. A mathematical description relies on an hypothetico-deductive type of reasoning: a mathematical structure (MS) is constructed by positing UKs, and this ultimately enables one to exhibit an EPDU. Let us unpack the most relevant elements we introduced. EDs formalize UKs, and therefore the former ground the MS. They are mathematical objects that are time-translationally invariant in the sense that a state or a change of state of the object cannot affect one of them. Yet, a class of UKs involves DADTs for some duration: their validity is not time-translationally invariant. Therefore, a mathematical description, which is constructed by positing at least a UK of this class, is no longer justified at some point in time.

[3] All initial conditions are set by things, except possibly in the discussion around the past hypothesis in physics (Chen, 2023). Note, however, that some of these things may not be described by mathematized objects.

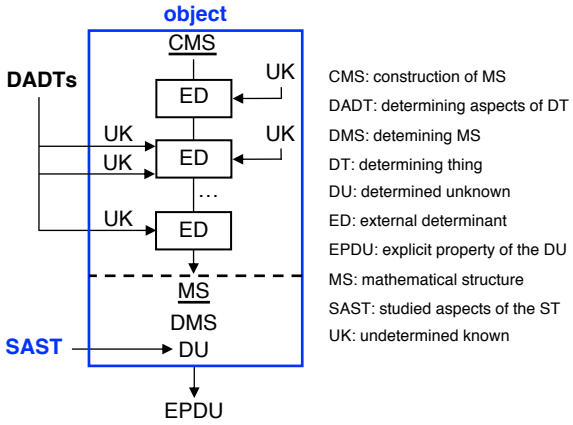


Figure 1: Elements and dependencies of an object defined by a mathematical description. The dashed line depicts the distinction between the process of modelization (CMS) and its product (MS). The object includes the elements inside the blue rectangle, and objectivizes the SAST. The class of UKs involving DADTs are on the left side. EDs formalize one or several UKs. The meaning of all those concepts and their articulation is discussed in the text.

1.5 Examples

We illustrate our conceptual framework with two examples from physics (Landau and Lifshitz, 1987; Johnson and Goody, 2011). The same analysis could be applied *mutatdis mutandis* in other disciplines.

1.5.1 Hagen-Poiseuille description of a liquid flow

We study the velocity of a liquid (SAST) flowing into a macroscopic and straight pipe, with a uniform internal surface. It flows thanks to a uniform pressure gradient between both sides of a pipe, which is ensured by a pump. The liquid fills the pipe, and the flow exits the pipe in the air on the other side. We aim at evaluating the steady state flow rate out of the pipe (EPDU). Let us proceed to the CMS.

We posit that the pipe can be approximated as a cylinder, with constant length L and radius R. The liquid particle mean free path λ is also invariant. We restrict our description to small Knudsen number: $L, R \gg \lambda$. This enables us to use the continuum assumption.

We use cylindrical coordinates $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, with the origin at the center of the entrance cross-section of the liquid in the pipe — $\mathbf{e}_z$ is oriented along the cylinder axis. The space is described by $r \in [0, R]$, $\theta \in [0, 2\pi]$, and $z \in [0, L]$; We note ρ the mass density of the liquid in the pipe and $\mathbf{v} = (v_r, v_\theta, v_z)$ the spatially resolved—e.g., $v_z = v_z\,(r, \theta, z)$— velocity vector (DU) whose possible values are in $\mathbb{R}^3$ (DMS-state space).

We assume that the pipe constantly isolates the liquid from external sources or sinks of mass from the surrounding on $r = R$, all along $\mathbf{e}_z$. Invoking the principle of mass conservation, the continuity equation reads: $\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{v}) = 0$. We posit that the Mach number of the liquid remains relatively low, and therefore posit that is incompressible: ρ is a constant parameter, and $\nabla \cdot \mathbf{v} = 0$ follows. We use the principle of momentum conservation, which is formalized by Navier-Stokes equation (DMS-equation).

We assume that the liquid is an isotropic Newtonian fluid (stress tensor $\boldsymbol{\tau} = \eta\nabla\mathbf{v}$) with constant viscosity η (and no second viscosity effect). The equation simplifies to: $\rho\frac{D\mathbf{v}}{Dt} = \rho\eta\nabla^2\mathbf{v} - \nabla p + \rho\mathbf{g}$, where $D \cdot /Dt$ is the material derivative, p the pressure of the fluid and $\mathbf{g}$ the gravitational field. As a heuristic, we posit the pressure gradient dominates over the gravitational force $\rho\mathbf{g} \ll \nabla p$ while the flow remains laminar $(\mathbf{v} \cdot \nabla)\,\mathbf{v} \ll \nabla^2\mathbf{v}$ — the convective term is negligible w.r.t. to the diffusion term.

We posit that the pressure is always uniformly applied by the pump of the entrance cross sections, so that the flow is invariantly axisymmetric: $\partial v_z/\partial\theta = 0$. Without a driving force along the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ direction, the principle of inertia supports the assumption that non-axial velocities are constantly trivial $v_r = 0$ and $v_\theta = 0$. We can measure the roughness of the internal surface of the pipe, and posit its constant translational invariance. We formalize this piece of knowledge by the translational symmetry of the axial velocity along the pipe's axis: $\partial v_z/\partial z = 0$. Our interest

UKs	EDs	DTs		
—	—	Liquid	Pipe	Pump
invariant cylinder shape	L, R		✓	
invariant dimensions			✓	
invariant mean free path of liquid	λ	✓		
continuum assumption	$L, R \gg \lambda$			
invariant isolation of pipe along $\mathbf{e}_z$	$\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0$		✓	
mass conservation principle				
invariant mass density of liquid	ρ	✓		
momentum conservation principle	Navier-Stokes equation			
invariant isotropy and viscosity	η	✓		
invariant newtonian behavior	$\tau = \eta \nabla \mathbf{v}$	✓		
negligible gravity heuristic	$\rho \mathbf{g} \ll \nabla p$			
laminar flow heuristic	$(\mathbf{v} \cdot \nabla)\, \mathbf{v} \ll \nabla^2 \mathbf{v}$			
non-axial velocities assumption	$v_\theta = 0,\, v_r = 0$			
invariant axisymmetry	$\partial v_z/\partial\theta = 0$			✓
invariant trans. symmetry of axial velocity	$\partial v_z/\partial z = 0$		✓	
restriction to steady-state	$\partial\,(...)\,/\partial t = 0$			
invariant boundary condition	$\frac{\partial v_z(r)}{\partial r}\vert_{r=R} = 0$	✓	✓	
invariant applied pressure gradient	ΔP			✓

Table 1: Summary of the CMS for the Hagen-Poiseuille description of a studied liquid flow. If a UK formalized by an ED involves a DT, a checkmarck is placed in the corresponding column. We detail the DADTs in the text.

lies in the steady state, so we posit $\partial\,(...)\,/\partial t = 0$ to focus on the EPDU.

It can be shown that $p = p\,(z)$, and that the DU is determined by the following simplified equation: $\eta\partial\,(r\partial v_z/\partial r)\,/\partial r = r\mathrm{d}p/\mathrm{d}z$. We know that the pipe is macroscopic, so the slip length of the liquid on the pipe can be ignored. The interaction between the pipe and the liquid is considered to be constantly purely steric, and this is formalized by the invariant no-slip boundary condition: $\frac{\partial v_z(r)}{\partial r}\vert_{r=R} = 0$. Finally, the constantly applied pressure gradient by the pump is formalized by a positive parameter ΔP[4]. The solution of this equation is a first EPDU:

$$v_z\,(r) = \frac{\Delta P}{4\eta L}\left(R^2 - r^2\right).$$

The flow rate out of the pipe, the EDPU of interest can be obtained by averaging over a cross section. Table 1 summarizes the UKs, the DTs, and the related EDs.

To conclude this example, let us give reasons why UKs involving DADTs can change. For UKs involving the liquid, they can change with temperature, especially if they crystallize. For the pump, the invariance of the pressure uniformity and intensity depends on the good functioning of the apparatus, which will eventually dysfunction. For the pipe, its geometry, dimensions, impermeability, internal roughness homogeneity, and interaction nature with the liquid are involved. Corrosion is one source of degradation if it is metallic. Other degradation phenomena involve the accumulation of impurities on its internal surface that may change its radius. A random fabrication defect may break the translational symmetry of the axial velocity. A leak can create a sink of liquid mass which invalidates the isolation of the flow, etc.

1.5.2 Michaelis-Menten description of a catalyzed chemical transformation

We study the catalyzed chemical transformation of a reactant (noted A) into a product (noted B) in a thermostated solvent situated in a container. On the flat bottom of the container, a rotating

[4]The pressure gradient can also be understood as the combination of two boundary conditions.

UKs	EDs	DTs				
—	—	1	2	3	4	5
invariant configurational symmetries	$X' \sim X$		✓			
postulate of equilibrium						
invariant kinetic constants	k_i	✓	✓	✓		
invariant volume of dilution	V		✓		✓	
invariant mixing	$[X]$				✓	✓
empirical law	rate equation					
invariant number of catalysts	N_C			✓		
mass conservation principle	$\mathrm{d}N_{\mathrm{tot}}/\mathrm{d}t = 0$					
initial conditions	N_A^0, N_B^0	✓				
restriction of parameter space	$N_C \ll N_{\mathrm{tot}}$					
heuristic steady state	$\mathrm{d}N_{AC}/\mathrm{d}t \simeq 0$					

Table 2: Summary of the CMS for the Michaelis-Menten description of a catalyzed chemical transformation. If a UK — formalized by an ED — involves a DT, a checkmark is placed in the corresponding column. We detail DADTs in the text. DTs are numbered as follows: 1) reactant and product, 2) solvent, 3) catalyst, 4) container, 5) magnetic stir bar.

magnetic stir bar mixes the solution. The studied things are the reactant and the product, and their studied aspects are their concentrations in the solvent (SASTs). The catalyst (noted C) can bind to the reactant to form a complex. The latter is then transformed into the product, while restoring the catalyst. Let us proceed to the CMS.

We suppose that the solvent constantly exchanges energy but no matter with the species (closed system). Using the postulate of thermodynamic equilibrium, we posit that if X' is a configuration of a species, it is symmetric to X, where X is the thermal equilibrium configuration. The standard representation of this reaction in chemical kinetics is:

$$A + C \underset{k_{-1}}{\overset{k_1}{\leftrightarrows}} AC \xrightarrow{k_2} B + C.$$

where k_i are kinetic constants for $i \in \{-1, 1, 2\}$, depending on the invariant temperature, dielectric properties of the solvent, and some aspects of the chemical structures of the reactant, catalyst and product.

We assume the volume of the solvent V in the container is invariant, because of the invariant impermeability of the container that isolates it from other sources of the liquid. The rotating magnetic stir bar constantly mixes the contained solvent, thus maintaining a spatial symmetry of the chemicals' distribution in the solvent. This enables the introduction of uniform concentrations, noted $[X] = N_X/V$, where N_X is the number of species X in the volume. The possible concentrations $([A], [B])$ (DU) form the state space $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ (DMS-state space). The empirical rate equation of chemical kinetics determines how the state of the object changes (DMS-equations).

We consider that the number of catalysts — irrespectively of its free or binding form — in the volume is invariant because it is not consumed by the reaction. It is defined by the parameter N_C. The principle of mass conservation is formalized as $\mathrm{d}N_{\mathrm{tot}}/\mathrm{d}t = 0$, where $N_{\mathrm{tot}} = N_A + N_B + N_{AC}$. Initial conditions $\left(N_A^0, N_B^0\right)$ stem from the initial number of reactants and products in the container. Restricting their possible values to the case for which $N_C \ll N_{\mathrm{tot}}$, we have $N_{AC} \ll N_A + N_B$, and therefore $\mathrm{d}N_A/\mathrm{d}t = -\mathrm{d}N_B/\mathrm{d}t$, and $N_{\mathrm{tot}} = N_A^0 + N_B^0$. Using as a heuristic that the complex concentration is in a steady state ($\mathrm{d}N_{AC}/\mathrm{d}t \simeq 0$), the equations determining the DU simplify to:

$$\frac{\mathrm{d}\,[B]}{\mathrm{d}t} = k_2\,[C]\,\frac{[A]}{K_{\mathrm{m}} + [A]} = -\frac{\mathrm{d}\,[A]}{\mathrm{d}t}.$$

where $K_{\mathrm{m}} = (k_{-1} + k_2)/k_1$. A relevant EPDU may be the ratio of B in the solvent: $[B]\,V/N_{\mathrm{tot}}$. Table 2 summarizes the UKs, the DTs they involve, and the EDs that formalize them.

To conclude this example, let us give reasons why UKs involving DADTs can change. The temperature of the solvent (and the kinetic constants) can change if a source of heat triggers this

change. The volume of solvent can change due to some leaks in the container, or due to long-term evaporation. The magnetic stir bar can stop rotating fast enough to justify the well-mixed assumption.

1.6 Conclusion

Those two examples reveal something essential regarding UKs of the second class. Recall that they involve DADTs for some duration: they rely on pieces of knowledge regarding some kind of invariance of DADTs. Yet, this invariance is undetermined as far as the submitted description is concerned. When this invariance ceases, the mathematical description is no longer valid. Moreover, we have no principled reasons to posit that we have more knowledge about the dynamics of the DADT than on the one of the SAST. A refined description of the global situation is therefore possible.

2 Relations of determination between objects

In the previous part, we introduced two classes of UKs: initial conditions and the UK involving a thing for some duration. We are interested in those two classes because they are the basis for the relations of determination between objects. The structure of this part of the paper follows the distinction between those two classes. We first present the motivations for introducing determining relations between objects in Sec. 2.1. Then, we define the diachronic determination in Sec. 2.2, and the synchronic one in Sec. 2.3. Finally, we show in Sec. 3 how both relations can be combined, notably with a detailed example.

2.1 Motivation

The questions that motivate the introduction of determining relations in this part are the following: Where do DADTs come from and what justifies their validity? What happens to DADTs during changes of the SAST (or *vice versa*)?

To objectivize the origin of DADTs or what happens to them during changes of the SAST, we resort to objects defined by mathematical descriptions. Our attention is focused on *relations* between those objects. Schematically, the general idea is the following. An object relies on UKs: they are not determined, but taken for granted without principled reasons. On the other hand, an object does determine an unknown (DU). A relation between two objects can naturally be a *relation of determination*: an object determines an unknown, which is then known but undetermined by the other. Building on the distinction between the two classes of UKs, we introduce two types of *oriented* relations.

The first relation is a relation of *diachronic determination* between two objects $\mathcal{X}$ and $\mathcal{Y}$, noted $\mathcal{Y} \longrightarrow\!\!\!\!\shortmid\ \mathcal{X}$. The *determining object* $\mathcal{Y}$ determines a UK of $\mathcal{X}$ that belongs to the first class. It explains in part the initial conditions of $\mathcal{X}$.

The second relation is a relation of *synchronic determination* between two objects $\mathcal{X}$ and $\mathcal{Y}$, noted $\mathcal{Y} \Longrightarrow \mathcal{X}$. It refers to a determination by $\mathcal{Y}$ of a UK of the *determined object* ($\mathcal{X}$) that belongs to the second class. The *determining object* ($\mathcal{Y}$) objectivizes what happens to some DADTs of $\mathcal{X}$ during some amount of time.

Before precisely defining those relations, we detail how we plan to answer the above-mentioned questions with those relations. Suppose $\mathcal{X}$ describes the SAST. What happens to DADTs of $\mathcal{X}$ during changes of the SAST? If $\mathcal{Y} \Longrightarrow \mathcal{X}$, then $\mathcal{Y}$ describes what happens to some of them. Where do those DADTs come from? If $\mathcal{Z} \longrightarrow\!\!\!\!\shortmid\ \mathcal{Y}$, then $\mathcal{Z}$ describes the origin of the initial conditions of $\mathcal{Y}$. Let us now turn to the formal definition of the diachronic determination.

2.2 Diachronic determination ($\mathcal{Y} \longrightarrow\!\!\!\!\shortmid\ \mathcal{X}$)

In this section, we define the relation of diachronic determination between two objects using the concepts introduced in Sect. 1.

2.2.1 Definition

Definition 1 ($\mathcal{Y} \longrightarrow\!\!\!\!\shortmid\ \mathcal{X}$) Let $\mathcal{X}$ and $\mathcal{Y}$ be two objects respectively defined by a mathematical description. $\mathcal{Y}$ is diachronically determining for $\mathcal{X}$ if and only if:

1. An EPDU of $\mathcal{Y}$ explains at least in part the initial conditions of $\mathcal{X}$.

2. What $\mathcal{X}$ describes happens after what $\mathcal{Y}$ describes.

For instance, an EPDU of $\mathcal{Y}$ may be the equilibrium concentration of a chemical, which explains at least in part the initial number of particles of this specie in $\mathcal{X}$, where $\mathcal{X}$ describes the kinetic reaction of this specie with some others that are absent from $\mathcal{Y}$, and in a different diluting volume.

Let us clarify some of the terms of the definition. First, as mentioned in the first part of this paper, the EPDU of an object is *not unique*. It requires the choice of the mathematical property of the DU to exhibit, and several may do the job. Here, the *explanandum* is the initial conditions of $\mathcal{X}$.

Second, an EPDU of $\mathcal{Y}$ may explain more than the initial conditions of $\mathcal{X}$. For example, the justification may invoke a kind of coarse-graining or dimensionality reduction from the EPDU to the initial conditions. Reciprocally, several determining objects may be required to explain the initial conditions of $\mathcal{X}$. This is reminiscent of the fact that a single ED may formalize several UKs. Therefore, a single determining object explains *at least in part* those initial conditions.

Three, the temporal ordering between the two descriptions must be clear. What happens *after* is described by the determined object, not the determining one. A diachronic determination relation between two objects cannot point in two different directions. For instance, if $\mathcal{X} \longrightarrow \mathcal{Y}$ also holds for some reason, we must write $\mathcal{Y} \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y}$. For instance, consider the Carnot cycle performed by a heat engine. We can introduce four objects $\mathcal{X}_{AB}$, $\mathcal{X}_{BC}$, $\mathcal{X}_{CD}$, and $\mathcal{X}_{DA}$ to describe the four thermodynamical transformations. The states of $\mathcal{X}_{AB}$ and $\mathcal{X}_{CD}$ are values of a variable pressure and volume (P, V), because they describe isothermal transformations. The states of $\mathcal{X}_{BC}$ and $\mathcal{X}_{DA}$ are values of a variable pressure, volume, and temperature (P, V, T), because they describe adiabatic transformations. Along a single cycle, we have $\mathcal{X}_{AB} \longrightarrow \mathcal{X}_{BC} \longrightarrow \mathcal{X}_{CD} \longrightarrow \mathcal{X}_{DA}$, because the final conditions of one object explain at least in part the initial conditions of the next. If the cycle is repeated, we must append the new cycle to this sequence as follows: $(... \longrightarrow \mathcal{X}_{DA}) \longrightarrow (\mathcal{X}_{AB} \longrightarrow ...)$.

Four, the second requirement implies that $\mathcal{Y}$ may describe a cause whose effect is described by $\mathcal{X}$. For instance, the speed of a stone falling into a pond described by $\mathcal{Y}$ may explain at least in part the initial conditions for the wave dynamics of the pond described by $\mathcal{X}$ (Rovelli, 2023).

Five, it is always possible to extend the definition of the state of an object $\mathcal{X}'$ so that its initial state takes into account what happened before the initial time $t = 0$. Indeed, suppose that $\mathcal{Y} \longrightarrow \mathcal{X}'$, with x_0 the initial state of $\mathcal{X}'$. We can define $\mathcal{X}$ with state $\left(x_{[t-\tau,t[}, x_t\right)$, where $x_{[t-\tau,t[}$ is a register for the trajectory of $x_{t'}$ for $t - \tau \leq t' < t$, and for *arbitrary* $\tau > 0$. Then, $\mathcal{Y} \longrightarrow \mathcal{X}$ holds if an EPDU of $\mathcal{Y}$ also explains at least in part $x_{[-\tau,0[}$[5].

Six, if an object diachronically depends on several objects, those objects need to explain the initial conditions of $\mathcal{X}$ simultaneously.

Finally, the initial condition of $\mathcal{X}$ is said to be a *diachronic ED* because it can be involved in the relation of the type $\mathcal{Y} \longrightarrow \mathcal{X}$.

2.2.2 Time resetting ($\mathcal{X} \longrightarrow \mathcal{X}$)

We here introduce a diagrammatic convention for resetting the clock defining the initial time for an object. It reads:

$$\mathcal{X} \longrightarrow \mathcal{X}. \tag{1}$$

From a formal perspective, it corresponds to a *change of value of the initial conditions* of $\mathcal{X}$. Suppose x_t is the state of an object $\mathcal{X}$, with x_0 its initial state at $t = 0$. We can choose a time $t = t_{\text{reset}} > 0$ for $\mathcal{X}$ as the determining object, which becomes the initial time $t = 0$ for $\mathcal{X}$ as the determined object. This is what Eq. 1 means, without making explicit t_{reset}. Note that this may have no impact from a physical point of view if the state $x_{t_{\text{reset}}}$ of $\mathcal{X}$ as the determining object is equal to the initial state x_0 for $\mathcal{X}$ as the determined object[6].

Note that Eq. 1 builds on a convention that we will repeatedly use. Let us first emphasize the difficulty. An object is defined notably by its EDs, which include its initial conditions. Therefore, it may be argued that two objects with different initial conditions are not the same object. Yet, initial conditions are usually defined generically because a generic EPDU can be derived for generic initial conditions. Our stance is that an infinitesimal change of value of an initial conditions (or of a parameter, see the equivalent convention in Sect. 3.1.2) should not systematically lead to the introduction of another object—hence the unambiguous time resetting convention in Eq. 1. The motivation for this conservative choice is to minimize the number of different objects in a

[5] The time evolution of the state $\left(x_{[t-\tau,t[}, x_t\right)$ of $\mathcal{X}$ is therefore partially determined by $\mathcal{Y}$ for $t \leq \tau$.

[6] Memory effects can be included as shown above.

description. The rationale supporting it is that a change of value of an initial condition does not require a revision of the CMS of an object.

2.3 Synchronic determination ($\mathcal{Y} \Longrightarrow \mathcal{X}$)

In this section, we define the synchronic determination relation between two objects using the concepts introduced in the Part. 1. The definition (Sect. 2.3.2) requires some preliminaries (Sect. 2.3.1). Subsequently, we introduce a partition of synchronic determinations in Sect. 2.4.2.

2.3.1 Relative invariance

UKs of the second class rely on pieces of knowledge regarding some kind of invariance of the DADTs of the object. In the next paragraphs, we show that this kind of invariance can be understood as a temporally bounded relative time-translational invariance or *relative invariance*, for short. Relative invariance is fundamental, because it is what enables the formalization of a physical determination between two aspects of things by having the first aspect determining an ED for the study of the second one. This gesture draws the boundaries between what is internal and what is external from a causal perspective, an opposition notoriously problematic that deserves careful attention. We therefore take a step back and scrutinize in detail the conditions of possibility of the introduction of an ED in the CMS.

To illustrate it for readers familiar with mathematical modeling, we start the other way around, by considering a well-known particular MS that, by hypothesis, describes some thing for which two different aspects can be designated. From this special case, we find a sufficient condition for one of those two aspects to be externalized (i.e., constitutes some DADT) while the second remains under focus (SAST). This will give us a crucial but partial understanding of the role of relative invariance in the CMS. By discussing the limit of this understanding, we will then see why relative invariance cannot be mathematically handled in the general case. In turn, this will motivate the formal definition that we posit in the next section.

Consider two dynamical variables x and y formalizing two aspects of things. Their changes in time are determined by the following coupled differential equations: $\mathrm{d}x/\mathrm{d}t = f\left(x,y\right)$ and $\mathrm{d}y/\mathrm{d}t = g\left(x,y\right)$, with initial conditions (x_0, y_0). We note $\mathcal{V}_x^\tau$ (resp. $\mathcal{V}_y^\tau$) the values that x_t (resp. y_t) take for $0 \leq t \leq \tau$. Let $C_{y,x}^\tau$ be the class of values of y that are equivalent to y_0 in the following sense:

$$y \sim y_0 \Leftrightarrow \forall x \in \mathcal{V}_x^\tau, \left\| f\left(x,y\right) - f\left(x,y_0\right) \right\| < \epsilon, \tag{2}$$

where ϵ is an arbitrarily small real number and $\|.\|$ an appropriate norm. If $\mathcal{V}_y^\tau \subset C_{y,x}^\tau$, the state of y is, as far as x is concerned, invariant during τ. Then we have a *relative time-translational invariance* of y *for* x *during* τ. Equivalently, we say that y is relatively invariant for x during τ. Let us now explain this denomination.

First, it is not the invariance of a quantity: y may vary in $C_{y,x}^\tau$ during τ. Moreover, x may quantitatively affect the state of y (via g) without affecting the effect of y on itself (via f). This relative invariance is temporally bounded if $\exists t > 0$, $y_t \notin C_{y,x}^\tau$. In practice, we can either emphasize the maximal duration of this relative invariance, or be agnostic to it. If τ_{estim} is the first moment in time at which y_t exits $C_{y,x}^\tau$, we say that y is relatively τ_{estim}-invariant for x. If we do not estimate this timescale, we may simply state that y is relatively invariant for x. Relative time-translational invariance, or relative ∞-invariance, corresponds to the limit case $\tau_{\text{estim}} \to \infty$.

The aspects that y formalize can be externalized (i.e., considered as DADT for the SAST formalized by x) thanks to this relative invariance. Before τ, the state space of x can be reduced to $\mathcal{V}_x^\tau$ and we can introduce $\tilde{f}$, the restriction of f to the domain $(\mathcal{V}_x^\tau, y_0)$. In turn, focusing on the SAST, we introduce the ED $\tilde{f}$, and we read $\mathrm{d}x/\mathrm{d}t = \tilde{f}\left(x\right)$. Note that externalizing y makes the final mathematical writing simpler and practical, but dangerous if not considered with care. First, it is written as if y was relatively ∞-invariant for x. Second, by "hiding" y, it seems that y is not determining for x. Of course, these considerations come on top of usual physico-mathematical ones such as resonance, where small causes can build up over time. Those difficulties are reminiscent of the fact that the aspects that y formalize are DADT for the object whose state is now given by x_t.

In trying to generalize relative invariance based on this particular case, one could argue the following. If all mathematical descriptions require EDs related to DADTs to be constructed, then the functions f and g governing the dynamics of (x,y) are already restrictions of other functions that include, say, a third externalized variable z. In sum, generalizing even further, one would argue that there are "hidden" variables in all mathematical descriptions, and that care should be dedicated to the duration of relative invariances and to the silent determinations at play in all cases.

We agree with this line of reasoning, but it only leads to a partial generalization, and therefore misses the crucial point. After all, from a mathematical point of view, one could consider an extended version of a state space in which variables and parameters are not distinguished. Even though there is an obvious practical difficulty in complexifying descriptions, there is no conceptual necessity in elaborating new modeling practices. However, UKs involving DADT are formalized by EDs that are not necessarily parameters. When, for instance, such EDs are symmetries, boundary conditions, or when they enable the introduction of a conserved quantity, they enable the very introduction of the state space or the equation—and do not have a generic *value*, like parameters do. In turn, the end of the relative invariance of some DADTs may invalidate the state space in which the DU is observed, or the equations by which it is determined. For instance, in describing a rolling bead on a table, one commonly introduces a two-dimensional state space for the position of the bead, leaving its causal context implicit. Yet, the table will break, and the state space ceases to be justified at some point. In turn, the bead may and, in this case, does access degrees of freedom that were not possible beforehand. Keeping the same state space underdetermines its position. In general, there is indetermination of the DU. Relative invariance is necessary to introduce *any* mathematical object formalizing assumptions involving things for some duration.

When describing a thing with two distinguishable aspects, if one of both is determining for the other but relatively invariant, it is therefore desirable to make explicit the externalization that enables the introduction of an ED. Instead of *unpicking* the SAST—leaving its causal context out of sight—, we may rather *unpack* the description by defining a relation of determination based on relative invariance. This idea is behind the introduction of constraints (Montévil and Mossio, 2015): constraints are relatively invariant entities (see Sect. 4.1). The next section formally defines such unpacking, using *two* mathematical descriptions.

2.3.2 Definition

In the CMS of an object $\mathcal{X}$, unpicking the described situation consists in leaving implicit the conditions of possibility of the formalization of some DADT by an ED of $\mathcal{X}$. On the contrary, unpacking the situation consists in making them explicit: it requires an object $\mathcal{Y}$ and a relation of determination between both objects.

Let us then consider a situation where we intend to describe a DADT of an object $\mathcal{X}$ with another object $\mathcal{Y}$. Such an explanation requires that for a duration τ, an EPDU of $\mathcal{Y}$ corresponds to the intended ED of $\mathcal{X}$, possibly up to a tolerance margin that will depend on the properties of the DMS of $\mathcal{X}$. Let us call this criterion E. In general, the states or trajectories of $\mathcal{Y}$ who meet E form a class, and the DU of $\mathcal{Y}$ have to be in this class to explain the intended ED. Then, the ED is invariant, but at the level of $\mathcal{Y}$ this invariance is relative, because it is defined *in relation to* $\mathcal{X}$.

This leads us to define the following relation of synchronic determination:

Definition 2 ($\mathcal{Y} \Longrightarrow \mathcal{X}$) Let $\mathcal{X}$ and $\mathcal{Y}$ be two objects respectively defined by a mathematical description. $\mathcal{Y}$ is synchronically determining for $\mathcal{X}$ if and only if:

1. An explicit property of a class of states $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ of $\mathcal{Y}$ explains at least in part an ED of $\mathcal{X}$ (typically by meeting a criterion E).

2. What $\mathcal{X}$ describes happens in an interval of time during which the state of $\mathcal{Y}$ has to remain in $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ for the explanation to hold.

Some terms regarding (1) have been explained above for the relation of diachronic determination. Let us make some complementary remarks. First, the synchronic determination does not explain the initial conditions of $\mathcal{X}$—in contrast to the diachronic one. This is implied by the definition because if $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ explains at least in part the initial conditions of $\mathcal{X}$, the state of $\mathcal{Y}$ does not have to remain in $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ for the explanation to hold. The synchronic determination refers to determinations that last, not to punctual determinations in time.

Second, $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ must be defined inside the state space of $\mathcal{Y}$ but depends on $\mathcal{X}$. Given an appropriate measure, its size with respect to the one of the state space quantifies the *specificity* of the determination. There are two limit cases, corresponding to extrema of specificity or genericity. On the one side of the spectrum, the class may be equal to the entire state space. On the other side of the spectrum, the class may be a singleton. Note that if $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ is not contiguous, the relevant size is the one of the subspace of $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ which is *accessible* from the state of $\mathcal{Y}$ at the beginning of the interval of time.

Third, for $\mathcal{Y}$ to effectively explain the ED of $\mathcal{X}$ for some duration, its state must be in $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ for some amount of time. The duration of this interval of time, $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$, is estimated based

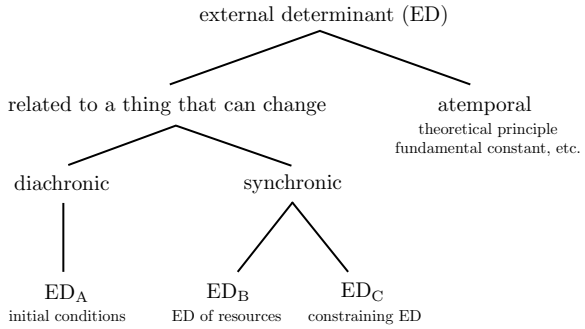


Figure 2: A taxonomy of EDs. The class under scrutiny in this paper is the one of EDs formalizing pieces of knowledge that involves DADTs. In this class, an ED may involve DADT for some duration (synchronic ED), or punctually (diachronic ED or ED_A). A synchronic ED may be an ED of resources (ED_B) or a constraining ED (ED_C). An ED of an object in class ED_A, ED_B, or ED_C can be explained by another object via a determining relation of the type $\longrightarrow\!\!\!\!\shortmid$, $\longrightarrow$, or $\rightsquigarrow$, respectively.

on the object $\mathcal{Y}$ only. Moreover, $\mathcal{X}$ and $\mathcal{Y}$ must describe some aspects of things that synchronously occur. If what $\mathcal{Y}$ describes happens after what $\mathcal{X}$ describes (such as in the case $\mathcal{Y} \longrightarrow\!\!\!\!\shortmid\ \mathcal{X}$), the explanation is not valid. Importantly, this does not mean that the state dynamics of both objects occur on the same timescale.

Four, if an object synchronically depends on several objects, those objects need to explain the EDs of $\mathcal{X}$ simultaneously. For example, suppose that $\mathcal{Y}_1 \Longrightarrow \mathcal{X}$ and $\mathcal{Y}_2 \Longrightarrow \mathcal{X}$. If the interval of time during which the state of $\mathcal{Y}_1$ is in $\mathscr{C}_{\mathcal{Y}_1 \Longrightarrow \mathcal{X}}$ does not overlap with the one during which the state of $\mathcal{Y}_2$ is in $\mathscr{C}_{\mathcal{Y}_2 \Longrightarrow \mathcal{X}}$ on a global shared timeline, then $\mathcal{Y}_1 \Longrightarrow \mathcal{X} \Longleftarrow \mathcal{Y}_2$ cannot hold. The reason is that there is no interval of time during which what $\mathcal{X}$ describes can possibly happen.

Finally, an ED of $\mathcal{X}$ that can be involved in a relation of the type $\mathcal{Y} \Longrightarrow \mathcal{X}$ is said to be a *synchronic ED*.

2.4 A partition of synchronic determinations ($\mathcal{Y} \longrightarrow \mathcal{X}$ and $\mathcal{Y} \rightsquigarrow \mathcal{X}$)

We have introduced oriented relations of determination (synchronic and diachronic) between two objects to justify an ED in one of them. Not all EDs of an object can be involved in such relations, because some of them may stem from principled theoretical considerations. Among those formalizing assumptions involving things, we have distinguished diachronic EDs from synchronic EDs. In short, a diachronic ED formalizes a UK that involves DADT punctually in time, whereas a synchronic ED formalizes a UK that involves DADT for some duration. Here, we introduce a last distinction among synchronic EDs, and summarize our categorization in Fig. 2.

2.4.1 EDs of resources and constraining EDs

This distinction is motivated by the fact that the relative invariance of some DADT does not necessarily imply the *physical* conservation of those DADT. This is indeed a conceptual difficulty that is typical when dealing with resources: some quantity can be altered, consumed, or produced while being objectivized by a synchronic ED and therefore assumed to be relatively invariant. Here, the mathematical handling of the phenomenon is seemingly at odds with the physical process it depicts. For instance, in the hydrodynamical model described in Part. 1, the angular momentum of the pump rotor is transferred to the liquid, but relatively invariant *thanks* to the electromagnetic force of the stator. In turn, a pressure gradient is effectively constant, although there is a continuous *transfer* of linear momentum between the rotor and the liquid. Moreover, any assumption involving infinite reservoirs builds on the same reasoning. The reservoir is invariantly infinite, although net fluxes of quantities can originate from it.

The rationale for distinguishing cases in which some relatively invariant DADT are conserved or transformed is the following. Recall that a synchronic ED *enables* the determination of the DU of an object. Now, suppose a synchronic ED of an object $\mathcal{X}$ refers to a *net* flux of some quantities. If this net flux were unaltered by what describes $\mathcal{X}$, it would not be *necessary* for the determination of the DU of $\mathcal{X}$. Thus, such ED ultimately refers to relatively invariant DADT that are *altered* by what $\mathcal{X}$ describes. We call this ED an *ED of resources*. In contrast, if an ED does not refer to a

net flux of some quantities, it refers to DADT that are assumed to be conserved in constructing $\mathcal{X}$. We call this ED a *constraining ED*.

Importantly, this conservation is generally approximate, and therefore depends on the assumptions posited to construct $\mathcal{X}$, notably the observational timescale. For example, wear may alter a relatively invariant DADT of $\mathcal{X}$, but if this alteration is considered negligible to construct $\mathcal{X}$, those DADT are involved in a UK formalized by a constraining ED of $\mathcal{X}$. The precision of the formulation of the UKs of $\mathcal{X}$ may here be particularly relevant. For instance, for the catalytic reaction described in Part. 1, the catalysts are altered during catalysis, but are regenerated by the chemical reaction. In turn, the *concentration* of the catalysts is assumed to be conserved, and this is formalized by a constraining ED. Note that phenomena like wear can also typically be decoupled from $\mathcal{X}$ and described by another object. The reason is that wear is a consequence of $\mathcal{X}$ but it is not intrinsic to the SAST — it is possible to assume that there is no wear as a limit case, whereas the lack of flux of a resource would not make sense.

To formalize this distinction between synchronic EDs, we build on the physical meaning of mathematical terms constituting the *continuity equation*, which grounds a large number of equations in different domains of physics. In this generic terminology, we may refer to net fluxes of conservative quantities, or net fluxes, sources, or sinks of nonconservative ones. A conservative quantity (e.g., mass-energy, momentum, charge, etc.) is a quantity that obeys an 'exact' conservation law, or equivalently, which is linked to a continuous symmetry by Noether's theorem. The flux may correspond to an increase or decrease of the conservative quantity. Non-conservative quantities are usually introduced using some kind of coarse-graining. In a control volume, one may consider that some entities are spontaneously generated (source), or destroyed (sink).

We submit the following definition.

Definition 3 (ED of resources and constraining ED) Let e be a synchronic ED of an object $\mathcal{X}$. If e refers to net flux of a conservative quantity, or a net flux, source, or sink of a non-conservative quantity, then e is an ED of resources, i.e., it formalizes a determination of some aspects of things that are altered by what $\mathcal{X}$ describes, according to the assumptions of $\mathcal{X}$. Otherwise, e is a constraining ED, i.e., it formalizes a determination of some aspects of things that are assumed to be unaltered by what $\mathcal{X}$ describes, according to the assumptions of $\mathcal{X}$.

2.4.2 Two classes of synchronic determinations

To make explicit this refinement at the level of synchronic determinations between objects, we note $\mathcal{Y} \longrightarrow \mathcal{X}$ if the ED of $\mathcal{X}$ involved in a relation $\mathcal{Y} \Longrightarrow \mathcal{X}$ is an ED of resources and $\mathcal{Y} \rightsquigarrow \mathcal{X}$ if it is an constraining ED.

As an example, consider the liquid flow described by the example in Part. 1 as the determined object $\mathcal{X}$. An object $\mathcal{Y}_1$ describes the dynamical liquid level h_t (DU) of an hydraulic dam, synchronically filling the pipe (an equivalent of the pump for the flow). The state of $\mathcal{Y}$ is in a class of altitudes of the liquid level $h = 10 \pm 1\text{m}$ (defining $\mathscr{C}_{\mathcal{Y}_1 \Longrightarrow \mathcal{X}}$), which explains the pressure gradient $\Delta P = 1\text{bar}$ in $\mathcal{X}$ for a finite amount of time. The ED ΔP refers to a flux of linear momentum, which is a conservative quantity. Therefore, $\mathcal{Y}_1 \longrightarrow \mathcal{X}$. Moreover, suppose that $\mathcal{Y}_2$ describes the long-term degradation of the pipe that canalizes the flow, e.g., using reliability physics (Birolini, 2014). $\mathscr{C}_{\mathcal{Y}_2 \Longrightarrow \mathcal{X}}$ includes states that are equivalent to the good state of repair of the pipe as far as the flow is concerned, and τ_{estim} is given by the first failure rate. The pipe is assumed to be unaltered by what $\mathcal{X}$ according to the assumptions of $\mathcal{X}$. Therefore, $\mathcal{Y}_2 \rightsquigarrow \mathcal{X}$. Synthesizing, we write the following diagram:

$$\begin{array}{ccc} \mathcal{Y}_1 & & \mathcal{Y}_2 \\ & \searrow \; \swarrow & \\ & \mathcal{X} & \end{array} \quad . \tag{3}$$

2.5 Summary

In Part. 1, we have identified elements of an object defined by a mathematical description. In this part of the paper, we have introduced two relations of determination between them (synchronic and diachronic). The pivotal remark justifying this move is that although mathematical descriptions rely on pieces of knowledge about the relative time-translational invariance of DADTs, the latter are temporally bounded. Fig. 2 provides a graphical representation of the submitted partition of the EDs of an object. In sum, we introduce three types of EDs (diachronic, synchronic-resources, and synchronic-constraining) that can be explained at least in part with three different relations of determinations ($\longrightarrow\!\!\!\!\!\!\mid$, $\longrightarrow$, or $\rightsquigarrow$) and another determining object. We now show how to combine those relations of determination.

3 Combining relations of determination

In the previous part, we have defined the diachronic ($\longrightarrow\!\!\!\!\!\dashv$) and synchronic ($\Longrightarrow$) oriented relation of determination between two objects. In this section, we show how to combine them. Note that we use the relation $\Longrightarrow$ when we do not distinguish $\longrightarrow$ from $\rightsquigarrow$.

We first introduce elementary combinations in Sect. 3.1 by scrutinizing the beginning, change, and end of a synchronic relation of determination $\mathcal{Y} \Longrightarrow \mathcal{X}$. We then provide an example in Sect. 3.2, before concluding this part.

3.1 Elementary combinations

The number of possible combinations of determining relations between objects grows rapidly with the number of relations between objects. We here focus on elementary combinations to depict the beginning, change, and end of a relation $\mathcal{Y} \Longrightarrow \mathcal{X}$, by limiting ourselves to three objects in total.

3.1.1 Beginning of ($\mathcal{Y} \Longrightarrow \mathcal{X}$)

Let us come back to the questions motivating the introduction of the relations of determination. Where does some DADT come from? What happens to some DADT during changes of the SAST? With $\mathcal{X}$ describing the SAST, we suggested the following relations: $\mathcal{Y} \Longrightarrow \mathcal{X}$, and $\mathcal{Z} \longrightarrow\!\!\!\!\!\dashv\ \mathcal{Y}$. Graphically, those two relations can be combined to describe the *beginning of* $\mathcal{Y} \Longrightarrow \mathcal{X}$:

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow\!\!\!\!\!\dashv & \mathcal{Y} \\ & & \Downarrow \\ & & \mathcal{X} \end{array}. \tag{4}$$

This diagram means that $\mathcal{Z}$ explains at least in part the initial conditions of $\mathcal{Y}$, and that a property of the latter explains at least in part a synchronic ED of $\mathcal{X}$. In other words, $\mathcal{Z}$ is *indirectly* determining for $\mathcal{X}$. Note that time resetting enables us to begin a synchronic relation $\mathcal{Y} \Longrightarrow \mathcal{X}$ using $\mathcal{Z} = \mathcal{Y}$. In turn, the relation of determination $\mathcal{Y} \Longrightarrow \mathcal{X}$ effectively begins at any time for $\mathcal{Y}$.

In Eq. 4, $\mathcal{Z}$ may describe a mechanism closing the aperture of a degrading closed box ($\mathcal{Y}$), which encloses a moving particle ($\mathcal{X}$)—in which case $\mathcal{Y} \Longrightarrow \mathcal{X}$ is $\mathcal{Y} \rightsquigarrow \mathcal{X}$.

Symmetrically, the beginning of $\mathcal{Y} \Longrightarrow \mathcal{X}$ may be due to a determination of the initial conditions of $\mathcal{X}$:

$$\begin{array}{ccc} & & \mathcal{Y} \\ & & \Downarrow \\ \mathcal{Z} & \longrightarrow\!\!\!\!\!\dashv & \mathcal{X} \end{array}. \tag{5}$$

For instance, $\mathcal{Z}$ may describe the lifting of a weight, which determines the initial altitude of a mass that experiences free fall on Earth—described by $\mathcal{X}$. The object $\mathcal{Y}$ may describe the Earth's mass variation in time.

In Part. 1, we put forward *the* object, i.e., the one describing the SAST. In combining relations of determination, the hierarchy between what objects describe holds locally but no longer globally. The method we submit in this paper puts all objects represented in a diagram on an equal footing. For instance, we could (and will) consider some DADT of $\mathcal{Y}$, and explain at least in part one of its ED by another synchronically determining object.

3.1.2 Change of ($\mathcal{Y} \Longrightarrow \mathcal{X}$)

Consider two objects $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{Y} \Longrightarrow \mathcal{X}$. What happens after $\tau_{\text{estim}}\,(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}})$, i.e. the timescale on which the state of $\mathcal{Y}$ exits the class $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ defining this relation? We here scrutinize a first scenario. In this scenario, the end of a first synchronic relation of determination gives rise to a *different* one that we *can describe with the same objects* $\mathcal{X}$ and $\mathcal{Y}$ using time resetting. Therefore, it is a *change* of $\mathcal{Y} \Longrightarrow \mathcal{X}$ which reads:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow\!\!\!\!\!\dashv & \mathcal{Y} \\ \mathscr{C}\Downarrow & & \Downarrow\mathscr{C}' \\ \mathcal{X} & \longrightarrow\!\!\!\!\!\dashv & \mathcal{X} \end{array}. \tag{6}$$

Eq. 6 shows that there is a time resetting for $\mathcal{X}$ and $\mathcal{Y}$, and that $\mathcal{Y}$ *causes a change of a class of values of a synchronic ED* of $\mathcal{X}$. After τ_{estim}, the state of $\mathcal{Y}$ enters another class of states $\mathscr{C}'_{\mathcal{Y}\Longrightarrow\mathcal{X}}$ which explains a different (class of) value of an ED of $\mathcal{X}$. This is manifested by the different

classes of states of $\mathcal{Y}$, $\mathscr{C}$ and $\mathscr{C}'$, that label the relations. In general, there is no relation between $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$ and $\tau_{\text{estim}}\left(\mathscr{C}'_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$.

Let us make three remarks. First, a change of state of $\mathcal{X}$ cannot affect its EDs, so the *cause* of the change of value of one of them is necessarily a change of state of $\mathcal{Y}$. Second, in this first scenario, we describe the new relation of synchronic determination with the same objects $\mathcal{X}$ and $\mathcal{Y}$. This is a necessary condition to say that it is a *change* of synchronic determination (and not an end). This change corresponds to a change of a class of values of an ED of $\mathcal{X}$ (which can be a singleton), typically of a *parameter* of $\mathcal{X}$. Parameters are defined by signs in mathematical descriptions, so their values are invariant but *generic*. We can use the same object with different classes of values of parameters. In contrast, if an ED grounds the definition of the state space or the structural form of the equation, it cannot change value (e.g., symmetry, conservation laws, etc.). We can introduce a shorter notation for Eq. 6, because it is not ambiguous:

$$\begin{array}{ccc} & \mathcal{Y} & \\ \mathscr{C} & \swarrow\!\!\swarrow \quad \searrow & \mathscr{C}'. \\ \mathcal{X} & \longrightarrow\!\!\shortmid & \mathcal{X} \end{array} \tag{7}$$

The time reset of $\mathcal{Y}$ is implicit in Eq. 7.

The class of value of an ED of an object $\mathcal{Y}$ can change without altering a relation $\mathcal{Y} \Longrightarrow \mathcal{X}$. For instance, consider a third object $\mathcal{Z}$, with a changing relation $\mathcal{Z} \Longrightarrow \mathcal{Y}$, and suppose that $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Z}\Longrightarrow\mathcal{Y}}\right) < \tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$. After $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Z}\Longrightarrow\mathcal{Y}}\right)$, the trajectory of $\mathcal{Y}$ *bifurcates* because of $\mathcal{Z}$, i.e., its DMS changes. Yet, this may leave $\mathcal{Y} \Longrightarrow \mathcal{X}$ invariant:

$$\begin{array}{ccc} & \mathcal{Z} & \\ \mathscr{C} & \swarrow\!\!\swarrow \quad \searrow & \mathscr{C}' \\ \mathcal{Y} & \longrightarrow\!\!\shortmid & \mathcal{Y} \\ & \searrow\!\!\swarrow & \\ & \mathcal{X} & \end{array} . \tag{8}$$

A change of $\mathcal{Z} \Longrightarrow \mathcal{Y}$ may cause an increase or decrease of the *duration* of $\mathcal{Y} \Longrightarrow \mathcal{X}$. Accordingly, a change of $\mathcal{Y} \Longrightarrow \mathcal{X}$ can ultimately be caused by a change of state of $\mathcal{Z}$ (which causes the one of $\mathcal{Y}$). In any case, the cause of a bifurcation of $\mathcal{X}$ (i.e., a change of a class of value of a parameter in this context) is external to $\mathcal{X}$.

3.1.3 End of $(\mathcal{Y} \Longrightarrow \mathcal{X})$

We here describe the *end* of a synchronic determination $\mathcal{Y} \Longrightarrow \mathcal{X}$. In this scenario, $\mathcal{Y}$ is no longer determining for $\mathcal{X}$, because its state exits $\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}$, and this does not correspond to a change of $\mathcal{Y} \Longrightarrow \mathcal{X}$, i.e., it does not correspond to a change of value of an ED of $\mathcal{X}$ (e.g., symmetry breaking). In other words, the mathematical description of $\mathcal{X}$ is no longer justified *because of a change of state of* $\mathcal{Y}$:

$$\begin{array}{ccc} \mathcal{Y} & & \\ \Downarrow & & \\ \mathcal{X} & \longrightarrow\!\!\shortmid & \mathcal{X}' \end{array} . \tag{9}$$

We may be able to describe what happens once the mathematical description of $\mathcal{X}$ is no longer justified. In this case, we describe it with *another* object $\mathcal{X}'$, so that $\mathcal{X} \longrightarrow\!\!\shortmid \mathcal{X}'$. The object $\mathcal{X}'$ may stem from a revision of some UKs of $\mathcal{X}$, or be a significantly different object. Let us consider only one scenario with an example.

In this scenario, another relation $\mathcal{Z} \Longrightarrow \mathcal{X}'$ begins. For instance, $\mathcal{X}$ describes a particle in a box whose degradation is described by $\mathcal{Y}$. The latter breaks at some point in time: boundary conditions for the particle are no longer justified. If we have pieces of knowledge regarding the room in which the box is placed, we can formalize them by boundary conditions in $\mathcal{X}'$. An object $\mathcal{Z}$ describing, for instance, the degradation of the room explains the ED involved in $\mathcal{Z} \Longrightarrow \mathcal{X}'$. Initial conditions for the particle in the room are determined by $\mathcal{X}$ when the box breaks:

$$\begin{array}{ccc} \mathcal{Y} & & \\ \Downarrow & & \\ \mathcal{X} & \longrightarrow\!\!\shortmid & \mathcal{X}'. \\ & & \Uparrow \\ \mathcal{Z} & \longrightarrow\!\!\shortmid & \mathcal{Z} \end{array} \tag{10}$$

For reasons identical to changes of $\mathcal{Y} \Longrightarrow \mathcal{X}$, the end of $\mathcal{Y} \Longrightarrow \mathcal{X}$ may happen *before* an estimated $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$ if $\mathcal{Z} \Longrightarrow \mathcal{Y}$ holds and $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Z}\Longrightarrow\mathcal{Y}}\right) < \tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$. In this case, it is because the mathematical description of $\mathcal{Y}$ is no longer justified that the one of $\mathcal{X}$ is also not: the cause for the end of $\mathcal{Y} \Longrightarrow \mathcal{X}$ is a change of state of $\mathcal{Z}$. For instance, $\mathcal{Z}$ may describe the degradation of the table on which the box enclosing the particle is placed. If the table breaks before the box does, the box falls and breaks on a timescale shorter than a timescale $\tau_{\text{estim}}\left(\mathscr{C}_{\mathcal{Y}\Longrightarrow\mathcal{X}}\right)$ given by the failure rate of the box.

Importantly, Eq. 9 does not necessarily imply that the mathematical description of $\mathcal{X}$ is no longer justified. For instance, combining Eq. 5 and Eq. 9, the altitude at $t \to \infty$ of the mass experiencing free fall ($\mathcal{X}$) can explain the initial altitude of the object describing its lifting ($\mathcal{Z}$)—or *vice versa*:

$$\begin{array}{ccc} & \mathcal{Y} & \\ \swarrow & & \searrow \\ \mathcal{X} & \longdashv & \mathcal{Z} \end{array} . \tag{11}$$

3.1.4 Summary

We have exhibited some elementary diagrams using the objects and determining relations we previously introduced. They cover the beginning, change, and end of a synchronic relation of determination between two objects $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{Y} \Longrightarrow \mathcal{X}$. The causal hierarchy between a determining and a determined object can be extended along *a chain of determination*. Along a chain, if the mathematical description of an object is no longer justified, neither are the ones that are determined by it.

3.2 Example

In this section, we provide an example that combines relations of determination, starting from the object (we note it $\mathcal{R}$) describing the concentrations of two reacting chemical species A and B (introduced in Part. 1). A UK of this object is the well-mixed assumption, which is formalized by a spatial symmetry of the chemicals' distribution (ED), and which enables the use of concentrations as DU. This UK involves the stir bar, and more precisely its rotation in the solvent (DADT). This prompt us to introduce an object $\mathcal{S}'$ describing the rotation of the stir bar.

The magnetic stir bar rotates because a magnetic stirrer applies a rotating magnetic field which exerts a torque on its magnetic moment. This compensates for the solvent's friction that hampers its rotation. Let us describe this rotation mathematically to construct $\mathcal{S}'$. We consider the stir bar as a rod with radius r and length l, with $r \ll l$. We suppose that its moment of inertia (I) and magnetization are constant. Because of the gravitational field, we consider that the altitude of its center of mass (COM) is invariant at the bottom of the container. We suppose that the container is on the magnetic stirrer, which fixes the lateral position of the COM of the stir bar. This enables us to introduce a horizontal plane and an axis of rotation perpendicular to it, that intersects at the position of the COM of the stir bar. The state of $\mathcal{S}'$ is defined by the value of two variables: the angle of the rod θ with respect to a reference, and its angular momentum L. The state space is given by $[0; 2\pi[\times \mathbb{R}$. We assume that the torque T_{ext} that the magnetic stirrer exerts is constant, because the apparatus delivers constant power. We consider that the viscosity of the solvent is fixed. Together with the constant dimensions of the rod, this enables us to introduce a constant friction coefficient λ. The dynamical equations for θ and L are given by the theory of classical mechanics. We posit that the initial conditions are (θ_0, L_0). Table 3 classifies the UKs, related EDs and the DTs they involve.

To posit that $\mathcal{S}'$ is determining for $\mathcal{R}$, a class of states of $\mathcal{S}'$ must justify at least in part the well-mixed assumption (symmetry). The criterion we posit is that the stir bar must have been rotating at a sufficient speed for about several seconds before $t = 0$, for the symmetry to be ensured at $t = 0$: $|L_t| > L^*, \forall t, -\tau \leq t \leq 0$. Here, L^* is the minimal angular momentum for a good mixing of the solution, and $\tau \sim$ seconds. We therefore add a register for L_t to the state of $\mathcal{S}'$ to define $\mathcal{S}$, whose state at some point in time is given by $(\theta_t, L_t, L_{[t-\tau,t[})$. The initial conditions of $\mathcal{S}$ are determined by $\mathcal{S}'$, which describes the transient regime when the torque is applied: $\mathcal{S}' \longdashv \mathcal{S}$. The initial conditions of $\mathcal{S}'$ correspond to a zero angular momentum: the stir bar is at rest. The EPDU of $\mathcal{S}'$ explaining the initial conditions of $\mathcal{S}$ that satisfy the well-mixed assumption of $\mathcal{R}$ is the steady-state angular momentum $L_{[-\tau,0]} = L^{\text{ss}}_{[-\tau,0]}$. The genericity of the equivalence class of trajectories $\mathscr{C}_{\mathcal{S}\Longrightarrow\mathcal{R}}$ of $\mathcal{S}$ depends on L^*. Both descriptions happen at the same time. By hypothesis (to be explained), $L^{\text{ss}} > L^*$. We have established that $\mathcal{S} \Longrightarrow \mathcal{R}$. The ED involved in this relation

UKs	EDs					
—	—	2	5	6	7	8
invariant moment of inertia	I			✓		
invariant cylindric shape	r, l			✓		
invariant dimensions				✓		
restriction rod	$r \ll l$					
invariant altitude of COM	state space		✓			✓
invariant lateral position of COM			✓		✓	
invariant viscosity of solvent	λ	✓				
invariant dimensions				✓		
invariant magnetization	T_{ext}			✓		
invariant power supply					✓	
fundamental principle of dynamics	Newton's equation					
initial conditions	(θ_0, L_0)			✓		

Table 3: Summary of the CMS for the rotating magnetic stir bar. If an ED is justified at least in part by a DT, a checkmark is placed in the corresponding column. Relevant DADTs can be found in the text. DTs are numbered as follows: 2) solvent, 5) container, 6) magnetic stir bar, 7) magnetic stirrer, 8) Earth. This table can be concatenated with Table 2.

is a spatial symmetry, and therefore does not refer to a net flux of a conservative quantity, or to a quantity that is consumed in $\mathcal{R}$, so $\mathcal{S} \rightsquigarrow \mathcal{R}$.

We now explain at least in part the initial conditions of $\mathcal{S}'$. The COM of the stir bar is at rest in the container, because it is dropped into it by an experimenter from a position (x_0, y_0, z_0) with velocity (v_0^x, v_0^y, v_0^z). We posit an object $\mathcal{F}$ describing the free fall of a point mass in a liquid. The EPDU of this object that we exhibit to posit $\mathcal{F} \longrightarrow\!\!\!\!\shortmid\ \mathcal{S}'$ is its state at $t \to \infty$, i.e., when it has reached the bottom of the container $(x_\infty, y_\infty, z_\infty = 0)$, at rest $(v_\infty^x, v_\infty^y, v_\infty^z) = 0$.

Let us widen the scope of the description. The magnetic stirrer is responsible for the external torque in $\mathcal{S}'$ and $\mathcal{S}$. It is an electrical device, in which a motor transducts electrical energy into mechanical energy. We assume that the torque on the stir bar is proportional to the one of the rotor of the DC motor, which is proportional to the electric current in the coil of wire that passes through it. We therefore introduce an object $\mathcal{W}$ describing the current flow in an electrical wire. The DU is the current, noted i, and the equation is given by Ohm's law: $i = \Delta V/R$, where ΔV (ED) is the electric potential difference (voltage), and R (ED) the resistance. $\mathcal{W}$ is a determining object both for $\mathcal{S}'$ and $\mathcal{S}$. The class $\mathscr{C}_{\mathcal{W} \Longrightarrow \mathcal{S}'} = \{i = 0\}$ explains an initial null torque $T_{\text{ext}} = 0$ for $\mathcal{S}'$, which is changed by the experimenter to a strictly positive value $T_{\text{ext}} > T_{\text{ext}}^*$ during the experiment by manipulating the potentiometer and therefore the resistance R. Another relevant class of state of $\mathcal{W}$ therefore reads $\mathscr{C}'_{\mathcal{W} \Longrightarrow \mathcal{S}'} = \mathscr{C}_{\mathcal{W} \longrightarrow \mathcal{S}} = \{i > i^*\}$. The torque is a mechanical force, which refers to a flux of a conservative quantity (mechanical energy). Thus, we have $\mathcal{W} \longrightarrow \mathcal{S}'$ and $\mathcal{W} \longrightarrow \mathcal{S}$.

We can synthesize the relations of determinations at play graphically as follows:

$$
\begin{array}{ccccccccc}
 & & & & \overset{\varphi}{\rightsquigarrow} & & & & \\
 & & & & R = \infty \quad R < R^* & & & & \\
 & & & & \mathcal{W} & \longrightarrow\!\!\!\!\shortmid & \mathcal{W} & & \\
 & & & & \mathscr{C} \downarrow & \mathscr{C}' \downarrow & \searrow & & \\
\varphi & \longrightarrow\!\!\!\!\shortmid & \mathcal{F} & \longrightarrow\!\!\!\!\shortmid & \mathcal{S}' & \longrightarrow\!\!\!\!\shortmid & \mathcal{S}' & \longrightarrow\!\!\!\!\shortmid & \mathcal{S} \\
 & & & & & & & & \vdots \\
 & & & & & & & & \varphi \longrightarrow\!\!\!\!\shortmid\ \mathcal{R}
\end{array}
\tag{12}
$$

In Eq. 12, we introduce a symbol φ which means that a human being is the determining factor, typically the experimenter. More precisely, in a determining relation with a determined object, the symbol φ signifies that the *teleological actions of a human being explain an ED of* $\mathcal{X}$. For instance, the experimenter chooses the initial number of chemicals (N_A^0, N_B^0), the positions at which the stir

bar is dropped in the container (x_0, y_0, z_0). She controls the speed of rotation of the stir bar via the potentiometer: it goes from $R \to +\infty$ to a finite value R^*[7]. This is what causes the change of value of the parameter T_{ext} in $\mathcal{S}$'. Notice that when φ is involved, the determined entities are not generic. The value of R has to be chosen *wisely* for the chemical reaction to occur as described (the class is specific); the stir bar has to be dropped *so as to* fall into the container; the number of products is typically negligible with respect to the one of reactant (i.e., $N_A^0 \gg N_B^0$). Finally, the overall relations of determination between the objects are in part *arranged* by the experimenter.

φ has a special role because it comes with a normative dimension. When its actual role does not fit the description, in a sense, the description remains correct, but the experimenter does not do what he ought to do: it is objective to call an error some initial conditions, to call a mistake a disarrangement, to say that a flux of a physical quantity is off-target, or that a lack of maintenance gives rise to a failure. φ also plays the role of a boundary term that stops the infinite regress of justification, and acts as an infinite source of specificity. We will discuss in details the symbol φ elsewhere.

The diagram in Eq. 12 highlights only a few DADTs of the objects we construct. For instance, ΔV refers to a net flux of a conservative quantity (charge). It is justified at least in part by the utility power supply to the apparatus, which is initially connected to a wire by the experimenter[8]. What compels us to objectivize some DADTs rather than others? We answer this question in the conclusion of this part.

3.3 Conclusion

The method we submit enables one to *consistently* widen the scope of study of the determinations around an object, and along three dimensions (two synchronic and one diachronic). For instance, (φ, $\mathcal{F}$, $\mathcal{S}$') belong to the *diachronic dimension of* $\mathcal{S}$. On the other hand, ($\mathcal{S}$, $\mathcal{W}$, φ) belong to the *synchronic plane of* $\mathcal{R}$ in Eq. 12.

In this setup, it is clear that the introduction of the state space of $\mathcal{R}$ is enabled by $\mathcal{S}$. Thus, a constraining ED does *not*, in general, reduce the degrees of freedom of the state of an object in a state space, which is there *a priori*. It is, in general, a reduction in the degrees of freedom due to a determination. The well-mixed assumption (formally, a symmetry) prevents unmixed configurations to be observed.

An extension of the scope of study cannot be exhaustive. Introducing an object calls for the introduction of other ones. Does the infinite regress of this approach drastically limit its practical applications? Our position is that we submit a method for scientific works taking place in a *theoretical context*. The theoretical framework and the modeler *select* the relevant *observables* in a given situation. The relevance of the method we submit is to be evaluated by its explanative power with respect to an *explanandum*, and more generally with respect to a theory.

A practical advantage of this method is that we can make use of mathematical descriptions whose explanatory power have already been demonstrated. The method aims at taking advantage of the extremely fruitful epistemology of physics. Objects are not altered, but articulated with one another to establish relations of determination.

4 Synchronically self-determining set of objects

In the previous parts, we have *i*) defined objects using mathematical descriptions, *ii*) introduced relations of determinations between those objects, and *iii*) shown how those objects and relations between them can be combined. Based on this work, we submit a formalization of a theoretical principle of biological organization: biological systems realize a *closure of constraints* (Mossio et al., 2016). First of all, we report the definition of a constraint by the authors and discuss our appropriation of this concept in a modeling context (Sect. 4.1). Then, following the line of the original article (Montévil and Mossio, 2015), we identify two different types of dependencies between constraints, and define a closure of constraining object (Sect. 4.2). This section does not discuss the biological relevance of the closure of constraints principle, but rather focuses on expressing it using mathematical descriptions. The reader unfamiliar to this framework is invited to read the original article and the subsequent works to put this last part into context.

[7] φ is not an object and therefore has no (class of) states. Therefore, we directly represent the class of values of the ED of the determined object in the diagram (see Eq. 12).

[8] We could add $\mathcal{G} \longrightarrow \mathcal{W}$ to the diagram. The state of $\mathcal{G}$ describes a fluctuating voltage with a distribution centered on $g_0 = 230$ V in the absence of random power cut. This value is in $\mathscr{C}_{\mathcal{G}\longrightarrow\mathcal{W}} = [220, 240]$ and explains at least in part ΔV. This voltage does not refer to ΔV, but for the apparatus to deliver ΔV to the wire, it must function.

4.1 Constraints

This section formalizes the main concept of the original article using objects defined by mathematical descriptions.

4.1.1 Original definition

The original article builds on a distinction between *processes* and *constraints*. For the authors, a process refers to a change involving the alteration, consumption, production and/or constitution of relevant entities. On the other hand, they define a constraint as follows:

> Given a process $A \to B$ (A becomes B), C is a constraint on $A \to B$, at a specific time scale τ , if and only if the following two conditions are fulfilled:
>
> 1. The situations $A \to B$ and $A_C \to B_C$ (i.e. $A \to B$ under the influence of C) are not, as far as B is concerned, symmetric at a timescale τ .
>
> *Note $C_{A\to B}$ those aspects of C which play a role in the above asymmetry between $A \to B$ and $A_C \to B_C$ at timescale τ.*
>
> 2. A temporal symmetry is associated with all aspects of $C_{A\to B}$ with respect to the process $A_C \to B_C$, at timescale τ.

A constraint is defined by its causal role *relatively to* a process, and a *timescale*. Its relevant aspects, as far as the process is concerned, exhibit a temporal symmetry on the timescale of the latter (i.e., by a translation of τ in time). The authors remark that a constraint is, from the appropriate viewpoint, not affected by the process it constrains. Moreover, in their words:

> In describing physical and chemical systems, constraints are usually introduced as external determinations (boundary conditions, parameters, restrictions on the configuration space, etc.), which contribute to determining the behaviour and dynamics of a system, although their existence does not depend on the dynamics on which they act.

We have introduced several concepts such as the ones of ED and unpacking to provide a framework for constraints in modeling practices. We now show how to understand the articulation between those two works.

4.1.2 Constraining objects

In general, the synchronic plane of a generic object $\mathcal{X}$ reads:

$$\begin{array}{ccccc} & & \{\gamma_\mathcal{X}\} & & \\ & & \vdots & & \\ \{\alpha_\mathcal{X}\} & \longrightarrow & \mathcal{X} & \longrightarrow & \{\beta_\mathcal{X}\}. \\ & & \vdots & & \\ & & \{\delta_\mathcal{X}\} & & \end{array} \tag{13}$$

where $\{\lambda_\mathcal{X}\}$ is a *set* of objects, $\lambda = \alpha, \beta, \gamma, \delta$. A third dimension could be introduced for diachronic determinations. Yet, the closure of constraints corresponds to a synchronic plane, so this treatment is beyond the scope of this article. We posit the following:

Definition 4 (Constraining object) An object $\mathcal{C}$ is constraining if and only if $\{\delta_\mathcal{C}\} \neq \varnothing$.

This definition needs some comments because the original definition requires exhibiting a process on which it acts and a timescale, which are absent in Def. 4.

First, the notion of *process* does not correspond to a precise kind of change as far as mathematical descriptions are concerned. It does not necessarily correspond to a change of state of an object. For example, in a steady state, the state of the object does not change, although there is some dynamics happening (e.g., the liquid flow in the pipe). A process is not necessarily the temporal evolution of the state of an object (i.e., a trajectory). For instance, Ohm's law $i = \Delta V/R$ does not ground a dynamical system, but it refers to a dynamical transport of charge. Moreover, a process includes (according to the original definition) changes of states caused by a change of values of EDs (e.g., a thermodynamical transformation from V_1 to V_2 using the perfect gas equation $P = Nk_BT/V$). Finally, the rotation of a stir bar does not involve the alteration, consumption, production and/or constitution of relevant entities, but it is a dynamical change (e.g., see example

Eq. 3.2). In sum, there is no general one-to-one mapping between what the authors call a process and a specific change in a mathematical description.

We opt for the most inclusive definition of a process, at the cost of extending the definition of constraints acting on seemingly static entities. It is all what one can physically express with a *single* object defined by a mathematical description. In general, this includes dynamical and non-dynamical systems, along with changes of (a class of) values of EDs. For instance, a box on a table can be said to experience a constrained process if the object describing it determines that its position is invariant (in time) on a two-dimensional plane. Moreover, a change of a relation of determination (e.g., describing a phase transition, see Eq. 7) is also a process in our framework, because it can be described with a single object.

Second, if a process generally corresponds to an object in our framework, a constraint is defined *relatively* to objects, in agreement with the fact that a constraint is defined relatively to a process. Constraints correspond to two distinct concepts in the framework of the current paper, which correspond to the two roles constraints play in diagrams: the causal role on the intended process and the constraint being generated by another process. The causal role corresponds to the concept of a constraining ED; that is to say, an element that shapes the intended object, is invariant, and does not correspond to a flux. However, EDs as such are merely mathematical; it is the associated UK, and, here, the associated thing and more precisely the DADT, that gives a physical or biological meaning to the constraint. The determining aspects of the determining thing (DADT) correspond exactly to $C_{A\to B}$ in the original definition. Identifying a constraint on a process first requires identifying some ED and DADT. One then objectivizes those aspects which, from a formal point of view, correspond here to a class of states of $\mathcal{C}$ having some explanatory properties.

Third, in both the frameworks of closure of constraints, a constraint C may informally correspond to a thing, here a DT. In both frameworks, this is not a sufficient identification in practice: what gives the status of constraint to a thing is the relative invariance of its determining aspects for a process. The notion of DT may be informally relevant, but always *a posteriori*, once its determining aspects have been identified. Further, the original definition underlies that a constraint plays a given causal role. Without this requirement, a stone at rest a mile away from a liquid flow would be a constraint on it. Their causal role is satisfied by construction because an ED of an object enables the determination of the DU, and therefore *causally contributes* to the process that it describes.

Four, the authors define a constraint with respect to a timescale. We have detailed the notion of relative invariance in Sect. 2.3.1 which grounds the objectivation of DADTs in mathematical descriptions. It is directly linked to the temporal symmetry of $C_{A\to B}$ mentioned by the authors in their definition. In the original article, the latter put an emphasis on timescales. Here, the notion that an ED justified by a DADT has limited validity is equally central because *there is no relative* ∞*-invariant DADT*; however, the timescale is no longer made explicit. The reason is that the constraining object being mathematized, it defines a time scale τ_{estim} for this justification (see Sect. 2.3.1). In general, physicists introduce *observational timescales* when making some mathematical description of the world (Feynman, 1972; Palmer, 1982; Beekman et al., 2019, 2.2.).

Let us note that the question of time scale for a constraint has two distinct meanings. On the one hand, there are time scales like τ_{estim} that are EPDU of constraining objects, they stem from the studied processes of degradation and maintenance. On the other hand, there are the *effective* time scales that are observed empirically. Theoretically, they depend on the given biological context, taking into account the organization and its milieu. For instance, going back to physics, estimating the mean failure rate of a box enclosing a particle has a physical meaning, but if the table on which it is placed breaks first, this estimated timescale is not the effective one (see also Sect. 3.1.3).

Another remark of the authors is that some constraints may change faster than the processes they constrain as long as there is a statistically significant number of them (e.g., catalysts on chemical reactions, collisions of fast degrees of freedom effectively performing a stochastic force on a colloid, etc.): the statistical average is the invariant property that explains the ED. In this framework, we do not consider a large number of constraints, but a single constraining object that expresses those statistical effects with the appropriate mathematical description.

Finally, as the authors point out in the original article (see quote), constraints are usually manifested by external determinations in mathematical descriptions. This is fully in line with our approach, which is based on the identification of EDs. However, in Def. 4, we posit that all EDs do not refer to constraints, but only a certain class of EDs that we label constraining EDs (ED_C in Fig. 2). The reason is that, EDs of resources formalize the causal role of DADTs that are consumed, produced, or more generally, altered *while* being relatively invariant. For instance, the pressure gradient ΔP in Eq. 3 is an ED of this type. If a liquid level in a dam explains ΔP, then the former is altered by the process, but this does not prevent it from being relatively invariant (it

can be assumed that this alteration is constantly compensated). A possible objection at this point is the following. In Table 1, we have considered that a pump explains this pressure gradient. This pump would not be altered by the liquid flow. Therefore, our reasoning is not valid in general. The response to this objection is that the explanation is too coarse-grained: a pump is a DT, and not some DADT. For a peristaltic pump, the *explanans* for the pressure gradient is the angular momentum of its rotor. It is altered by the liquid flow, because momentum is transferred to the liquid, but it is nevertheless kept invariant by the electromagnetic force of the stator.

4.1.3 Resourcing objects

In the original article, the authors emphasize the fact that processes generally occur in *non-equilibrium open systems*. This terminology belongs to the field of thermodynamics, which points to the most general physical system that exchanges energy and matter with its environment. We generalize this idea to make this statement independent of the underlying particular theory of physics.

We build on the distinction introduced in Def. 3, and define the dual object of a constraining object.

Definition 5 (Resourcing object) An object $\mathcal{A}$ is resourcing if and only if $\{\beta_{\mathcal{A}}\} \neq \varnothing$.

What is the advantage of generalizing non-equilibrium open systems to any control volume in which net fluxes, sources, or sinks are involved? It increases substantially the number of possible mathematical descriptions that can provide explanations. Consequently, we can provide more refined explanations of complex processes. For instance, in the original paper, the authors illustrate their framework with the constrained hydrodynamical process of blood flow in a vessel, generated by a pumping heart. The relevant process they identify is the transformation of the chemical energy of cardiomyocytes into the kinetic energy of the blood, i.e., a thermodynamical process— as required. In turn, the hydrodynamical nature of the process is only implicit. The generalization we submit encompasses the transfer of linear momentum (a conservative quantity), and therefore enables the explicit consideration of the hydrodynamical model in this particular case. How the chemical energy of cardiomyocytes produces some linear momentum can then be subject to explanations using intermediary mathematical descriptions.

4.2 Closure of constraining objects

In this section, following the original article on the closure of constraints, we introduce two kinds of dependencies between constraining objects: *direct* and *indirect* (Sect. 4.2.1). Then, we define the closure of constraining objects in Sect. 4.2.3, and posit rules to identify relevant closures in a generic diagram.

4.2.1 Dependencies between constraining objects

Definition 6 (Direct dependency between two constraining objects) A constraining object $\mathcal{C}_2$ directly depends on another constraining object $\mathcal{C}_1$ if and only if $\mathcal{C}_2 \in \{\delta_{\mathcal{C}_1}\}$.

Diagrammatically, a direct dependency of $\mathcal{C}_2$ on $\mathcal{C}_1$ corresponds to the identification of the following pattern:

$$\begin{array}{c} \mathcal{C}_1 \\ \vdots \\ \mathcal{C}_2, \\ \vdots \\ \mathcal{X} \end{array} \tag{14}$$

where $\mathcal{X}$ is an object which is not necessarily a constraining or resourcing object. Fig. 3 (top) details and summarizes the elements involved in a direct dependency of $\mathcal{C}_2$ on $\mathcal{C}_1$ on a particular case — where $\mathcal{X} = \mathcal{C}_3$ and $\mathcal{A}_2 \longrightarrow C_2$ also holds.

Definition 7 (Indirect dependency between two constraining objects) A constraining object $\mathcal{C}_2$ indirectly depends on another constraining object $\mathcal{C}_1$ if and only if there exists a resourcing object $\mathcal{A}$ such that $\mathcal{A} \in \{\delta_{\mathcal{C}_1}\}$ and $\mathcal{C}_2 \in \{\beta_{\mathcal{A}}\}$.

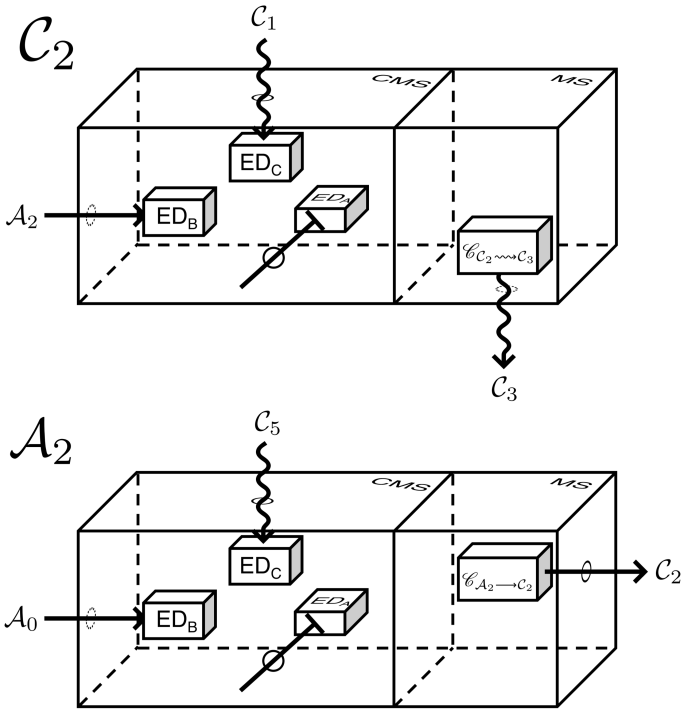


Figure 3: Sketches of dependencies between objects. The representation of objects is based on the distinction between the CMS and MS of an object introduced in Part 1 (Fig. 1 is a projection of the volume in this representation). EDs of different classes are represented in the CMSs, following the partition depicted in Fig. 2. The three dimensions (one diachronic and two synchronic) correspond to the respective relations of determinations and classes of EDs. A detailed treatment of diachronic determinations is beyond the scope of this part of the paper. Relations of determination relate to *different elements* of two different objects. A class of states of constraining objects explains at least in part constraining EDs ($\mathrm{ED_C}$), while a class of states of resourcing objects explains at least in part EDs of resources ($\mathrm{ED_B}$). Classes of states are part of the MS, because they are part of the state space. **(Top)** Sketch of a direct dependency of $\mathcal{C}_2$ on $\mathcal{C}_1$ (two constraining objects). $\mathcal{C}_2$ is constrained by $\mathcal{C}_1$ and resourced by $\mathcal{A}_2$. A class of states $\mathscr{C}_{\mathcal{C}_2 \rightsquigarrow \mathcal{C}_3}$ of $\mathcal{C}_2$ explains at least in part a constraining ED of $\mathcal{C}_3$. **(Bottom)** Sketch of a dependency of resources of $\mathcal{C}_2$ on $\mathcal{A}_2$. $\mathcal{A}_2$ is constrained by $\mathcal{C}_5$, resourced by $\mathcal{A}_0$, and provides resources to $\mathcal{C}_2$. A class of states $\mathscr{C}_{\mathcal{A}_2 \longrightarrow \mathcal{C}_2}$ of $\mathcal{A}_2$ explains at least in part an ED of resources of $\mathcal{C}_2$.

Diagrammatically, an indirect dependency of $\mathcal{C}_2$ on $\mathcal{C}_1$ corresponds to the identification of the following pattern:

$$\begin{array}{ccc} \mathcal{C}_1 & & \\ \wr & & \\ \mathcal{A} & \longrightarrow & \mathcal{C}_2, \\ & & \wr \\ & & \mathcal{X} \end{array} \tag{15}$$

where $\mathcal{A}$ is a resourcing object and $\mathcal{X}$ is an object which is not necessarily constraining or resourcing. Fig. 3 (top and bottom) details and summarizes the elements involved in an indirect dependency of $\mathcal{C}_2$ on $\mathcal{C}_5$ on a particular case — where $\mathcal{X} = \mathcal{C}_3$, $\mathcal{A} = \mathcal{A}_2$, and $\mathcal{A}_0 \longrightarrow \mathcal{A}_2$ also holds.

4.2.2 Organizational closure

The closure of constraints (Montévil and Mossio, 2015) formalizes the notion of organizational closure, which has been the focus of philosophers and scientists that put the emphasis on the fact that organisms exists first and foremost because they are organized entities (Kant, 1790; Bernard, 1865; Cannon, 1929; Schrödinger, 1944; Wiener, 1948; Von Bertalanffy, 1952; Piaget, 1967; Waddington, 1968; Maturana and Varela, 1973; Rosen, 1991; Kauffman, 2000). From a thermodynamical perspective, biological systems are open systems whose parts are maintained out-of-equilibrium. Those maintenance processes are necessarily constrained by the biological system itself if we refuse extrinsic teleology. Therefore, their parts can be considered as constraints whose determining aspects are maintained relatively invariant to enable those maintenance processes.

Now, if we observe a biological system on a sufficiently long time scale, we can show that, for a set of biological constraints to be *observable* on this time scale (i.e., maintained on this time scale), a subset C of them must realize closure. Each constraint C in C has determining aspects for at least one process maintaining another element of C, and is maintained by a process constrained by at least one other element of C. Constraints realizing closure are mutually dependent and their relative invariance depends on their determining role on processes maintaining other constraints of C. In this theoretical framework, observables of a biological system are constraints whose maintenance out-of-equilibrium is made possible by the biological system itself.

The causal circularity at play in closure is not the one of a present effect acting on its past cause—which would be highly problematic. Rather, if (P1) a cause is invariant by a translation in time (from past to future), and if (P2) the (present) effect of the past cause is a necessary condition for this symmetry, then it is objective to say that the effect contributes to determining its cause. From a physical point of view, the symmetry in (P1) corresponds to the relative invariance of the determining aspects of a constraint, and the necessary condition in (P2) corresponds to the enablement of the maintenance of the constraint by what the latter determine.

The premise (P2) implies that closure happens *during* an interval of time, because it involves a temporal symmetry (be it continuous, discrete, bounded in time, or not). The duration of this interval of time, τ_{closure}, is necessarily finite, but it may be arbitrarily long in the general case. At the end of this interval of time, the explanation for the relative invariance of the determining aspects of the constraint *must* involve their effect; for instance, invoking the thermalization timescale of the constraint is not sufficient to satisfy (P2). A closure of constraints is therefore a peculiar object, in the sense that it is irreducibly extended in time: it is not a point on a timeline because it has a thickness of τ_{closure}. In turn, in practice, a closure is to be observed on an observational timescale τ_{obs} such that $\tau_{\text{obs}} \gg \tau_{\text{closure}}$. However, on a timescale τ_{obs}, the organization can change; we will come back to this point in the general conclusion.

The circularity that arises from thermodynamical considerations can be transposed and abstracted from this particular physical theory in the present modeling framework. Justifying the assumptions of a mathematical description is a recursive operation. Logically, this infinite regress can be stopped if we encounter an object describing a thing which is stable on any biologically relevant time scale, *or* if we find a circular relation between objects. However, there is no biological thing whose determining aspects are intrinsically invariant on any biologically relevant timescale. We therefore posit that recursively justifying the assumptions involved in the partial mathematical descriptions of a biological system ultimately leads to a circular relation of synchronic determination between objects. This modeling practice unveils the biological organization of the self-maintaining biological system. Here, the observables are a subset of what the *explicitly justified* mathematical descriptions describe.

4.2.3 Definition

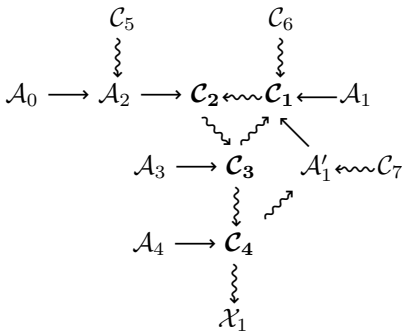
Following this line of reasoning, and given our previous work, we submit the following definition.

Definition 8 (Closure of constraining objects) Let $\mathfrak{C} = \{\mathcal{C}_i\}_i$ be a set of constraining objects. $\mathfrak{C}$ realizes a *closure of constraining objects* if the following conditions are fulfilled:

1. For all $\mathcal{C}_i \in \mathfrak{C}$, there exists $\mathcal{C}_j \in \mathfrak{C}$ such that $\mathcal{C}_i$ directly depends on $\mathcal{C}_j$.
2. For all $\mathcal{C}_i \in \mathfrak{C}$, there exists $\mathcal{C}_j \in \mathfrak{C}$ such that $\mathcal{C}_j$ directly or indirectly depends on $\mathcal{C}_i$.
3. For all $\mathcal{C}_i \in \mathfrak{C}$, for all $\mathcal{C}_j \in \mathfrak{C}$, there exists a chain of dependencies from $\mathcal{C}_i$ to $\mathcal{C}_j$.

When $\mathfrak{C}$ realizes a *closure of constraining objects*, we say that $\mathfrak{C}$ is a *closed set* or a *closure*, for short. The first two requirements are translations of the requirements of the original article. The last one corresponds to the notion of strict closure in the original article, even though its definition is slightly different. It prevents the set $\mathfrak{C}$ from being separable into two independent closed sets. Moreover, suppose that a constraining object $\mathcal{C}$ bridges two closed sets $\mathfrak{C}$ and $\mathfrak{C}'$: $\mathfrak{C} \rightsquigarrow \mathcal{C} \rightsquigarrow \mathfrak{C}'$. Here the notation means that $\mathcal{C}$ is constrained by an object of $\mathfrak{C}$ and constrains an object of $\mathfrak{C}'$. Without the last requirement, the set $\{\mathfrak{C}, \mathcal{C}, \mathfrak{C}'\}$ is closed. The last requirement therefore ensures the *mutual dependency* between the constraining objects of the set. A chain of dependencies from $\mathcal{C}_i$ to $\mathcal{C}_j$ is a sequence of dependencies (direct or indirect) between constraints that starts from $\mathcal{C}_i$ and ends at $\mathcal{C}_j$.

Consider the following diagram as an example:



(16)

We highlight in bold elements of a closed set. Only constraining objects can be part of it. $\mathcal{C}_5$, $\mathcal{C}_6$, and $\mathcal{C}_7$ do not depend on another constraint, so they cannot be part of it. They typically describe stable non-biological entities, e.g. the mass variation of the Earth, the wavelength distribution of the Sun's photons, etc. $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are directly dependent of one another, so the set $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ is closed. Moreover, $\mathcal{C}_1$ indirectly depends on $\mathcal{C}_4$ which directly depends on $\mathcal{C}_3$, so $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ is also closed.

4.2.4 Nested levels of closures

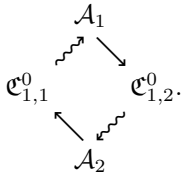
In this example, we face a situation in which we identify two closures. It is therefore desirable to unambiguously refer to one of them when scrutinizing this diagram. Identifying *nested levels of closures* is of interest in biology (Moreno and Mossio, 2015). Notably, it provides a biologically meaningful *distinction* between levels of *individuality*—i.e., of what cannot be divided. We introduce a systematic classification of a nested closed set: closures are labeled by a *level* and another closed set into which they are *nested* (except for the largest one).

Definition 9 (Identification of closures) The identification of closed sets starts by writing down all closures in a given diagram. Then, we label them according to the following recursive procedure:

1. The largest closed set is the closure of level 0, which is unique, not nested, and noted $\mathfrak{C}_{0,1}$ or $\mathfrak{C}_0$ for short in this special case.
2. Let us consider the j^{th} closure of level n nested in the i^{th} closure of level $n-1$, noted $\mathfrak{C}_{n,j}^{n-1,i}$, with $n > 0$. The largest closed strict subsets of $\mathfrak{C}_{n,j}^{n-1,i}$ that do not contain one another are of level $n+1$. We note them $\mathfrak{C}_{n+1,l}^{n,j}$.

In the previous example (Eq. 16), we identify two closures at two different levels: $\mathfrak{C}_0$ and $\mathfrak{C}_1^0$ with $\mathfrak{C}_0 = \mathfrak{C}_1^0 \cup \{\mathcal{C}_4\}$ and $\mathfrak{C}_1^0 = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$.

Now, let us consider the following example:



(17)

We identify three closures at two different levels: $\mathfrak{C}_0$, $\mathfrak{C}_{1,1}^0$ and $\mathfrak{C}_{1,2}^0$. The notation $\mathcal{A} \longrightarrow \mathfrak{C}$ indicates that $\mathcal{A}$ resources one constraint of the closed set $\mathfrak{C}$.

The two examples (Eq. 16, and Eq. 17) illustrate two roles that fluxes of resources have on biological systems. First, the maintenance of a closure depends on the relative invariance of some fluxes of resources: biological systems are organizationally closed but thermodynamically open. Resources are crucial for the observability of a closure even if they need to be incorporated (i.e., being transformed somehow into relatively stable constraints) to be part of the organization. Second, fluxes of resources may constitute indirect dependencies between distinct biological systems, thus forming lower-order closures. This would be typical of *symbiotic* systems, or *ecosystems* in general. For instance, the maintenance of $\mathfrak{C}_{1,2}^0$ indirectly depends on the one of $\mathfrak{C}_{1,1}^0$ and *vice versa* in Eq. 17.

4.3 Conclusion

In this last part, we have shown how the introduced objects and relations of determination between them could formalize a principle of organization in biology: biological systems realize a closure of constraints. To do so, we have introduced constraining and resourcing objects. In modeling practices, exhibiting a closure of constraint amounts to justifying the validity of a set of mathematical descriptions, using other mathematical descriptions. The infinite regress is stopped by closing a set of objects, or more precisely by exhibiting a *synchronically self-determining set of constraining objects*. Crucially, the justification of the validity of all mathematical descriptions in a closed set is a *necessary condition* for one of them be justified. In general, there are determining objects not part of a closure (see e.g., $\mathcal{C}_6$ and $\mathcal{A}_4$ in Eq. 16). Closures can be nested in one another at the different levels to distinguish biologically meaningful levels of *individuality*.

General conclusion

Let us first summarize the main points of this article. An object (of physics) is constructed by positing assumptions to determine some unknowns, and therefore relies on a hypothetico-deductive type of reasoning. One constructs a mathematical structure by formalizing those assumptions using external determinants (EDs), which are mathematical objects that cannot be affected by the state or a change of state of the object. An assumption usually relies on a piece of knowledge regarding the relative invariance of some determining aspects of a determining thing (DADT), so its validity is bounded in time. Consequently, so does the one of the mathematical description defining the object (see Fig. 1).

The main idea of this paper is to justify assumptions involving DADTs in mathematical descriptions, using other objects. We introduce two relations of determinations between objects. The diachronic one explains at least in part initial conditions, and the synchronic one explains at least in part EDs whose determining roles last in time. Regarding synchronic EDs, we distinguish constraining from resourcing EDs depending on whether or not they refer to net fluxes of quantities. With these distinctions, we introduce in total three distinct oriented relations between objects, with their respective symbols (see Fig. 2).

Next, we exhibit elementary combinations of those relations. The focus is set on a generic beginning, change, and end of a synchronic relation. We provide an example to illustrate how this framework could be used to describe a common laboratory experiment. This example allows us to emphasize that, in our terminology, a constraining object may restrict the degrees of freedom in a state space otherwise defined, or it may contribute at generating this space. The reduction in the degrees of freedom in a constraining relation is due to a determination. Notably, the determination of a state space is always already a reduction in the degrees of freedom regarding the possible states of the constrained object, and therefore of the observables related to it. This determination

stems from the modeler's assumptions—pieces of knowledge regarding the relative invariances of DADTs—, and from theoretical principles (see the quote of A. Einstein in the introduction). Theoretical symmetries ground the most generic state spaces in physics (Bailly and Longo, 2011), but symmetries stemming from assumptions have the same determining effect. In both cases, a symmetry breaking is therefore naturally associated to the fact that "what was thought to be a non observable turns out to be actually an observable" (Lee, 1981, p. 181).

The last part of this paper focuses on synchronic relations to specify the framework of closure of constraints in the case of mathematical descriptions (Montévil and Mossio, 2015). It introduces resourcing objects along with constraining objects, as well as direct and indirect dependencies between constraining objects (see Fig. 3). The bulk of this paper does not discriminate between the different sciences whose objects of study are defined by mathematical descriptions. In general, a method to consistently extend a scope of study (diachronically and synchronically) and/or explain assumptions in models can only be beneficial from an epistemic perspective. Apart from biology, whether this modeling method can be adopted in theoretical frameworks is an open question. In any case, this work gives a better understanding about the objects of physics, and the epistemological difficulties we build on.

The end of this article formalizes a modeling practice framed by the biological theory briefly presented in the introduction. It posits organizational closure and organizational variation as fundamental invariants. Closure is required to explain the relative invariances of DADTs in biological systems on a time scale τ_{closure}. Without the self-maintenance of the biological system, those far from equilibrium things would degrade, and the biological system would rest in pieces. In modeling parts of biological systems using mathematical descriptions, recursively justifying assumptions involving pieces of knowledge regarding the relative invariance of DADTs unveils the organizational closure of the biological system. In modeling practices, the closure of constraints becomes a closure of constraining objects. From a biological perspective, this circularity means that biological systems are capable of *self-determination*. By no means does this imply that they are not subject to external constraints, such as the gravity of Earth. Notably, despite being *organizationally closed*, biological systems are *thermodynamically open* so they are also externally determined by resourcing objects (see Eq. 16) that they can sometimes modulate by displaying agency, for example locomotion.

The organizational closure of biological systems grounds the naturalization of ubiquitous and problematic biological concepts (Moreno and Mossio, 2015). For instance, in a closed set of constraining objects, the validity of a mathematical description necessarily requires the validity of all the other ones. This implies that the mathematical description defining an object in a closed set is valid in part *because of* the determining properties of one of its *own* class of states. Explaining the membership of a thing to a system by its effect on it paves the way for an account of *functions* (Wright, 1973). It can therefore be argued that the explanatory property of the relevant class of states of a constraining object in a closed set define a *biological function* (Mossio et al., 2009; Garson, 2016). Another example is the naturalization of the concept of *individual*, and how distinguishable levels of individuality can be nested into one another, or into the concept of agency hinted at above.

As mentioned in the introduction, a single mathematical description could not be a model of (a part of) a biological system undergoing open-ended biological evolution, because it would assume the relative invariance of its causal context. In contrast, the principle of organization, by positing *closure* as a fundamental invariant of biological systems, leaves the constituting constraining objects of the closed set free to change on $\tau_{\text{obs}} \gg \tau_{\text{closure}}$— and possibly in number. If closure is an invariant, it is with respect to organizational variations. From a diachronic perspective, the principle of organization is a condition of possibility for unbounded *functional variations* and *complexifications* to occur on ontogenetic and phylogenetic timescales (two relevant τ_{obs})—as long as they can be physically explained by a closure.

Now, in light of this modeling practice, what generates those variations? In justifying the assumptions of an object on τ_{closure}, we resort to an organization of mathematical descriptions, whose validity are mutually dependent. In turn, the number of assumptions effectively involved in the validity of a single mathematical description is multiplied. On τ_{obs}, the number of possible causes for the change of a causal context of an object is therefore also multiplied. Paradoxically, a closure of constraints exhibits an invariant regime of organization on τ_{closure} which is *critically* fragile on τ_{obs} (Longo and Montévil, 2020).

The epistemological challenge entailed by this move is due to the absence of laws governing the dynamics along this diachronic dimension (Darwin, 1859; Gould, 1989; Beatty, 1995; Longo et al., 2012; Montévil, 2019b). It calls for the integration of historical conceptions in modeling practices. Without it, the latter still do not match empirical ones in biology (Montévil, 2019a).

This work contributes to the operationalization of a theory that builds on the hybridization of two epistemologies (Montévil and Mossio, 2020).

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References

Achinstein, P. (1964, October). Models, Analogies, and Theories. *Philosophy of Science 31*(4), 328–350. 1.2.2

Bailly, F. and G. Longo (2011). *Mathematics and the Natural Sciences: The Physical Singularity of Life*. Number v. 7 in Advances in Computer Science and Engineering: Texts. London : Hackensack, NJ: Imperial College Press ; Distributed by World Scientific Pub. (document), 1.1.3, 4.3

Beatty, J. (1995). The evolutionary contingency thesis. *Concepts, theories, and rationality in the biological sciences 45*, 81. 4.3

Beekman, A. J., L. Rademaker, and J. van Wezel (2019). An Introduction to Spontaneous Symmetry Breaking. *SciPost Physics Lecture Notes*, 11. (document), 4.1.2

Bernard, C. (1865). *Introduction à l'étude de La Médecine Expérimentale*. Number 2. JB Baillière. (document), 4.2.2

Bich, L., M. Mossio, and A. M. Soto (2020). Glycemia Regulation: From Feedback Loops to Organizational Closure. *Frontiers in Physiology 11*. (document)

Birolini, A. (2014). *Reliability Engineering: Theory and Practice*. Berlin, Heidelberg: Springer Berlin Heidelberg. 2.4.2

Britton, N. F. (2003). *Essential Mathematical Biology*. Springer Undergraduate Mathematics Series. London: Springer. (document)

Cannon, W. B. (1929). Organization for physiological homeostasis. *Physiological reviews 9*(3), 399–431. (document), 4.2.2

Chen, E. K. (2023). The past hypothesis and the nature of physical laws. In B. Loewer, B. Weslake, and E. B. Winsberg (Eds.), *The Probability Map of the Universe: Essays on David Albert?s* Time and Chance, pp. 204–248. Harvard University Press. 3

Darwin, C. (1859). *On the Origin of Species by Means of Natural Selection, or the Preservation of Favoured Races in the Struggle for Life*. London: John Murray. (document), 4.3

El-Hani, C. N., F. R. G. de Lima, and N. d. F. Nunes-Neto (2024). From the Organizational Theory of Ecological Functions to a New Notion of Sustainability. In M. Mossio (Ed.), *Organization in Biology*, pp. 285–328. Cham: Springer International Publishing. (document)

Feynman, R. (1967). *The Character of Physical Law*. MIT press. (document)

Feynman, R. P. (1972). *Statistical Mechanics: A Set of Lectures*. Westview Press. (document), 4.1.2

Garson, J. (2016). *A Critical Overview of Biological Functions*. Springer. 4.3

Gould, S. J. (1989). *Wonderful Life: The Burgess Shale and the Nature of History*. WW Norton & Company. 4.3

Heisenberg, W. (1971). *Physics and Beyond*. Allen & Unwin London. (document)

International Commission on Zoological Nomenclature (1999). *International Code of Zoological Nomenclature* (4th ed. ed.), Volume 1999. London: International Trust for Zoological Nomenclature, c/o Natural History Museum. 1.1.3

Johnson, K. A. and R. S. Goody (2011, October). The Original Michaelis Constant: Translation of the 1913 Michaelis–Menten Paper. *Biochemistry 50*(39), 8264–8269. 1.5

Kant, I. (1790). *Critique of Judgment*. Hackett Publishing (1987). (document), 4.2.2

Kauffman, S. A. (2000). *Investigations*. Oxford University Press. (document), 4.2.2

Kauffman, S. A. (2019). *A World Beyond Physics: The Emergence and Evolution of Life*. New York: Oxford University Press. (document)

Knuuttila, T. (2011, June). Modelling and representing: An artefactual approach to model-based representation. *Studies in History and Philosophy of Science Part A 42*(2), 262–271. 1.2.2

Landau, L. D. and E. M. Lifshitz (1987). *Fluid Mechanics: Volume 6*, Volume 6. Elsevier. 1.5

Lee, T. D. (1981). *Particle Physics and Introduction to Field Theory*. Number v. 1 in Contemporary Concepts in Physics. Chur, Schweiz ; New York: Harwood Academic Publishers. 4.3

Lillacci, G. and M. Khammash (2010, March). Parameter Estimation and Model Selection in Computational Biology. *PLOS Computational Biology 6*(3), e1000696. (document)

Longo, G. (2018, September). How Future Depends on Past and Rare Events in Systems of Life. *Foundations of Science 23*(3), 443–474. (document)

Longo, G. and M. Montévil (2014). *Perspectives on Organisms: Biological Time, Symmetries and Singularities*. Lecture Notes in Morphogenesis. Berlin, Heidelberg: Springer. (document), 1.1.3

Longo, G. and M. Montévil (2020). From physics to biology by extending criticality and symmetry breakings: An update. *Acta Europeana Systemica 9*, 77–92. 4.3

Longo, G., M. Montévil, and S. Kauffman (2012, July). No entailing laws, but enablement in the evolution of the biosphere. In *Proceedings of the 14th Annual Conference Companion on Genetic and Evolutionary Computation*, GECCO '12, New York, NY, USA, pp. 1379–1392. Association for Computing Machinery. 4.3

Longo, G. and A. M. Soto (2016, October). Why do we need theories? *Progress in biophysics and molecular biology 122*(1), 4–10. (document)

Maturana, H. and F. J. Varela (1973). *De máquinas y seres vivos; una teoría sobre la organización biológia* (1. ed ed.). Cormoran. El Mundo de la cienca, 11. Santiago de Chile]: Editorial Universitaria. (document), 4.2.2

Mohan, S. and D. Dwivedi (2018). *Gandhi and philosophy: On theological anti-politics*. Bloomsbury Publishing. 1.1.1

Montévil, M. (2018). A Primer on Mathematical Modeling in the Study of Organisms and Their Parts. In M. Bizzarri (Ed.), *Systems Biology*, Volume 1702, pp. 41–55. New York, NY: Springer New York. (document)

Montévil, M. (2019a). Measurement in biology is methodized by theory. *Biology & Philosophy 34*(3), 35. (document), 4.3

Montévil, M. (2019b, November). Possibility spaces and the notion of novelty: From music to biology. *Synthese 196*(11), 4555–4581. (document), 4.3

Montévil, M. (2020). Historicity at the heart of biology. *Theory in Biosciences 141*(2), 165–173. (document)

Montévil, M. (2022, March). Disruption of biological processes in the Anthropocene: The case of phenological mismatch. Preprint, In Review. (document)

Montévil, M. and M. Mossio (2015). Biological organisation as closure of constraints. *Journal of Theoretical Biology 372*, 179–191. (document), 2.3.1, 4, 4.2.2, 4.3

Montévil, M. and M. Mossio (2020). The Identity of Organisms in Scientific Practice: Integrating Historical and Relational Conceptions. *Frontiers in Physiology 11*, 611. (document), 4.3

Montévil, M., M. Mossio, A. Pocheville, and G. Longo (2016). Theoretical principles for biology: Variation. *Progress in Biophysics and Molecular Biology 122*(1), 36–50. (document)

Montévil, M., L. Speroni, C. Sonnenschein, and A. M. Soto (2016, October). Modeling mammary organogenesis from biological first principles: Cells and their physical constraints. *Progress in Biophysics and Molecular Biology 122*(1), 58–69. (document)

Moreno, A. and M. Mossio (2015). *Biological Autonomy: A Philosophical and Theoretical Enquiry*, Volume 12 of *History, Philosophy and Theory of the Life Sciences*. Dordrecht: Springer Netherlands. 4.2.4, 4.3

Mossio, M. (2024). Introduction: Organization as a Scientific Blind Spot. In M. Mossio (Ed.), *Organization in Biology*, pp. 1–22. Cham: Springer International Publishing. (document)

Mossio, M., M. Montévil, and G. Longo (2016, October). Theoretical principles for biology: Organization. *Progress in Biophysics and Molecular Biology 122*(1), 24–35. (document), 4

Mossio, M., C. Saborido, and A. Moreno (2009). An Organizational Account of Biological Functions. *British Journal for the Philosophy of Science 60*(4), 813–841. 4.3

Murray, J. D. (2002). *Mathematical Biology: I. An Introduction*, Volume 17 of *Interdisciplinary Applied Mathematics*. New York, NY: Springer. (document)

National Research Council of the National Academies (2009). *A new biology for the 21st century*. National Academies Press. (document)

O'Malley, M. A. and J. Dupré (2005, December). Fundamental issues in systems biology. *BioEssays 27*(12), 1270–1276. (document)

Palmer, R. (1982). Broken ergodicity. *Advances in Physics 31*(6), 669–735. (document), 4.1.2

Piaget, J. (1967). *Biologie et Connaissance: Essai Sur Les Relations Entre Les Régulations Organiques et Les Processus Cognitifs*. L'Avenir de La Science, 42. Paris: Gallimard. (document), 4.2.2

Pomian, K. and S. Amsterdamski (1990). *La Querelle Du Déterminisme: Philosophie de La Science D'Aujourd'hui*. Editions Gallimard. (document)

Reiner, M. (1964, January). The Deborah Number. *Physics Today 17*(1), 62. (document)

Rosen, R. (1991). *Life Itself: A Comprehensive Inquiry into the Nature, Origin, and Fabrication of Life*. Columbia University Press. (document), 4.2.2

Rovelli, C. (2023, December). How Oriented Causation Is Rooted into Thermodynamics. *Philosophy of Physics 1*(1), 11. 2.2.1

Sarti, A., G. Citti, and D. Piotrowski (2022). Differential Heterogenesis. In A. Sarti, G. Citti, and D. Piotrowski (Eds.), *Differential Heterogenesis: Mutant Forms, Sensitive Bodies*, pp. 55–96. Cham: Springer International Publishing. (document)

Schrödinger, E. (1944). *What Is Life?: With Mind and Matter and Autobiographical Sketches*. Canto. Cambridge: Cambridge University Press. (document), 4.2.2

Schweigert, C. (2023). Mathematical Structures in Physics. 1.2.2

Soros, L. and K. Stanley (2014, July). Identifying Necessary Conditions for Open-Ended Evolution through the Artificial Life World of Chromaria. In *ALIFE 14: The Fourteenth International Conference on the Synthesis and Simulation of Living Systems*, pp. 793–800. MIT Press. (document)

Soto, A. M., G. Longo, D. Noble, N. Perret, M. Montévil, C. Sonnenschein, M. Mossio, A. Pocheville, P.-A. Miquel, and S.-Y. Hwang (2016, October). From the century of the genome to the century of the organism: New theoretical approaches. *Progress in Biophysics and Molecular Biology, Special issue 122*(1), 1–82. (document)

Van Fraassen, B. C. (1989). *Laws and Symmetry*. Clarendon Press. (document)

Von Bertalanffy, L. (1952). *Problems of Life. An Evaluation of Modern Biological Thought*. John Wiley & Sons, New York. (document), 4.2.2

Waddington, C. H. (1968). The basic ideas of biology. In *Towards a Theoretical Biology: Prolegomena* (New York: Atheneum ed.)., pp. 1–41. (document), 4.2.2

Weyl, H. (1952). *Symmetry*, Volume 104. Princeton, New Jersey: Princeton University Press. (document)

Wiener, N. (1948). *Cybernetics: Or, Control and Communication in the Animal and the Machine*. Cambridge: MIT press. (document), 4.2.2

Wright, L. (1973). Functions. *Philosophical Review 82*(2), 139–168. 4.3

Zalamea, F. (2015). The Mathematical Description of a Generic Physical System. *Topoi 34*(2), 339–348. 1.1.3, 1.2.2