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EXTENDED BAXTER RELATIONS AND QQ -SYSTEMS FOR QUANTUM AFFINE ALGEBRAS

EDWARD FRENKEL AND DAVID HERNANDEZ

ABSTRACT. Generalized Baxter's TQ -relations and the QQ -system are systems of algebraic relations in the category \mathcal{O} of representations of the Borel subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, which we established in our earlier works [FH1, FH2]. In the present paper, we conjecture a family of analogous relations labeled by elements of the Weyl group W of \mathfrak{g} , so that the original relations correspond to the identity element. These relations are closely connected to the W -symmetry of q -characters established in [FH3]. We prove these relations for all $w \in W$ if \mathfrak{g} has rank two, and we prove the extended TQ -relations if w is a simple reflection. We also generalize our results and conjectures to the shifted quantum affine algebras.

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1. INTRODUCTION

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} and I the set of vertices of its Dynkin diagram. Let $U_q(\widehat{\mathfrak{g}})$ be the corresponding quantum affine algebra of level 0 and $U_q(\widehat{\mathfrak{b}})$ its Borel subalgebra, defined in terms of the Drinfeld-Jimbo generators of $U_q(\widehat{\mathfrak{g}})$. In [HJ], M. Jimbo and one of the authors introduced the category \mathcal{O} whose objects are representations of $U_q(\widehat{\mathfrak{b}})$ (possibly infinite-dimensional) that have a weight space decomposition with finite-dimensional weight components satisfying a certain condition (see Definition 2.10). It contains the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ (of type 1) as a subcategory.

In this paper we study various algebraic relations in the (completed) Grothendieck ring $K_0(\mathcal{O})$ of the category \mathcal{O} , and the corresponding relations on q -characters. They can be viewed as extensions of the *generalized Baxter TQ-relations* and the *QQ-system*. Here's a brief summary of these relations and their connection to the spectra of commuting quantum Hamiltonians of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}})$.

(1) The *TQ-relations* are generalizations of the celebrated Baxter *TQ-relation* of the quantum XXZ model corresponding to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$:

$$(1.1) \quad T_a = u \frac{Q_{aq^2}}{Q_a} + u^{-1} \frac{Q_{aq^{-2}}}{Q_a}.$$

It can be viewed (see e.g. [FH1]) as a relation in $K_0(\mathcal{O})$, with $T_a = [L(Y_{aq^{-1}})]$, the class of the two-dimensional representation of the Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ (the restriction of the fundamental representation of $U_q(\widehat{\mathfrak{sl}}_2)$); $Q_a = [L(\Psi_a)]$, the class of its *prefundamental representation*; and u is the class of a one-dimensional representation (these notions will be introduced below).

There are analogues of this relation in the ring $K_0(\mathcal{O})$ for an arbitrary Lie algebra \mathfrak{g} and any finite-dimensional irreducible representation V of $U_q(\widehat{\mathfrak{g}})$, which were proved in [FH1]. The starting point is the observation that the RHS of formula (1.1) can be obtained by replacing the variable Y_a appearing in the q -character of the two-dimensional representation $L(Y_{aq^{-1}})$,

$$\chi_q(L(Y_{aq^{-1}})) = Y_{aq^{-1}} + Y_{aq}^{-1},$$

with the ratio $u^{-1} \frac{Q_{aq^{-1}}}{Q_{aq}}$.

Likewise, in general one should replace each variable $Y_{i,a}, i \in I$, appearing in the q -character $\chi_q(V)$ of a finite-dimensional representation V of $U_q(\widehat{\mathfrak{g}})$ with the ratio $u_i^{-1} \frac{Q_{i,aq_i^{-1}}}{Q_{i,aq_i}}$, where $Q_{i,a} = [L(\Psi_{i,a})]$, the class of the i th prefundamental representation in $K_0(\mathcal{O})$, $q_i = q^{d_i}$ with d_i being the length of the i th simple root, and u_i is the class of a one-dimensional representation. By equating the resulting expression with $[V]$, we obtain a relation in the field of fractions of $K_0(\mathcal{O})$ (and clearing the denominators, we obtain a polynomial relation in $K_0(\mathcal{O})$). This is the *generalized Baxter TQ-relation* associated to the representation V of $U_q(\widehat{\mathfrak{g}})$ proved in [FH1] (following a conjecture of N. Reshetikhin and one of the authors in [FR]).

(2) Using the universal R -matrix of $U_q(\widehat{\mathfrak{g}})$, for any finite-dimensional representation U of $U_q(\widehat{\mathfrak{g}})$ we obtain a homomorphism from $K_0(\mathcal{O})$ to $\text{End}_{\mathbb{C}}(U)[[z]]$ sending a representation W to the corresponding transfer-matrix $t_W(z)|_U$ acting on U . Then the above TQ -relations become the relations on the joint eigenvalues of these transfer-matrices on U , which are the commuting quantum Hamiltonians of the XXZ-type integrable model associated to $U_q(\widehat{\mathfrak{g}})$.

In [FH1], we proved that if U is a tensor product of finite-dimensional irreducible representations of $U_q(\widehat{\mathfrak{g}})$, then all eigenvalues of the transfer-matrix corresponding to the i th prefundamental representation $L_{i,a}$ are polynomials in z (the degrees of these polynomials can be found from the weight of the corresponding eigenvector) up to a universal function that is determined by U . This generalizes Baxter's description of the spectrum of the commuting quantum Hamiltonians of the XXZ model (which corresponds to $\mathfrak{g} = \mathfrak{sl}_2$). For other representations, the eigenvalues will have different analytic properties (see e.g. [BLZ1, BLZ2, BLZ3]).

(3) Next, we discuss the QQ -system. To motivate it, let us rewrite the relation (1.1) for $\mathfrak{g} = \mathfrak{sl}_2$ as a second-order q^2 -difference equation satisfied by Q_a (we set $u = 1$ to simplify the formula)

$$(1.2) \quad (\mathbf{D}^2 - T_{aq^2}\mathbf{D} + 1)Q_a = 0,$$

where $\mathbf{D} \cdot f_a = f_{aq^2}$. There is a unique second solution \tilde{Q}_a of equation (1.2) such that the q^2 -difference Wronskian of Q_a and \tilde{Q}_a is equal to 1:

$$(1.3) \quad \tilde{Q}_{aq}Q_{aq^{-1}} - \tilde{Q}_{aq^{-1}}Q_{aq} = 1.$$

It is a remarkable fact that equation (1.3) can also be interpreted as a relation in $K_0(\mathcal{O})$, with $\tilde{Q} = [L(\Psi_a^{-1})]$, the class of another representation of the Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ which we denote by $L(\Psi_a^{-1})$, up to a normalization factor. This is the QQ -system for $\mathfrak{g} = \mathfrak{sl}_2$ (it consists of a single equation in this case).

For a general simple Lie algebra \mathfrak{g} , an analogous system was discovered by D. Masoero, A. Raimondo, and D. Valeri in [MRV1, MRV2] in the context of affine opers, which were introduced in [FF]. In [FH2], motivated by [MRV1, MRV2] and the Langlands duality conjecture of [FF] expressing the spectra of the quantum KdV Hamiltonians in terms of affine opers, we interpreted the QQ -system of [MRV1, MRV2] as a relation in $K_0(\mathcal{O})$. It has the form

$$(1.4) \quad \tilde{Q}_{i,aq_i} Q_{i,aq_i^{-1}} - a_i^{-1} \tilde{Q}_{i,aq_i^{-1}} Q_{i,aq_i}$$

$$= \prod_{j, C_{i,j}=-1} \mathbf{Q}_{j,a} \prod_{j, C_{i,j}=-2} \mathbf{Q}_{j,aq} \mathbf{Q}_{j,aq^{-1}} \prod_{j, C_{i,j}=-3} \mathbf{Q}_{j,aq^2} \mathbf{Q}_{j,a} \mathbf{Q}_{j,aq^{-2}},$$

where $\mathbf{Q}_{i,a}$ is the class of $L(\Psi_{i,a})$ divided by its ordinary character and $\tilde{\mathbf{Q}}_{i,a}$ is the class of another simple representation divided by its ordinary character and by the factor $(1 - a_i^{-1})$ (where a_i^{-1} is the class of a one-dimensional representation corresponding to the negative i th simple root). (Note that in [MRV1, MRV2] and [FH2], this system was referred to as the $Q\tilde{Q}$ -system.) Hence we can again interpret it as a universal system of relations in $K_0(\mathcal{O})$, from which the QQ -systems satisfied by the joint eigenvalues of the corresponding transfer-matrices follow automatically.

In [FKSZ] (see also [KZ]), the QQ -system for simply-laced \mathfrak{g} was interpreted in terms of the Miura (G, q) -opers (for non-simply laced \mathfrak{g} , the QQ -system arising from (G, q) -opers corresponds to the folded quantum integrable system of XXZ type introduced in [FHR]).

(4) There is also a similar system (with additive shifts instead of multiplication by q^n) for the XXX-type model associated to the Yangian $Y(\mathfrak{g})$, which similarly implies the QQ -systems on the joint eigenvalues of the corresponding transfer-matrices. It was first constructed in [MV] in the framework of discrete Miura opers. In the framework of representation theory of Yangians, the QQ -system for $\mathfrak{g} = \mathfrak{gl}_n$ was proved in [BFLMS] following [T]. In the case of $\mathfrak{g} = \mathfrak{so}_{2n}$, analogous results were obtained recently in [FFK, EV]. These results have been generalized (conjecturally) to all simply-laced Lie algebras in [ESV].

Remark 1.1. *In both cases (quantum affine algebras and Yangians), the QQ -system implies (under a genericity assumption) that the roots of the polynomial parts of the $Q_{i,a}$'s satisfy the so-called Bethe Ansatz equations. These equations were previously used to describe the eigenvalues of the transfer-matrices. However, we consider the QQ -system as more fundamental than Bethe Ansatz equations because the eigenvalues of the transfer-matrices automatically satisfy the QQ -system, as explained above, whereas Bethe Ansatz equations (for a general Lie algebra \mathfrak{g}) originally appeared as an educated guess from analyzing explicit formulas for eigenvectors and eigenvalues. It was indeed a good guess since we can now link solutions of these Bethe Ansatz equations to the eigenvalues of the transfer-matrices via the QQ -system (under a genericity assumption).*

(5) In [MRV1, MRV2], the QQ -system (1.4) was obtained on the “dual side” of the Langlands correspondence proposed in [FF]; namely, from solutions of differential equations defined by affine opers. In the same paper, it was shown that in fact, there is a whole family of coupled QQ -systems labeled by elements of the Weyl group W of \mathfrak{g} . Namely, there are natural variables $\mathbf{Q}_{w(\omega_i),a}$, where $w \in W, i \in I, a \in \mathbb{C}^\times$, such that the old variables $\mathbf{Q}_{i,a}$ and $\tilde{\mathbf{Q}}_{i,a}$ are identified with $\mathbf{Q}_{\omega_i,a}$ (i.e. $w = e$) and $\mathbf{Q}_{s_i(\omega_i),a}$ (i.e. $w = s_i$, the i th simple reflection from W), respectively. These variables satisfy the equations

$$\begin{aligned} & \mathbf{Q}_{ws_i(\omega_i),aq_i} \mathbf{Q}_{w(\omega_i),aq_i^{-1}} - \mathbf{Q}_{ws_i(\omega_i),aq_i^{-1}} \mathbf{Q}_{w(\omega_i),aq_i} \\ = & \prod_{j, C_{i,j}=-1} \mathbf{Q}_{w(\omega_j),a} \prod_{j, C_{i,j}=-2} \mathbf{Q}_{w(\omega_j),aq} \mathbf{Q}_{w(\omega_j),aq^{-1}} \prod_{j, C_{i,j}=-3} \mathbf{Q}_{w(\omega_j),aq^2} \mathbf{Q}_{w(\omega_j),a} \mathbf{Q}_{w(\omega_j),aq^{-2}}, \end{aligned}$$

(here we have suppressed some coefficients analogous to the a_i^{-1} appearing in relations (1.4); the precise form of these relations can be found in Theorem 5.6 below).

These equations also appear naturally from the Miura (G, q) -opers [FKSZ], and their additive analogues appear from the discrete Miura opers [MV]. The latter have been interpreted as relations on representations of Yangians in [T, BFLMS, FFK, EV, ESV].

Thus, one obtains a family of QQ-systems labeled by $w \in W$, with the original equations corresponding to $w = e$ (note that these equations are coupled to each other in a non-trivial way). It is called the *extended QQ-system*.

It is natural to ask whether the equations of the extended QQ-system can be interpreted as relations in $K_0(\mathcal{O})$. In particular, what representations from the category \mathcal{O} correspond to the variables $\mathbf{Q}_{w(\omega_i),a}$ with $w \neq 1$? This was our first motivating question for the present paper.

Our second motivating question was: are there w -analogues of the TQ -systems with $\mathbf{Q}_{i,a}$ replaced by $\mathbf{Q}_{w(\omega_i),a}$, $w \in W$?

In this paper we tackle these, and some other related, questions. We conjecture that the answer to each of these two questions is affirmative. In particular, we conjecture (and prove for \mathfrak{g} of rank up to 2) that if we set $\mathbf{Q}_{w(\omega_i),a} = [L(\Psi_{w(\omega_i),a})]$, the class of a particular representation from the category \mathcal{O} defined below, times a constant normalization factor, then these $\mathbf{Q}_{w(\omega_i),a}$'s will indeed satisfy the extended QQ-system (see Conjecture 5.9).

(6) While investigating these questions, we asked whether there is a natural *action of the Weyl group* on the ring where the q -characters of representations from \mathcal{O} take values, such that when we apply $w \in W$ to the (renormalized) q -character of $L(\Psi_{i,a})$, we obtain the q -character of $L(\Psi_{w(\omega_i),a})$. In our previous paper [FH3] we showed that one can indeed define such an action.

More precisely, we defined an action of the Weyl group W of \mathfrak{g} on a direct sum of completions of the ring $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ (this is the ring where the q -characters of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ take values), such that the subring of invariants of the action of W in \mathcal{Y} is spanned by the q -characters of finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ (see Theorem 3.13 below). Thus, the Grothendieck ring of the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ is realized as the ring of invariants of an action of the Weyl group, much like the Grothendieck ring of the category of finite-dimensional representations of the Lie algebra \mathfrak{g} .

(7) The Weyl group action that we constructed in [FH3] has led us to conjecture a *q -character formula* for the representations $L(\Psi_{w(\omega_i),a})$ for all $w \in W$ (Conjecture 4.8). Namely, $\chi_q(L(\Psi_{w(\omega_i),a}))$ satisfies the q_i^2 -difference equation

$$(1.5) \quad [w(\omega_i)] \frac{\chi_q(L(\Psi_{w(\omega_i),aq_i^{-1}}))}{\chi_q(L(\Psi_{w(\omega_i),aq_i}))} = \Theta_w(Y_{i,a}),$$

where Θ_w is the action of the element $w \in W$ that we have constructed in [FH3] and $[w(\omega_i)]$ corresponds to an invertible one-dimensional representation defined below. This fixes $\chi_q(L(\Psi_{w(\omega_i),a}))$ up to a constant (i.e. spectral parameter-independent) factor.

We have previously proved formula (1.5) for $w = e$ in [FH1] (see formula (3.5) below) and $w = s_i$ in [FH2, FHR] (see Theorem 3.11 below). In the present paper we prove it for all $w \in W$ for Lie algebras \mathfrak{g} of rank 2.

In Theorem 4.5 we compute $\Theta_w(Y_{i,a})$, which enables us to pin down the q -character of $L(\Psi_{w(\omega_i),a})$ up to a constant factor. The W -invariance of the q -characters of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ proved in [FH3] then gives us (modulo Conjecture 4.8) the sought-after extended Baxter TQ -relations in $K_0(\mathcal{O})$ (see Conjecture 3.9 and Corollary 4.11). Namely, let V be an irreducible finite-dimensional representation of $U_q(\widehat{\mathfrak{g}})$. Equating $[V]$ with the expression obtained by replacing in the q -character $\chi_q(V)$ every $Y_{i,a}, i \in I$, with

$$Y_{i,a} \mapsto [w(\omega_i)] \frac{[L(\Psi_{w(\omega_i),aq_i^{-1}})]}{[L(\Psi_{w(\omega_i),aq_i})]}$$

(and clearing the denominators), we obtain the extended Baxter TQ -relation corresponding to V and $w \in W$.

(8) The third motivating question for this paper was whether the eigenvalues of the transfer-matrices corresponding to the representations $L(\Psi_{w(\omega_i),a}), w \in W$, acting on a tensor product U of finite-dimensional irreducible representations of $U_q(\widehat{\mathfrak{g}})$ are *polynomials* up to an overall factor determined by U . We will address this question in a follow-up paper [FH4].

(9) Our results have also led us (in Section 7) to formulate conjectural q -character formulas for certain simple representations of *shifted quantum affine algebras* introduced in [FT, He]. We prove that they are satisfied for $w = e$ and $w = s_i$ and arbitrary $w \in W$ for Lie algebras of rank 2. We expect analogous formulas to hold for representations of shifted Yangians which have recently been studied in [BrK, KWWY, NW]; in particular, in the context of the quantization of transverse slices to Schubert varieties in the affine Grassmannian and Coulomb branches of quiver gauge theories.

The paper is organized as follows. In Section 2 we recall some basic facts about quantum affine algebras $U_q(\widehat{\mathfrak{g}})$ and their representations. We also introduce the category \mathcal{O} of representations of the Borel subalgebra $U_q(\widehat{\mathfrak{b}})$ of $U_q(\widehat{\mathfrak{g}})$ defined in [HJ]. In Section 3, we conjecture the extended TQ -relations (Conjecture 3.9) which involve classes of simple representations from the category \mathcal{O} whose highest ℓ -weights are obtained using a generalization of Chari's braid group action [C]. We then show how the Weyl group action introduced in [FH3] was motivated by this picture. In Section 4, we use the Weyl group action to prove Conjectures 3.9 and 3.10 when $w \in W$ is a simple reflection. We give a conjectural q -character formula for $L(\Psi_{w(\omega_i),a})$ in Conjecture 4.8 and show that it implies the extended Baxter TQ -relations (see Corollary 4.11). In Section 5, we formulate the extended QQ -system in $K_0(\mathcal{O})$ (Conjecture 5.9). In Section 6 we introduce the shifted quantum affine algebras and formulate analogues of our conjectures for representations of these algebras. In Section 7, we prove our conjectures for Lie algebras of rank 2.

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2. BACKGROUND ON QUANTUM AFFINE ALGEBRAS

In this section we collect definitions and results on quantum affine algebras and their representations that we will need in the main body of the paper. We refer the reader for more details to the textbook [CP] and the more recent surveys on this topic [CH, L]. We also discuss representations of the Borel subalgebra of a quantum affine algebra (see [HJ, FH1] for more details).

2.1. Lie algebra and Weyl group. Let \mathfrak{g} be a finite-dimensional simple Lie algebra of rank n and \mathfrak{h} its Cartan subalgebra. We will use the notation and conventions of [Kac]. We denote by $C = (C_{i,j})_{i,j \in I}$ its Cartan matrix, where $I = \{1, \dots, n\}$. We denote by h^\vee (resp. d) the dual Coxeter number (resp. the lacing number) of \mathfrak{g} .

Let $\{\alpha_i\}_{i \in I}$, $\{\alpha_i^\vee\}_{i \in I}$, $\{\omega_i\}_{i \in I}$, $\{\omega_i^\vee\}_{i \in I}$ be the sets of simple roots, simple coroots, fundamental weights, and fundamental coweights, respectively. We set

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \quad Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i, \quad P = \bigoplus_{i \in I} \mathbb{Z}\omega_i, \quad P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\omega_i, \quad P^\vee = \bigoplus_{i \in I} \mathbb{Z}\omega_i^\vee,$$

and let $P_{\mathbb{Q}} := P \otimes \mathbb{Q}$, $P_{\mathbb{Q}}^\vee := P^\vee \otimes \mathbb{Q}$.

We have the set of roots $\Delta \subset Q$, which is decomposed as $\Delta = \Delta_+ \sqcup \Delta_-$ where $\Delta_\pm = \pm(\Delta \cap Q^+)$. Let $D = \text{diag}(d_1, \dots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and d_i 's are relatively prime positive integers.

We will use the partial ordering on $P_{\mathbb{Q}}$ defined by the rule $\omega \leq \omega'$ if and only if $\omega' - \omega \in Q^+$. The vector space of invariant symmetric bilinear forms on \mathfrak{g} is known to be one-dimensional. There is a unique such form κ_0 with the property that the bilinear form $(\cdot, \cdot) := (\kappa_0|_{\mathfrak{h}})^{-1}(\cdot, \cdot)$ on \mathfrak{h}^* dual to the restriction $\kappa_0|_{\mathfrak{h}}$ of κ_0 to $\mathfrak{h} \subset \mathfrak{g}$ satisfies $(\alpha_i, \alpha_i) = 2d_i$ (κ_0 is the form introduced in [Kac] divided by d). We have for $i, j \in I$,

$$(\alpha_i, \alpha_j) = d_i C_{i,j} = d_j C_{j,i} \quad \text{and} \quad (\alpha_i, \omega_j) = d_j \delta_{i,j}.$$

Hence the bases $(\alpha_i)_{i \in I}$, $(\omega_i/d_i)_{i \in I}$ are dual to each other with respect to this form and $\alpha_i = \sum_{j \in I} C_{j,i} \omega_j$.

Let W be the Weyl group of \mathfrak{g} , generated by the simple reflections $s_i, i \in I$. For $w \in W$, we denote by $l(w)$ its length. The group W acts on P , with the action determined by the formula

$$s_k(\omega_j) := \omega_j - \delta_{k,j} \alpha_j, \quad j, k \in I.$$

The form (\cdot, \cdot) is invariant under this action, i.e. for any $\lambda, \mu \in P$, we have

$$(2.1) \quad (w\lambda, w\mu) = (\lambda, \mu) \text{ for any } w \in W.$$

The Weyl group invariance implies that for $i, k \in I$ and $w \in W$ we have

$$(w\omega_i, \alpha_k) = (\omega_i, w^{-1}\alpha_k)$$

which is equal to d_i multiplied by the multiplicity of α_i in the decomposition of the root $w^{-1}\alpha_k$. This root is positive or negative, hence the sign of $(w\omega_i, \alpha_k)$ does not depend on $i \in I$.

Now consider a reduced decomposition $w = s_{i_1} \cdots s_{i_N}$ of a Weyl group element w . Then $w^{-1} = s_{i_N} \cdots s_{i_1}$ is a reduced decomposition of w^{-1} and $s_{i_N} \cdots s_{i_L}$ is also a reduced decomposition for all $1 \leq L \leq N$.

Lemma 2.1. *The multiplicity of ω_{i_M} in*

$$(s_{i_{M+1}} \cdots s_{i_N})(\omega_i)$$

is non-negative for any $i \in I$.

Proof. This follows from the formula (see [Kac, Lemma 3.11])

$$(s_{i_N} \cdots s_{i_M})(\alpha_{i_M}) \in \Delta_-.$$

□

The following lemma is also well-known. Let W_i be the subgroup of W generated by s_j , $j \neq i$.

Lemma 2.2. *Let $w \in W$ and $i \in I$. Then*

- (1) $w(\alpha_i) \in \Delta_+$ if and only if $l(ws_i) = l(w) + 1$.
- (2) $w(\omega_i) = \omega_i$ if and only if $w \in W_i$, so the W -orbit of ω_i is in bijection with W/W_i .
- (3) The W -orbits $\{w(\omega_i)\}_{w \in W}$ of the fundamental weights are disjoint.

Proof. Part (1) follows, for example, from [Hu, Lemma 1.6, Corollary 1.7].

Part (2) is a particular case of [Hu, Proposition 1.15]. Indeed $(\omega_i, \alpha_i) > 0$ and

$$\omega_i \in C = \{\lambda \mid (\lambda, \alpha_j) = 0 \text{ for any } j \neq i\}.$$

This proposition in the case of the subset $\{1, 2, \dots, n\} \setminus \{i\} \subset I$ implies the result.

To prove part (3), note that $w(\omega_i) = \omega_j$ implies that $\omega_j \preceq \omega_i$ as ω_i is the highest weight of a fundamental representation of \mathfrak{g} . □

Denote the W -orbit of ω_i in P by \mathcal{P}_{ω_i} . This is the set of extremal weights of the i th fundamental representation \mathcal{V}_{ω_i} of \mathfrak{g} . Each of them has multiplicity one. If these are all the weights of \mathcal{V}_{ω_i} , then \mathcal{V}_{ω_i} is called minuscule (for example, this is so for all i if \mathfrak{g} is of type A).

Denote by w_0 the longest element of the Weyl group. We have the bar involution of the set I defined by $w_0(\alpha_i) = -\alpha_{\bar{i}}$, $i \in I$. Note that this implies $w_0(\omega_i) = -\omega_{\bar{i}}$, $i \in I$. This is the lowest weight of the fundamental representation \mathcal{V}_{ω_i} .

2.2. Quantum affine algebras and their Borel subalgebras. Let $\hat{\mathfrak{g}}$ be the untwisted affine Kac-Moody Lie algebra associated to \mathfrak{g} and $(C_{i,j})_{0 \leq i,j \leq n}$ the corresponding indecomposable Cartan matrix. Let a_0, \dots, a_n be the Kac labels defined in [Kac], pp. 55-56. We have $a_0 = 1$ and we set $\alpha_0 = -(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$.

Throughout this paper, we fix a non-zero complex number q which is not a root of unity. We also choose $h \in \mathbb{C}$ such that $q = e^h$ and define $q^r \in \mathbb{C}^\times$ for any $r \in \mathbb{Q}$ as e^{hr} . Since q is not a root of unity, the map $\mathbb{Q} \rightarrow \mathbb{C}^\times$ sending $r \mapsto q^r$ is injective.

We will use the standard symbols for q -integers

$$[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \prod_{j=1}^m [j]_z, \quad \begin{bmatrix} s \\ r \end{bmatrix}_z = \frac{[s]_z!}{[r]_z! [s-r]_z!}.$$

We set $q_i = q^{d_i}$.

The quantum affine algebra of level 0 (also known as the quantum loop algebra) $U_q(\widehat{\mathfrak{g}})$ is the \mathbb{C} -algebra with the generators $e_i, f_i, k_i^{\pm 1}$ ($0 \leq i \leq n$) and the following relations for $0 \leq i, j \leq n$:

$$\begin{aligned} k_i k_j &= k_j k_i, & k_0^{a_0} k_1^{a_1} \cdots k_n^{a_n} &= 1, & k_i e_j k_i^{-1} &= q_i^{C_{i,j}} e_j, & k_i f_j k_i^{-1} &= q_i^{-C_{i,j}} f_j, \\ [e_i, f_j] &= \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-C_{i,j}} (-1)^r e_i^{(1-C_{i,j}-r)} e_j e_i^{(r)} &= 0 \quad (i \neq j), & \sum_{r=0}^{1-C_{i,j}} (-1)^r f_i^{(1-C_{i,j}-r)} f_j f_i^{(r)} &= 0 \quad (i \neq j). \end{aligned}$$

Here $x_i^{(r)} = x_i^r / [r]_{q_i}!$ ($x_i = e_i, f_i$). The algebra $U_q(\widehat{\mathfrak{g}})$ has a Hopf algebra structure such that

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i \quad \text{for } 0 \leq i \leq n.$$

The algebra $U_q(\widehat{\mathfrak{g}})$ can also be presented in terms of the Drinfeld generators (see [Dr, Be] for details)

$$(2.2) \quad x_{i,r}^{\pm} \quad (i \in I, r \in \mathbb{Z}), \quad \phi_{i,\pm m}^{\pm} \quad (i \in I, m \in \mathbb{Z}), \quad k_i^{\pm 1} \quad (i \in I).$$

We will use the generating series ($i \in I$)

$$(2.3) \quad \phi_i^{\pm}(z) = \sum_{m \in \mathbb{Z}} \phi_{i,m}^{\pm} z^m = k_i^{\pm 1} \exp \left(\pm (q_i - q_i^{-1}) \sum_{r > 0} h_{i,\pm r} z^{\pm r} \right).$$

In particular, $\phi_{i,\pm m}^{\pm} = 0$ for $m < 0$, $i \in I$, and $\phi_{i,0}^{\pm} = k_i^{\pm 1}$, so $\phi_{i,0}^+ \phi_{i,0}^- = 1$.

The algebra $U_q(\widehat{\mathfrak{g}})$ has a \mathbb{Z} -grading defined by $\deg(e_i) = \deg(f_i) = \deg(k_i^{\pm 1}) = 0$ for $i \in I$ and $\deg(e_0) = -\deg(f_0) = 1$. It satisfies $\deg(x_{i,m}^{\pm}) = \deg(\phi_{i,m}^{\pm}) = m$ for $i \in I, m \in \mathbb{Z}$. For $a \in \mathbb{C}^{\times}$, there is an automorphism

$$\tau_a : U_q(\widehat{\mathfrak{g}}) \rightarrow U_q(\widehat{\mathfrak{g}})$$

such that $\tau_a(g) = a^m g$ for any element g of degree $m \in \mathbb{Z}$.

Definition 2.3. *The Borel subalgebra $U_q(\widehat{\mathfrak{b}})$ is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by e_i and $k_i^{\pm 1}$ with $0 \leq i \leq n$.*

This is a Hopf subalgebra of $U_q(\widehat{\mathfrak{g}})$. The algebra $U_q(\widehat{\mathfrak{b}})$ contains the Drinfeld generators $x_{i,m}^+, x_{i,r}^-, k_i^{\pm 1}, \phi_{i,r}^+$ with $i \in I, m \geq 0$ and $r > 0$.

Remark 2.4. *For $\mathfrak{g} = \mathfrak{sl}_2$, the algebra $U_q(\widehat{\mathfrak{b}})$ is generated by the Drinfeld generators, but for other Lie algebras \mathfrak{g} this is not the case (see [HJ, Section 2.3] and [Be]).*

2.3. Category \mathcal{O} for representations of Borel subalgebras.

Definition 2.5. *We define $\mathfrak{t}^{\times} := \text{Maps}(I, q^{\mathbb{Q}}) = (q^{\mathbb{Q}})^I$, where $q^{\mathbb{Q}} := \{e^{rh}, r \in \mathbb{Q}\}$, as a group with pointwise multiplication.*

We define a group homomorphism $\bar{\cdot} : P_{\mathbb{Q}} \rightarrow \mathfrak{t}^{\times}$ by setting

$$\overline{r\omega_i}(j) := q_i^{r\delta_{i,j}} = e^{rd_i h \delta_{i,j}}, \quad i, j \in I, r \in \mathbb{Q}.$$

Since we have assumed that q is not a root of unity, this is actually an isomorphism, with the inverse map $\overline{\varpi} : \mathfrak{t}^\times \rightarrow P_{\mathbb{Q}}$ defined by the formula

$$\psi_i = e^{d_i h(\overline{\varpi}(\boldsymbol{\psi}))(\alpha_i^\vee)}, \quad \forall \boldsymbol{\psi} = (\psi_i)_{i \in I} \in \mathfrak{t}^\times.$$

The partial ordering on P induces a partial ordering on \mathfrak{t}^\times .

For a $U_q(\widehat{\mathfrak{b}})$ -module V and $\omega \in \mathfrak{t}^\times$, we set

$$(2.4) \quad V_\omega := \{v \in V \mid k_i v = \omega(i)v \ (\forall i \in I)\},$$

and call it the weight subspace of weight ω . For any $i \in I$, $r \in \mathbb{Z}$ we have $\phi_{i,r}^\pm(V_\omega) \subset V_\omega$ and $x_{i,r}^\pm(V_\omega) \subset V_{\omega \alpha_i^{\pm 1}}$. We say that V is Cartan-diagonalizable if $V = \bigoplus_{\omega \in \mathfrak{t}^\times} V_\omega$.

Definition 2.6. A series $\boldsymbol{\Psi} = (\Psi_{i,m})_{i \in I, m \geq 0}$ of complex numbers such that $\Psi_{i,0} \in q^{\mathbb{Q}}$ for all $i \in I$ is called an ℓ -weight.

We denote by \mathfrak{t}_ℓ^\times the set of ℓ -weights. We have a natural projection $j : \mathfrak{t}_\ell^\times \rightarrow \mathfrak{t}^\times$ sending $(\Psi_{i,m})_{i \in I, m \geq 0}$ to $(\Psi_{i,0})_{i \in I}$ and a natural inclusion $\iota : \mathfrak{t}^\times \rightarrow \mathfrak{t}_\ell^\times$.

We will identify the collection $(\Psi_{i,m})_{m \geq 0}$ with its generating series and use this to represent every ℓ -weight as an I -tuple of formal power series in z :

$$\boldsymbol{\Psi} = (\Psi_i(z))_{i \in I}, \quad \Psi_i(z) := \sum_{m \geq 0} \Psi_{i,m} z^m.$$

Since each $\Psi_i(z)$ is an invertible formal power series, \mathfrak{t}_ℓ^\times has a natural group structure. We also have a surjective homomorphism of groups $\varpi : \mathfrak{t}_\ell^\times \rightarrow P_{\mathbb{Q}}$ defined as the composition of the projection $j : \mathfrak{t}_\ell^\times \rightarrow \mathfrak{t}^\times$ and the map $\overline{\varpi} : \mathfrak{t}^\times \rightarrow P_{\mathbb{Q}}$ defined above.

Definition 2.7. A $U_q(\widehat{\mathfrak{b}})$ -module V is said to be of highest ℓ -weight $\boldsymbol{\Psi} \in \mathfrak{t}_\ell^\times$ if there is $v \in V$ such that $V = U_q(\widehat{\mathfrak{b}})v$ and the following equations hold:

$$e_i v = 0 \quad (i \in I), \quad \phi_{i,m}^+ v = \Psi_{i,m} v \quad (i \in I, m \geq 0).$$

The ℓ -weight $\boldsymbol{\Psi} \in \mathfrak{t}_\ell^\times$ is uniquely determined by V . It is called the highest ℓ -weight of V . The vector v is said to be a highest ℓ -weight vector of V .

Lemma 2.8. [HJ, Section 3.2] For any $\boldsymbol{\Psi} \in \mathfrak{t}_\ell^\times$, there exists a simple highest ℓ -weight module $L(\boldsymbol{\Psi})$ over $U_q(\widehat{\mathfrak{b}})$ of highest ℓ -weight $\boldsymbol{\Psi}$. This module is unique up to isomorphism.

Definition 2.9. [HJ] For $i \in I$ and $a \in \mathbb{C}^\times$, let

$$(2.5) \quad L_{i,a}^\pm = L(\boldsymbol{\Psi}_{i,a}) \quad \text{where} \quad (\boldsymbol{\Psi}_{i,a})_j(z) = \begin{cases} (1 - za)^{\pm 1} & (j = i), \\ 1 & (j \neq i). \end{cases}$$

We call $L_{i,a}^+$ (resp. $L_{i,a}^-$) a positive (resp. negative) prefundamental representation.

For $\lambda \in P_{\mathbb{Q}}$, we define the 1-dimensional representation

$$[\lambda] = L(\boldsymbol{\Psi}_\lambda) \quad \text{with} \quad \boldsymbol{\Psi}_\lambda := \iota(\overline{\lambda}),$$

where $\iota : \mathfrak{t}^\times \rightarrow \mathfrak{t}_\ell^\times$ is the inclusion.

For $a \in \mathbb{C}^\times$, the subalgebra $U_q(\widehat{\mathfrak{b}})$ is stable under τ_a . Denote its restriction to $U_q(\widehat{\mathfrak{b}})$ by the same letter. Then the pullbacks of the $U_q(\widehat{\mathfrak{b}})$ -modules $L_{i,b}^\pm$ by τ_a is $L_{i,ab}^\pm$.

For $\lambda \in \mathfrak{t}^\times$, let $D(\lambda) = \{\omega \in \mathfrak{t}^\times \mid \omega \leq \lambda\}$. Consider the category \mathcal{O} introduced in [HJ].

Definition 2.10. *A $U_q(\widehat{\mathfrak{b}})$ -module V is said to be in category \mathcal{O} if*

- i) V is Cartan-diagonalizable;*
- ii) for all $\omega \in \mathfrak{t}^\times$ we have $\dim(V_\omega) < \infty$;*
- iii) there exist a finite number of elements $\lambda_1, \dots, \lambda_s \in \mathfrak{t}^\times$ such that the weights of V are in $\bigcup_{j=1, \dots, s} D(\lambda_j)$.*

The next result follows directly from the definition.

Lemma 2.11. *The category \mathcal{O} is a monoidal category.*

Definition 2.12. *We define \mathfrak{r} as the subgroup of \mathfrak{t}_ℓ^\times consisting of Ψ such that $\Psi_i(z)$ is an expansion in positive powers of z of a rational function in z for any $i \in I$.*

Theorem 2.13. [HJ] *Let $\Psi \in \mathfrak{t}_\ell^\times$. The simple module $L(\Psi)$ is in \mathcal{O} if and only if $\Psi \in \mathfrak{r}$.*

Definition 2.14. *We define \mathcal{E} as the additive group of maps $c : P_{\mathbb{Q}} \rightarrow \mathbb{Z}$ whose support*

$$\text{supp}(c) = \{\omega \in P_{\mathbb{Q}}, c(\omega) \neq 0\}$$

is contained in a finite union of sets of the form $D(\mu)$. For $\omega \in P_{\mathbb{Q}}$, we define $[\omega] = \delta_{\omega, \cdot} \in \mathcal{E}$.

For V in the category \mathcal{O} we define the character of V to be the following element of \mathcal{E}

$$(2.6) \quad \chi(V) = \sum_{\omega \in \mathfrak{t}^\times} \dim(V_\omega)[\omega].$$

In the same way as in the case of the ordinary category \mathcal{O} of \mathfrak{g} -modules (see [Kac, Section 9.6]), one shows that the multiplicity of any simple module in a given module from the category \mathcal{O} is well-defined. One uses this fact in the following definition [HL, Section 3.2].

Definition 2.15. *We define $K_0(\mathcal{O})$ as the completion of the Grothendieck ring of the category \mathcal{O} whose elements are sums (possibly infinite)*

$$\sum_{\Psi \in \mathfrak{r}} \lambda_\Psi [L(\Psi)],$$

such that the $\lambda_\Psi \in \mathbb{Z}$ satisfy

$$\sum_{\Psi \in \mathfrak{r}, \omega \in P_{\mathbb{Q}}} |\lambda_\Psi| \dim((L(\Psi))_\omega)[\omega] \in \mathcal{E}.$$

We naturally identify \mathcal{E} with the Grothendieck ring of the subcategory of the category \mathcal{O} whose objects are representations with constant ℓ -weights. Its simple objects are the $[\omega]$, $\omega \in P_{\mathbb{Q}}$. Thus as in [Kac, Section 9.7] we will regard elements of \mathcal{E} as formal sums

$$c = \sum_{\omega \in \text{Supp}(c)} c(\omega)[\omega].$$

The multiplication is given by $[\omega][\omega'] = [\omega + \omega']$ and \mathcal{E} is regarded as a subring of $K_0(\mathcal{O})$.

The character defines a ring homomorphism $\chi : K_0(\mathcal{O}) \rightarrow \mathcal{E}$ which is not injective.

2.4. Monomials and finite-dimensional representations. Following [FR], we give the following definition.

Definition 2.16. Define the ring of Laurent polynomials $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ in the variables $\{Y_{i,a}\}_{i \in I, a \in \mathbb{C}^\times}$. Define \mathcal{M} as the multiplicative group of monomials in \mathcal{Y} .

For $i \in I, a \in \mathbb{C}^\times$, define $A_{i,a} \in \mathcal{M}$ by the formula

$$(2.7) \quad A_{i,a} := Y_{i,aq_i^{-1}} Y_{i,aq_i} \left(\prod_{\{j \in I | C_{j,i} = -1\}} Y_{j,a} \prod_{\{j \in I | C_{j,i} = -2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j \in I | C_{j,i} = -3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right)^{-1}.$$

Define an injective group homomorphism $\mathcal{M} \rightarrow \mathfrak{r}$ by

$$Y_{i,a} \mapsto [\omega_i] \Psi_{i,aq_i^{-1}} \Psi_{i,aq_i}^{-1}.$$

Note that $\varpi(Y_{i,a}) = \omega_i$ and $\varpi(A_{i,a}) = \alpha_i$.

We will identify a monomial $m \in \mathcal{M}$ with its image in \mathfrak{r} . In particular, for any $m \in \mathcal{M}$, we will denote by $L(m)$ the corresponding simple $U_q(\widehat{\mathfrak{b}})$ -module.

Let \mathcal{C} be the category of type 1 finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. For brevity, we will omit ‘‘type 1’’ and refer to objects of the category \mathcal{C} simply as finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. For $j \in I$, a monomial in $m \in \mathcal{M}$ is said to be j -dominant (resp. j -antidominant) if the powers of the variables $Y_{j,b}$, $b \in \mathbb{C}^\times$, occurring in this monomial are all positive (resp. negative). A monomial $m \in \mathcal{M}$ is said to be *dominant* if it is j -dominant for any $j \in I$. We will denote by \mathcal{M}^+ the set of dominant monomials. This is a subsemi-group of \mathcal{M} .

Part (1) of the following theorem was established in [CP] following [Dr]. Parts (2) and (3) were proved in [HJ] using the result of part (1).

Theorem 2.17. (1) Every simple module in the category \mathcal{C} is of the form $L(m), m \in \mathcal{M}^+$.

(2) For $m \in \mathcal{M}$, the simple $U_q(\widehat{\mathfrak{b}})$ -module $L(m)$ is finite-dimensional if and only if $m \in \mathcal{M}^+$. Moreover, the action of $U_q(\widehat{\mathfrak{b}})$ on $L(m)$ can be uniquely extended to an action of $U_q(\widehat{\mathfrak{g}})$ in this case.

(3) Every finite-dimensional simple module in the category \mathcal{O} is of the form $L(m) \otimes [\omega], m \in \mathcal{M}^+, \omega \in P_{\mathbb{Q}}$.

In particular, the finite-dimensional fundamental representation $L(Y_{i,a}), i \in I, a \in \mathbb{C}^\times$, is called the i th fundamental representation. The classes of these representations freely generate the Grothendieck ring $K_0(\mathcal{C})$.

2.5. q -characters.

Definition 2.18. For a $U_q(\widehat{\mathfrak{b}})$ -module V and $\Psi \in \mathfrak{t}_\ell^\times$, the subspace of V ,

$$(2.8) \quad V_\Psi := \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi_{i,m}^+ - \Psi_{i,m})^p v = 0\}$$

is called the ℓ -weight subspace of V of ℓ -weight Ψ .

Theorem 2.19. [HJ] For V in category \mathcal{O} , $V_\Psi \neq 0$ implies $\Psi \in \mathfrak{r}$.

We define \mathcal{E}_ℓ as the additive group of maps $c : \mathfrak{r} \rightarrow \mathbb{Z}$ so that the image of its support

$$\text{supp}(c) = \{\Psi \in \mathfrak{r}, c(\Psi) \neq 0\}$$

by ϖ is contained in a finite union of sets of the form $D(\mu)$. The group \mathcal{E}_ℓ is a completion of the group ring of \mathfrak{r} and the ring structure on the latter extends to \mathcal{E}_ℓ by continuity. Moreover, the homomorphism ϖ naturally extends to a surjective homomorphism

$$(2.9) \quad \varpi : \mathcal{E}_\ell \twoheadrightarrow \mathcal{E},$$

where \mathcal{E} is defined in Definition 2.14. We also have a natural inclusion $\mathcal{E} \hookrightarrow \mathcal{E}_\ell$.

Since \mathcal{Y} is the group ring of \mathcal{M} (see Definition 2.16), the injective group homomorphism $\mathcal{M} \rightarrow \mathfrak{r}$ extends to an injective ring homomorphism $\mathcal{Y} \rightarrow \mathcal{E}_\ell$.

For $\Psi \in \mathfrak{r}$, we define $[\Psi] = \delta_{\Psi, \cdot} \in \mathcal{E}_\ell$.

Let V be a $U_q(\widehat{\mathfrak{b}})$ -module in category \mathcal{O} . Following [FR, HJ], we define the q -character of V as

$$(2.10) \quad \chi_q(V) = \sum_{\Psi \in \mathfrak{r}} \dim(V_\Psi) [\Psi] \in \mathcal{E}_\ell.$$

The following is proved in [FR] and [HJ, Section 3.4].

Proposition 2.20. *Formula 2.10 defines an injective ring homomorphism*

$$\chi_q : K_0(\mathcal{O}) \rightarrow \mathcal{E}_\ell.$$

Note that we have $\chi(V) = \varpi(\chi_q(V))$ for any representation V from the category \mathcal{O} .

Slightly abusing notation, we will write $[\omega]$ for both the one-dimensional representation in the category \mathcal{O} corresponding to $\omega \in P_{\mathbb{Q}}$ and its q -character.

Recall that \mathcal{M} is the multiplicative group of monomials of \mathcal{Y} and the injective group homomorphism $\mathcal{M} \rightarrow \mathfrak{r}$ from Definition 2.16, as well as the injective ring homomorphism $\mathcal{Y} \rightarrow \mathcal{E}_\ell$. We will identify \mathcal{M} and \mathcal{Y} with their images in \mathfrak{r} and \mathcal{E}_ℓ , respectively, under these homomorphisms.

Theorem 2.21. [FR, FM]

(1) *For any finite-dimensional representation V of $U_q(\widehat{\mathfrak{g}})$, all ℓ -weights appearing in V are in the image of \mathcal{M} in \mathfrak{r} .*

(2) *The q -character $\chi_q(V)$ of V is an element of $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} \subset \mathcal{E}_\ell$.*

(3) *For any simple finite-dimensional representation $L(m)$ of $U_q(\widehat{\mathfrak{g}})$, we have*

$$\chi_q(L(m)) \in m(1 + \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}).$$

Let $\varphi : \mathcal{M} \rightarrow P$ be the restriction of the homomorphism ϖ to \mathcal{M} . Explicitly, φ sends a monomial $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}(m)}$ in \mathcal{M} to the corresponding \mathfrak{g} -weight

$$\varphi(m) := \sum_{i \in I, a \in \mathbb{C}^\times} u_{i,a}(m) \omega_i \in P.$$

For example, for $i \in I, a \in \mathbb{C}^\times$, we have $\varphi(Y_{i,a}) = \omega_i$ and $\varphi(A_{i,a}) = \alpha_i$, where $A_{i,a} \in \mathcal{M}$ is defined by formula (2.7). In particular, if $m \in \mathcal{M}^+$, then $\varphi(m) \in P^+$.

The next result follows directly from the definitions.

Lemma 2.22. (1) For $m \in \mathcal{M}^+$, let $L(m)$ be the corresponding simple $U_q(\widehat{\mathfrak{g}})$ -module, and set $\lambda := \varphi(m) \in P^+$. The restriction of $L(m)$ to $U_q(\mathfrak{g})$ is isomorphic to a direct sum of simple $U_q(\mathfrak{g})$ -modules of the form

$$(2.11) \quad \mathcal{V}_\lambda^q \oplus \left(\bigoplus_{\mu < \lambda} (\mathcal{V}_\mu^q)^{\oplus c_{\lambda\mu}} \right),$$

where $\mathcal{V}_\mu^q, \mu \in P^+$, denotes the simple $U_q(\widehat{\mathfrak{g}})$ -module with highest weight μ , and we use the standard partial ordering $<$ on P^+ (see Section 2.1).

(2) For any finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module V , the ordinary character of its restriction to $U_q(\mathfrak{g})$ is equal to $\varphi(\chi_q(V))$.

We will also need the following result.

Theorem 2.23. [FH1] For any $a \in \mathbb{C}^\times$, $i \in I$ we have

$$\chi_q(L_{i,a}^+) = [\Psi_{i,a}] \chi(L_{i,a}^+)$$

and

$$\chi(L_{i,a}^+) = \chi(L_{i,a}^-).$$

2.6. Chari's braid group action and extremal monomials. Consider the simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module $L(m), m \in \mathcal{M}^+$, and set $\lambda = \varphi(m) \in P^+$. Lemma 2.22 implies that the ordinary character of the restriction of $L(m)$ to $U_q(\mathfrak{g})$ is invariant under the action of the Weyl group W . Furthermore, Lemma 2.22 implies that for any $w \in W$, the dimension of the weight subspace of $L(m)$ of weight $w(\omega)$ is equal to 1, and it is spanned by an extremal weight vector (in the sense of [Kas]) which belongs to the $U_q(\mathfrak{g})$ -submodule \mathcal{V}_λ^q generated by the highest weight vector of $L(m)$. We will denote this vector by v_w . (In particular, if $w = e$, the identity element of the Weyl group W , the vector v_e is the highest weight vector of $L(m)$.) Hence this weight subspace is also an ℓ -weight subspace, and v_w is an ℓ -weight vector.

V. Chari introduced in [C] a braid group action on $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ and used it to compute the monomial in \mathcal{Y} expressing the ℓ -weight of v_w .

Namely, Chari defined the operators $T_i, i \in I$, on \mathcal{Y} acting by the formula

$$(2.12) \quad T_i(Y_{i,a}) = Y_{i,a} A_{i,aq_i}^{-1} \text{ and } T_i(Y_{j,a}) = Y_{j,a} \text{ if } j \neq i$$

for all $i, j \in I$ and $a \in \mathbb{C}^\times$. The operator T_i is not an involution, but it was established in [C] (a closely related result was proved earlier in [BP]) that the T_i 's satisfy the relations of the braid group corresponding to \mathfrak{g} . Hence, for any $w \in W$, there is a well-defined operator T_w acting on \mathcal{Y} by the formula

$$T_w = T_{i_1} T_{i_2} \dots T_{i_k}$$

for any reduced decomposition $w = s_{i_1} \dots s_{i_k}$.

Theorem 2.24. [C, CM] The ℓ -weight of $v_w \in L(m)$ is $T_w(m)$. In particular, $T_w(m)$ occurs in $\chi_q(L(m))$ with multiplicity 1.

Recall the set $\mathcal{P}_{\omega_i} = W \cdot \omega_i \subset P$. By Lemma 2.2,(ii), $\mathcal{P}_{\omega_i} \simeq W/W_i$, where W_i is the subgroup of W generated by $s_j, j \neq i$.

Lemma 2.25. *For fixed $i \in I$ and $a \in \mathbb{C}^\times$, the monomial $T_w(Y_{i,a})$, $w \in W$, only depends on the class of w in W/W_i . Hence there is a one-to-one correspondence*

$$\{T_w(Y_{i,a})\}_{w \in W} \simeq \mathcal{P}_{\omega_i} \quad T_w(Y_{i,a}) \leftrightarrow w(\omega_i).$$

Proof. It follows from Proposition 3.1 in [C] that $T_w(Y_{i,a})$ depends only on $w(\omega_i) \in P$. \square

Lemmas 2.25 and 2.2,(3) imply that the following notation is well-defined:

$$(2.13) \quad Y_{w(\omega_i),a} := T_w(Y_{i,a}).$$

Example 2.26. (i) For $w = e$, $Y_{\omega_i,a} = Y_{i,a}$.

(ii) Let $w = s_i$ for some $i \in I$. For $k \in I$ and $a \in \mathbb{C}^\times$, we have

$$T_i(Y_k) = Y_{s_i(\omega_k),a} = Y_{k,a} A_{i,aq_i}^{-\delta_{i,k}}.$$

(iii) By [FM], we have $Y_{w_0(\omega_i),a} = Y_{i,aq^{dh^\vee}}^{-1}$.

It is well-known that the subgroup $\bigoplus_{i \in I} \mathbb{Z}(d+1-d_i)\omega_i \subset P$ is stable under the action of the Weyl group W . Similarly, we obtain the following result. For $i \in I$ and $a \in \mathbb{C}^\times$ let us set

$$(2.14) \quad W_{i,a} = \begin{cases} Y_{i,a} & \text{if } d_i = d, \\ Y_{i,aq^{-1}} Y_{i,aq} & \text{if } d_i = d-1, \\ Y_{i,aq^{-2}} Y_{i,a} Y_{i,aq^2} & \text{if } d_i = d-2. \end{cases}$$

Lemma 2.27. *The subring*

$$\mathbb{Z}[W_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} \subset \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} = \mathcal{Y}$$

is stable under the action of the operators T_w , $w \in W$.

Proof. It suffices to check this for the operators T_i , $i \in I$. We have

$$T_i(W_{j,a}) = W_{j,a} U_{i,aq_i}^{-\delta_{i,j}}$$

where

$$(2.15) \quad U_{i,a} = \begin{cases} A_{i,a} & \text{if } d_i = d, \\ A_{i,aq^{-1}} A_{i,aq} & \text{if } d_i = d-1, \\ A_{i,aq^{-2}} A_{i,a} A_{i,aq^2} & \text{if } d_i = d-2. \end{cases}$$

By inspection of these formulas and the formulas for the monomials $A_{i,a}$, we check that the $U_{i,a}$'s belong to $\mathbb{Z}[W_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$. This completes the proof. \square

In particular, we obtain the following result.

Proposition 2.28. *For any dominant monomial $m \in \mathbb{Z}[W_{i,a}]_{i \in I, a \in \mathbb{C}^\times}$, the extremal ℓ -weights of $L(m)$ belong to $\mathbb{Z}[W_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$.*

It implies the following result which will also be useful for our purposes.

Corollary 2.29. *Let $i, k \in I$ so that $C_{k,i} < -1$. For any $w \in W$, $Y_{w(\omega_i),1}$ is a Laurent monomial in the $W_{j,b}$, $j \in I$, $b \in \mathbb{C}^\times$.*

Proof. This follows from Proposition 2.28 as d_i is equal to the lacing number d in this case, and so $Y_{i,1} = W_{i,1}$. \square

3. EXTENDED TQ -RELATIONS

The plan of this section is as follows. In Section 3.1 we will recall the generalized Baxter TQ -relations in the Grothendieck ring of the category \mathcal{O} that we established in [FH1] (Theorem 3.1) and their q -character versions. Next, in Section 3.2 we will extend the Chari braid group action [C] from the ring \mathcal{Y} (see Section 2.6 above) to a larger ring $\mathcal{Y}' \supset \mathcal{Y}$. Then in Section 3.3 we will state the Extended TQ -relations Conjecture 3.9. Extended TQ -relations are labeled by elements of the Weyl group W of \mathfrak{g} . The relation corresponding to $w \in W$ involves classes of simple representations from the category \mathcal{O} whose highest ℓ -weights are labeled by the weights $w(\omega_i), i \in I$, and are obtained using the generalized Chari's action which we construct in Section 3.2. In Section 3.4 we will consider in more detail the case $w = s_i$. In Section 3.5 we will motivate the Weyl group action introduced in [FH3] by the results of Section 3.4. Finally, in Section 3.6 we will introduce some homomorphisms that we will use in the next section.

3.1. Recollections. We recall the TQ -relations which were conjectured in [FR] and proved in [FH1]. Recall the ring $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$.

Denote by $\tilde{K}_0(\mathcal{O})$ the localization of the ring $K_0(\mathcal{O})$ with respect to the elements $[L(\Psi_{i,a})]$, $i \in I, a \in \mathbb{C}^\times$. Define a homomorphism

$$(3.1) \quad \mathcal{B} : \mathcal{Y} \rightarrow \tilde{K}_0(\mathcal{O})$$

by the formula

$$(3.2) \quad Y_{i,a} \mapsto [\omega_i] \frac{[L(\Psi_{i, aq_i^{-1}})]}{[L(\Psi_{i, aq_i})]}.$$

Theorem 3.1. [FH1] *Let V be a finite-dimensional simple representation of $U_q(\widehat{\mathfrak{g}})$ and $\chi_q(V) \in \mathcal{Y}$ its q -character. Then $\mathcal{B}(\chi_q(V)) = [V]$ in $\tilde{K}_0(\mathcal{O})$.*

By clearing the denominators in the relation $\mathcal{B}(\chi_q(V)) = [V]$ we obtain an algebraic relation in $K_0(\mathcal{O})$. This is the TQ -relation in $K_0(\mathcal{O})$.

The following is an equivalent statement. Let $\tilde{\mathcal{E}}_\ell$ be the localization of the ring \mathcal{E}_ℓ with respect to the elements $\chi_q(L(\Psi_{i,a}))$, $i \in I, a \in \mathbb{C}^\times$. Define a homomorphism

$$(3.3) \quad \mathcal{B}' : \mathcal{Y} \rightarrow \tilde{\mathcal{E}}_\ell$$

by the formula

$$(3.4) \quad Y_{i,a} \mapsto [\omega_i] \frac{\chi_q(L(\Psi_{i, aq_i^{-1}}))}{\chi_q(L(\Psi_{i, aq_i}))}.$$

Recall that \mathcal{Y} is a subring of \mathcal{E}_ℓ . Hence it is a subring of $\tilde{\mathcal{E}}_\ell$.

Theorem 3.2. [FH1] *Let V be a finite-dimensional simple representation of $U_q(\widehat{\mathfrak{g}})$ and $\chi_q(V) \in \mathcal{Y}$ its q -character. Then we have $\mathcal{B}'(\chi_q(V)) = \chi_q(V)$ in $\tilde{\mathcal{E}}_\ell$.*

By clearing the denominators in the relation $\mathcal{B}'(\chi_q(V)) = \chi_q(V)$ we obtain an algebraic relation in \mathcal{E}_ℓ . This is the TQ -relation expressed in terms of q -characters.

For readers convenience, we recall the proof of these two theorems given in [FH1].

Proof of Theorems 3.1 and 3.2. Recall from Theorem 2.23 that

$$(3.5) \quad \chi(L(\Psi_{i,a})) = \chi_q(L_{i,a}^+) = \Psi_{i,a} \chi(L_{i,a}^+).$$

Therefore the relation of Theorem 3.2 can be described as follows: we replace each $Y_{i,a}, i \in I$, appearing in $\chi_q(V)$ with $[\omega_i] \chi_q(L(\Psi_{i,aq_i^{-1}})) (\chi_q(L(\Psi_{i,aq_i}))^{-1}$ and equate the result with $\chi_q(V)$. But according to formula (3.5), this replacement is equivalent to replacing each $Y_{i,a}, i \in I$, with $[\omega_i] \Psi_{i,aq_i^{-1}} \Psi_{i,aq_i}^{-1}$. According to Definition 2.16, the latter ratio is equal to $Y_{i,a}$, so this replacement tautologically gives $\chi_q(V)$. Thus, we obtain the statement of Theorem 3.2. Recalling the injectivity of the q -character homomorphism (Proposition 2.20), we obtain the TQ -relations of Theorem 3.1. \square

Remark 3.3. *We had noted in [FH1, Remark 4.10] that there is an analogous TQ -relation in terms of the simple representations $L(\Psi_{i,a}^{-1})$. Both relations appear as particular cases of our extended TQ -relations below, corresponding to the identity element and the longest element of the Weyl group, respectively.*

3.2. Generalization of Chari's braid group action. Recall the Chari braid group action on the ring \mathcal{Y} , which we described in Section 2.6. Introduce the following extension \mathcal{Y}' of the ring \mathcal{Y} :

$$(3.6) \quad \mathcal{Y}' := \mathbb{Z}[\Psi_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} \otimes_{\mathbb{Z}} \mathbb{Z}(P) = \mathbb{Z}[\Psi_{i,a}^{\pm 1}, y_j^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times; j \in I} \supset \mathcal{Y},$$

where we have used formula (3.22).

We are going to define ring automorphisms $T'_i : \mathcal{Y}' \rightarrow \mathcal{Y}'$, $i \in I$, satisfying the braid group relations, whose restrictions to $\mathcal{Y} \subset \mathcal{Y}'$ are the Chari braid group operators $T_i, i \in I$, on \mathcal{Y} given by formula (2.12). Thus, we will obtain an extension of the Chari braid group action from \mathcal{Y} to \mathcal{Y}' .

Recall the elements $\tilde{\Psi}_{i,a} \in \mathcal{Y}', i \in I$, which we introduced in [FH2] :

$$(3.7) \quad \tilde{\Psi}_{i,a} := \Psi_{i,a}^{-1} \left(\prod_{j \in I, C_{i,j} = -1} \Psi_{j,aq_i} \right) \left(\prod_{j \in I, C_{i,j} = -2} \Psi_{j,a} \Psi_{j,aq^2} \right) \left(\prod_{j \in I, C_{i,j} = -3} \Psi_{j,aq^{-1}} \Psi_{j,aq} \Psi_{j,aq^3} \right).$$

Define a ring automorphism σ of \mathcal{Y}' by the formula

$$(3.8) \quad \sigma(\Psi_{i,a}) = \Psi_{i,a^{-1}}, \quad i \in I, \quad a \in \mathbb{C}^\times; \quad \sigma([\omega]) = [\omega], \quad \omega \in P.$$

Define a ring automorphism $T'_i : \mathcal{Y}' \rightarrow \mathcal{Y}'$, $i \in I$, by the formulas

$$(3.9) \quad T'_i(\Psi_{i,a}) = \sigma(\tilde{\Psi}_{i,a^{-1}q_i^{-2}}^{-1}) = \Psi_{i,aq_i^2}^{-1} \left(\prod_{j \in I, C_{i,j} = -1} \Psi_{j,aq_i} \right) \left(\prod_{j \in I, C_{i,j} = -2} \Psi_{j,a} \Psi_{j,aq^2} \right) \left(\prod_{j \in I, C_{i,j} = -3} \Psi_{j,aq^{-1}} \Psi_{j,aq} \Psi_{j,aq^3} \right).$$

$$(3.10) \quad T'_i(\Psi_{j,a}) = \Psi_{j,a}, \quad j \neq i; \quad T'_i[\omega] = [s_i(\omega)], \quad \omega \in P.$$

This is indeed an automorphism, with the inverse given by

$$(T'_i)^{-1}(\Psi_{i,a}) = \tilde{\Psi}_{i,aq_i^{-2}}, \quad (T'_i)^{-1}(\Psi_{j,a}) = \Psi_{j,a}, \quad j \neq i; \quad (T'_i)^{-1}[\omega] = [s_i(\omega)], \quad \omega \in P.$$

Theorem 3.4. (1) *The operators $T'_i, i \in I$, generate an action of the braid group associated to \mathfrak{g} on \mathcal{Y}' .*

(2) *The restriction of T'_i to $\mathcal{Y} \subset \mathcal{Y}'$ is equal to the Chari operator T_i (see formula (2.12)).*

Proof. It suffices to prove that the braid relations are satisfied on the generators of \mathcal{Y}' . Since the $T_i, i \in I$, acting on \mathcal{Y} , satisfy the braid relations by Chari's theorem [C], we know that the $T'_i, i \in I$, satisfy the braid relations on the generators $Y_{j,a}, j \in I$, of $\mathcal{Y} \subset \mathcal{Y}'$. This is also clear for the generators $y_j, j \in I$, of \mathcal{Y}' because T'_i acts on $\mathbb{Z}[y_j^{\pm 1}]_{j \in I} = \mathbb{Z}(P)$ in the same way as $s_i \in W$ acts on $\mathbb{Z}(P)$.

Hence the braid relations $\mathcal{B}_{i,k}, i, k \in I$, are satisfied on the monomials $\Psi_{j,aq_j^{-1}} \Psi_{j,aq_j}^{-1} = Y_{j,a}[-\omega_i], j \in I$. It remains to show that they are also satisfied on $\Psi_{j,a}^{\pm 1}, j \in I$.

Consider first the simplest non-trivial case: $C_{i,k} = C_{k,i} = -1$. In this case $\mathcal{B}_{i,k}$ is the relation $(T'_k)^{-1}(T'_i)^{-1}(T'_k)^{-1}T'_i T'_k T'_i = \text{Id}$. Since it is satisfied on $\Psi_{j,aq_j^{-1}} \Psi_{j,aq_j}^{-1}$, we have

$$(T'_k)^{-1}(T'_i)^{-1}(T'_k)^{-1}T'_i T'_k T'_i(\Psi_{j,aq_j^{-1}} \Psi_{j,aq_j}^{-1}) = \Psi_{j,aq_j^{-1}} \Psi_{j,aq_j}^{-1}.$$

Therefore,

$$\Psi_{j,aq_j^{-1}}^{-1}(T'_k)^{-1}(T'_i)^{-1}(T'_k)^{-1}T'_i T'_k T'_i(\Psi_{j,aq_j^{-1}}) = \Psi_{j,aq_j}^{-1}(T'_k)^{-1}(T'_i)^{-1}(T'_k)^{-1}T'_i T'_k T'_i(\Psi_{j,aq_j^{-1}}).$$

This implies that $\Psi_{j,aq_j^{2m}}^{-1}(T'_k)^{-1}(T'_i)^{-1}(T'_k)^{-1}T'_i T'_k T'_i(\Psi_{j,aq_j^{2m}})$ does not depend on $m \in \mathbb{Z}$.

But this expression is a Laurent monomial in $\Psi_{l,aq_l^r}^{\pm 1}, l \in I, r \in \mathbb{Z}$. Therefore, it must be equal to 1. Hence we obtain that the braid relation $\mathcal{B}_{i,k}$ holds on $\Psi_{j,a}$ in this case. The braid relations for other values of $C_{i,k}$ and $C_{k,i}$ are proved in a similar way. \square

Recall the monomials $Y_{w(\omega_i),a} \in \mathcal{Y}, w \in W, i \in I, a \in \mathbb{C}^\times$ introduced in Section 2.6 (see equation (2.13)):

$$Y_{w(\omega_i),a} := T_w(Y_{i,a}).$$

Here T_w is the Chari operator on \mathcal{Y} corresponding to w , viewed as an element of the braid group via its reduced decomposition. According to Corollary 2.29, if $C_{k,i} < -1$, then $Y_{w(\omega_i),a}$ is a Laurent polynomial in $W_{k,b}$.

Now we compute the ℓ -weights $\sigma \circ T'_w \circ \sigma(\Psi_{i,a}), w \in W$, where T'_w is obtained from a reduced decomposition of w viewed as an element of the braid group and σ is the ring automorphism of \mathcal{Y}' introduced in (3.8).

Definition 3.5. *For $i \in I, a \in \mathbb{C}^\times$, and $w \in W$, define an ℓ -weight $\Psi_{w(\omega_i),a}$ by the following formulas.*

The ℓ -weight $\Psi_{w(\omega_i),1}$ is defined from the factorization of $Y_{w(\omega_i),1}$ as a product of the variables $Y_{k,b}^{\pm 1}$ by replacing

$$\begin{aligned} Y_{k,b} &\text{ by } \Psi_{k,b^{-1}} \text{ if } d_i = d_k, \\ Y_{k,b} &\text{ by } \Psi_{k,b^{-1}q} \Psi_{k,b^{-1}q^{-1}} \text{ if } C_{i,k} = -2, \\ Y_{k,b} &\text{ by } \Psi_{k,b^{-1}q^2} \Psi_{k,b^{-1}} \Psi_{k,b^{-1}q^{-2}} \text{ if } C_{i,k} = -3, \end{aligned}$$

$W_{k,b}$ by $\Psi_{k,b^{-1}}$ if $C_{k,i} < -1$.
 Finally, we set

$$\Psi_{w(\omega_i),a} := \tau_a(\Psi_{w(\omega_i),1}).$$

Proposition 3.6. *We have*

$$\Psi_{w(\omega_i),a} = \sigma \circ T'_w \circ \sigma(\Psi_{i,a}), \quad w \in W,$$

where $T'_w = T'_{i_1} T'_{i_2} \dots T'_{i_k}$ for any reduced decomposition $w = s_{i_1} \dots s_{i_k}$ and σ is given by formula (3.8).

This statement is a consequence of Proposition 4.7 below and its proof is given at the end of Section 4.2.

Remark 3.7. *If \mathfrak{g} is simply-laced (i.e. $d_i = 1$ for all $i \in I$), then we can obtain $\Psi_{w(\omega_i),1}$ for all $i \in I$ by replacing $Y_{k,b}$ by $\Psi_{k,b^{-1}}$ in $Y_{w(\omega_i),1}$ for all k and b . However, if \mathfrak{g} is not simply-laced, then this is only true for those k for which the lengths of α_i and α_k are the same. If the length of α_i is shorter than the length of α_k , we replace $Y_{k,b}$ by a product of two or three Ψ 's (depending on the lacing number of \mathfrak{g}). And if the length of α_i is longer than the length of α_k , then we have to replace $W_{k,b}$, which is the product of two or three Y 's (depending on the lacing number of \mathfrak{g}), by $\Psi_{k,b^{-1}}$ (this procedure is well-defined by Corollary 2.29). Under this construction, the Cartan matrix gets replaced by its transpose, which we can see as a non-trivial manifestation of Langlands duality.*

Example 3.8. (i) For $\mathfrak{g} = \mathfrak{sl}_2$ we have

$$Y_{\omega_1,1} = Y_{1,1} \text{ and } Y_{w_0(\omega_1),1} = Y_{1,q^2}^{-1}$$

and so

$$\Psi_{\omega_1,a} = \Psi_{1,a} \text{ and } \Psi_{w_0(\omega_1),a} = \Psi_{1,aq^{-2}}^{-1}.$$

(ii) It is known from [FM] that $Y_{w_0(\omega_i),1} = Y_{i,q^{dh^\vee}}^{-1}$. Hence

$$\Psi_{w_0(\omega_i),a} = \Psi_{i,aq^{-dh^\vee}}^{-1}.$$

3.3. Extended TQ-relations Conjecture. We now propose the conjectural extended TQ-relations labeled by elements w of the Weyl group W , so that the original TQ-relations of Theorem 3.1 correspond to the identity element of W .

Recall that for any monomial Ψ in $\Psi_{i,b}^{\pm 1}$, $i \in I$, $b \in \mathbb{C}^\times$ (such as $\Psi_{w(\omega_i),a}$), there is a unique simple highest ℓ -weight module $L(\Psi)$ in the category \mathcal{O} (see Lemma 2.8).

Fix $w \in W$, and denote by $\tilde{K}_0^w(\mathcal{O})$ the localization of the ring $K_0(\mathcal{O})$ with respect to the elements $[L(\Psi_{w(\omega_i),a})]$, $i \in I$, $a \in \mathbb{C}^\times$. Define a homomorphism

$$(3.11) \quad \mathcal{B}_w : \mathcal{Y} \rightarrow \tilde{K}_0^w(\mathcal{O})$$

by the formula

$$(3.12) \quad Y_{i,a} \mapsto [w(\omega_i)] \frac{[L(\Psi_{w(\omega_i),aq_i^{-1}})]}{[L(\Psi_{w(\omega_i),aq_i})]}.$$

Note that $\tilde{K}_0^e(\mathcal{O}) = \tilde{K}_0(\mathcal{O})$ introduced in Section 3.1 and \mathcal{B}_e is \mathcal{B} given by formulas (3.1) and (3.2).

Conjecture 3.9. *Let V be a finite-dimensional simple representation of $U_q(\widehat{\mathfrak{g}})$ and $\chi_q(V) \in \mathcal{Y}$ its q -character. Then $\mathcal{B}_w(\chi_q(V)) = [V]$ in $\widetilde{K}_0^w(\mathcal{O})$.*

By clearing the denominators in the relation $\mathcal{B}_w(\chi_q(V)) = [V]$ we obtain an algebraic relation in $K_0(\mathcal{O})$. This is the conjectural extended TQ -relation in $K_0(\mathcal{O})$ corresponding to $w \in W$.

Informally, Conjecture 3.9 can be interpreted as follows: let us make the following substitution for every variable $Y_{i,a}$, $i \in I$, appearing in the q -character $\chi_q(V)$:

$$Y_{i,a} \mapsto [w(\omega_i)] \frac{[L(\Psi_{w(\omega_i), aq_i^{-1}})]}{[L(\Psi_{w(\omega_i), aq_i})]}.$$

Equating the resulting expression with $[V]$, we obtain the extended TQ -relation in $\widetilde{K}_0^w(\mathcal{O})$ corresponding to $w \in W$ (and by clearing the denominators, in $K_0(\mathcal{O})$).

Since the q -character homomorphism is injective, Conjecture 3.9 is equivalent to the following statement. Let $\widetilde{\mathcal{E}}_\ell^w$ be the localization of the ring \mathcal{E}_ℓ with respect to the elements $\chi_q(L(\Psi_{w(\omega_i), a}))$, $i \in I$, $a \in \mathbb{C}^\times$. Define a homomorphism

$$(3.13) \quad \mathcal{B}'_w : \mathcal{Y} \rightarrow \widetilde{\mathcal{E}}_\ell^w$$

by the formula

$$(3.14) \quad Y_{i,a} \mapsto [w(\omega_i)] \frac{\chi_q(L(\Psi_{w(\omega_i), aq_i^{-1}}))}{\chi_q(L(\Psi_{w(\omega_i), aq_i}))}.$$

Note that $\widetilde{\mathcal{E}}_\ell^e = \widetilde{\mathcal{E}}_\ell$ introduced in Section 3.1 and \mathcal{B}'_e is \mathcal{B}' given by formulas (3.3) and (3.4).

Recall that \mathcal{Y} is a subring of \mathcal{E}_ℓ and hence of $\widetilde{\mathcal{E}}_\ell^w$.

Conjecture 3.10. *Let V be a finite-dimensional simple representation of $U_q(\widehat{\mathfrak{g}})$ and $\chi_q(V) \in \mathcal{Y}$ its q -character. Then we have $\mathcal{B}'_w(\chi_q(V)) = \chi_q(V)$ in $\widetilde{\mathcal{E}}_\ell^w$.*

By clearing the denominators in the relation $\mathcal{B}'_w(\chi_q(V)) = \chi_q(V)$ we obtain an algebraic relation in \mathcal{E}_ℓ . This is the extended TQ -relation corresponding to $w \in W$ expressed in terms of q -characters.

Informally, this statement can be interpreted as follows: in $\widetilde{\mathcal{E}}_\ell^w$, the q -character $\chi_q(V)$ is equal to the expression obtained by replacing every $Y_{i,a}$, $i \in I$, appearing in $\chi_q(V)$ with

$$[w(\omega_i)] \frac{\chi_q(L(\Psi_{w(\omega_i), aq_i^{-1}}))}{\chi_q(L(\Psi_{w(\omega_i), aq_i}))}$$

(by clearing the denominators, we obtain a relation in \mathcal{E}_ℓ).

3.4. The case of simple reflections. For $w = e$, Conjecture 3.10 is proved in Theorem 3.2. In this case, we have the q -character formula which follows from Theorem 2.23:

$$\chi_q(L(\Psi_{i,a})) = [\Psi_{i,a}] \chi(L_{i,a}^+).$$

Next, consider the case when w is a simple reflection, $w = s_i$, $i \in I$. In this case, we have $\Psi_{s_i(\omega_i), a} = \widetilde{\Psi}_{i, aq_i^{-2}}$ (see Example 3.8, (ii)), and the q -character formula for $L(\Psi_{s_i(\omega_i), a})$ is given in the following theorem.

Theorem 3.11. [FH2, FHR] *The q -character of $L(\Psi_{s_i(\omega_i),a}) = L(\tilde{\Psi}_{i,aq_i^{-2}})$ is equal to*

$$\chi_q(L(\Psi_{s_i(\omega_i),a})) = [\tilde{\Psi}_{i,aq_i^{-2}}] \chi(L(\tilde{\Psi}_{i,aq_i^{-2}})) (1 - [-\alpha_i]) \Sigma_{i,aq_i^{-2}}^e$$

where

$$(3.15) \quad \Sigma_{i,a}^e := \sum_{k \geq 0} \prod_{0 < j \leq k} A_{i,aq_i^{-2j+2}}^{-1} = 1 + A_{i,a}^{-1} (1 + A_{i,aq_i^{-2}}^{-1} (1 + \dots)).$$

3.5. Weyl group action. Conjecture 3.9 and Theorem 3.11 suggest defining an operator Θ_i sending $Y_{i,a}$ to

$$(3.16) \quad \Theta_i(Y_{i,a}) := [s_i(\omega_i)] \frac{\chi_q(L(\tilde{\Psi}_{i,aq_i^{-3}}))}{\chi_q(L(\tilde{\Psi}_{i,aq_i^{-1}}))} = Y_{i,a} A_{i,aq_i^{-1}} \frac{\Sigma_{i,aq_i^{-3}}^e}{\Sigma_{i,aq_i^{-1}}^e}$$

and posing the question whether these operators satisfy the relations of the Weyl group W .

To make sense of this statement, we need to deal with the fact that $\Sigma_{i,a}^e$ lies in a completion of

$$(3.17) \quad \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$$

(and so in particular Θ_i does not preserve \mathcal{Y}). In [FH3] we resolved this problem by extending the operators Θ_i to a direct sum of completions of \mathcal{Y} labeled by $w \in W$.

The key point here is that $\Sigma_{i,a}^e$ is a solution of the q^{2d_i} -difference equation

$$(3.18) \quad \Sigma_{i,a} = 1 + A_{i,a}^{-1} \Sigma_{i,aq^{-2d_i}}.$$

In order to define the operators Θ_i we also need another solution of this equation; namely,

$$(3.19) \quad \Sigma_{i,a}^{s_i} := -A_{i,aq_i^2} (1 + A_{i,aq_i^4} (1 + \dots) \dots).$$

Using these two solutions, we defined in [FH3] operators $\Theta_i, i \in I$, acting on the direct sum

$$\Pi := \bigoplus_{w \in W} \tilde{\mathcal{Y}}^w$$

of topological completions $\tilde{\mathcal{Y}}^w$ of \mathcal{Y} with respect to a partial ordering \prec_w on the multiplicative group \mathcal{M} of monomials in \mathcal{Y} . It is defined as follows.

First, define the partial ordering \prec_w on P by the rule $\omega \prec_w \omega'$ is and only $w(\omega) < w(\omega')$, where $<$ denotes the standard partial ordering (see Section 2.1). Next, recall the multiplicative group \mathcal{M} of monomials in the variables $Y_{i,a}^{\pm 1}$ and the homomorphism $\varphi : \mathcal{M} \rightarrow P$ introduced in Section 2.5. Taking the pull-back of the partial ordering \prec_w under φ , we obtain a partial ordering \prec_w on \mathcal{M} . Finally, define a topology on the ring \mathcal{Y} invariant under addition, in which the base of open neighborhoods of the zero element of \mathcal{Y} consists of the sets $U_M, M \in \mathcal{M}$, of all finite linear combinations of monomials $m_i \in \mathcal{M}$ satisfying $m_i \prec_w M$. By definition, $\tilde{\mathcal{Y}}^w$ is the completion of \mathcal{Y} with respect to this topology. Since $m_1 \prec_w M_1$ and $m_2 \prec_w M_2$ implies $m_1 m_2 \prec_w M_1 M_2$, the ring structure on \mathcal{Y} naturally extends to $\tilde{\mathcal{Y}}^w$. Thus, $\tilde{\mathcal{Y}}^w$ is a complete topological ring.

The algebra \mathcal{Y} embeds diagonally in Π and we have canonical projections $E_w : \Pi \rightarrow \tilde{\mathcal{Y}}^w, i \in I$.

Definition 3.12. [FH3] For each $i \in I$, introduce the operator $\Theta_i : \Pi \rightarrow \Pi$,

$$\Theta_i = (\Theta_i^v : \tilde{\mathcal{Y}}^v \rightarrow \tilde{\mathcal{Y}}^{vs_i})_{v \in W}$$

with $\Theta_i^v, v \in W$, given by the formulas $\Theta_i^v(Y_{j,a}) = Y_{j,a}$ if $j \neq i$ and

$$(3.20) \quad \Theta_i^v(Y_{i,a}) = Y_{i,a} A_{i,aq^{-d_i}}^{-1} \frac{\Sigma_{i,aq^{-3d_i}}^{vs_i}}{\Sigma_{i,aq^{-d_i}}^{vs_i}}$$

where $\Sigma_{i,a}^{vs_i} \in \tilde{\mathcal{Y}}^{vs_i}$ is one of the two solutions of the q^{2d_i} -difference equation (3.18):

$$(3.21) \quad \Sigma_{i,a}^{vs_i} := \begin{cases} \Sigma_{i,a}^e & \text{if } l(vs_i) < l(v), \\ \Sigma_{i,a}^{s_i} & \text{if } l(vs_i) > l(v). \end{cases}$$

Recall that $\Sigma_{i,a}^e$ is given by formula (3.15) and $\Sigma_{i,a}^{s_i}$ is given by formula (3.19).

Theorem 3.13. [FH3] (1) There is an action of the Weyl group W of \mathfrak{g} on Π , with the simple reflection $s_i \in W$ ($i \in I$) acting as Θ_i .

(2) The subring \mathcal{Y}^W of W -invariants in \mathcal{Y} is equal to the image of the q -character homomorphism $\chi_q : K_0(\mathcal{C}) \rightarrow \mathcal{Y}$, and hence \mathcal{Y}^W is isomorphic to $K_0(\mathcal{C})$ (here \mathcal{C} is the category of type 1 finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$).

Theorem 3.13,(1) allows us to define an operator $\Theta_w : \Pi \rightarrow \Pi$ for an arbitrary element $w \in W$ using a reduced decomposition of w in terms of the simple reflections s_i . This operator acts on Π as follows: $\Theta_w = (\Theta_w^v : \tilde{\mathcal{Y}}^v \rightarrow \tilde{\mathcal{Y}}^{vw^{-1}})_{v \in W}$.

3.6. Some useful homomorphisms. Let us recall the homomorphisms ϖ from [FH3, Section 2.5] and Λ from [FH3, Section 6.4] which we will use below.

Recall also the partial ordering $<_w$ on P introduced above in Section 3.5. Since

$$(3.22) \quad \mathbb{Z}(P) = \mathbb{Z}[y_i^{\pm 1}]_{i \in I}$$

where the variable y_i correspond to the fundamental weight $\omega_i \in P$, we obtain a partial ordering on the set of monomials in $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ that we will also denote by $<_w$. We define the topological completion $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w$ of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ with respect to this partial ordering in the same way in which we defined the completion $\tilde{\mathcal{Y}}^w$ of \mathcal{Y} in Section 3.5 (with respect to the partial ordering $<_w$).

Consider the ring

$$\pi := \bigoplus_{w \in W} (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w.$$

with the diagonal embedding $\mathbb{Z}[y_i^{\pm 1}] \hookrightarrow \pi$. The assignment $\varpi_w(Y_{i,a}) = y_i$ extends to a ring homomorphism

$$\varpi_w : \tilde{\mathcal{Y}}^w \rightarrow (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w.$$

Thus, we obtain a map

$$\varpi = (\varpi_w)_{w \in W} : \Pi \rightarrow \pi.$$

The standard action of the simple reflection $s_i, i \in I$, on $\mathbb{Z}[y_j^{\pm 1}]_{j \in I}$, naturally extends to a collection of $s_i^w : (\mathbb{Z}[y_j^{\pm 1}]_{j \in I})^w \rightarrow (\mathbb{Z}[y_j^{\pm 1}]_{j \in I})^{ws_i}$ and we obtain a well-defined action of the Weyl group W on π .

Lemma 3.14. [FH3] *The projection $\varpi : \Pi \rightarrow \pi$ intertwines the actions of W on the two rings.*

Now recall the multiplicative group \mathcal{M} of monomials in the variables $Y_{i,a}^{\pm 1}$ and the homomorphism $\varphi : \mathcal{M} \rightarrow P$ introduced in Section 2.5. Taking the pull-back of the partial ordering $<_w$ under φ , we obtain a partial ordering \prec_w on \mathcal{M} which we used in Section 3.5 to define the completion $\tilde{\mathcal{Y}}^w$ of \mathcal{Y} .

A subgroup $\overline{\mathcal{M}}$ of the group of invertible elements in Π is introduced [FH3]. For our purposes, it is enough to recall that $\overline{\mathcal{M}}$ contains \mathcal{M} and that it is stable by the Weyl group action. We have the truncation homomorphism

$$\Lambda : \overline{\mathcal{M}} \rightarrow \mathcal{M}$$

which assigns to $P \in \overline{\mathcal{M}}$ the leading monomial of $E_e(P)$.

Recall the ring automorphism σ of \mathcal{Y}' given by formula (3.8). We will denote by the same symbol its restrictions to $\mathcal{Y} \subset \mathcal{Y}'$ and to $\mathcal{M} \subset \mathcal{Y}$ (the latter is a group automorphism). Both are defined by the formula $\sigma(Y_{i,a}) = Y_{i,a^{-1}}$ for $i \in I$ and $a \in \mathbb{C}^\times$.

We proved in [FH3, Section 6.4] that for any $i \in I$, the Chari operator T_i of section 2.6 is also equal to

$$(3.23) \quad T_i = \sigma \circ \Lambda \circ \Theta_i \circ \sigma.$$

This will be generalized below for $w \in W$ by using homomorphism

$$\Lambda_w : \overline{\mathcal{M}} \rightarrow \mathcal{M}.$$

It is the leading term, relatively to the partial ordering \prec_w , of the expansion in $\tilde{\mathcal{Y}}^w$ of an element of $\overline{\mathcal{M}}$. For $w = e$, we have $\Lambda_e = \Lambda$.

Example 3.15. *For $i \in I$ and $a \in \mathbb{C}^\times$, we have*

$$\Lambda_e(\Sigma_{i,a}) = 1 \text{ and } \Lambda_{s_i}(\Sigma_{i,a}) = -A_{i,aq_i^2}.$$

For $w \in W$, we will also consider the group homomorphism:

$$T_w = \Lambda_w \circ \Theta_w : \mathcal{M} \rightarrow \mathcal{M}.$$

3.7. Extension of Π . We define the following extension of Π :

$$\Pi' := \mathcal{Y}' \otimes_{\mathcal{Y}} \Pi$$

where \mathcal{Y}' is defined by formula (3.6). Thus,

$$\Pi' = \bigoplus_{w \in W} \tilde{\mathcal{Y}}'^w$$

where

$$(3.24) \quad \tilde{\mathcal{Y}}'^w := \mathcal{Y}' \otimes_{\mathcal{Y}} \tilde{\mathcal{Y}}^w$$

is the completion of \mathcal{Y}' consisting of finite sums of elements of the form $P(\Psi_{i,a}^{\pm 1}, y_j^{\pm 1}) \cdot R(Y_{i,a}^{\pm 1})$, where P is a Laurent polynomial and R is an element of $\tilde{\mathcal{Y}}'^w$ which, as we explained in Section 3.5, is the completion of \mathcal{Y} with respect to the partial ordering \prec_w defined in Section 3.6.

Now we define another completion $\tilde{\mathcal{Y}}'^w$ as follows: it consists of finite sums of elements of the form $P(\Psi_{i,a}^{\pm 1}) \cdot R(Y_{i,a}^{\pm 1}, y_j^{\pm 1})$, where P is a Laurent polynomial and R is an element of the

completion of \mathcal{Y} with respect to the partial ordering obtained by combining the orderings \prec_w and \prec_w defined in Section 3.5. We then set

$$\bar{\Pi}' := \bigoplus_{w \in W} \bar{\mathcal{Y}}'^w.$$

Observe that $\bar{\mathcal{Y}}'^e$ is naturally realized as a subring of \mathcal{E}_ℓ . The following result follows from [HJ] and [W1, Theorem 4.19] (see Definition 2.15 for the definition of $K_0(\mathcal{O})$).

Proposition 3.16. *The q -character homomorphism $\chi_q : K_0(\mathcal{O}) \rightarrow \mathcal{E}_\ell$ takes values in $\bar{\mathcal{Y}}'^e$ and the resulting homomorphism $K_0(\mathcal{O}) \rightarrow \bar{\mathcal{Y}}'^e$ is an isomorphism.*

Next, we extend the Weyl group action on Π to a braid group action on Π' and $\bar{\Pi}'$. Note that this action is also discussed in [GHL].

Proposition 3.17. *The operators Θ'_i , $i \in I$ given by the formulas*

$$\Theta'_i([\omega]) = [s_i(\omega)], \quad \Theta'_i(\Psi_{j,a}) = \begin{cases} \Psi_{j,a} & \text{if } j \neq i, \\ \tilde{\Psi}_{i, aq_i^{-2} \Sigma_{i, aq_i^{-2}}} & \text{if } j = i. \end{cases}$$

generate an action of the braid group associated to \mathfrak{g} on Π' and $\bar{\Pi}'$.

Proof. The braid relations are proved from the rank 2 case in the same way as for the operators Θ_i in [FH3]. For example, in type A_2 , we have for $i \neq j$

$$\Theta'_j \Theta'_i(\Psi_{i,a}) = \Psi_{j, aq^{-3} \Sigma_{ji, aq^{-2}}^{-1}}$$

which is invariant under Θ'_i . The proof for types B_2 and G_2 is similar, using [FH3, Sections 4.7, 4.8]. \square

It is clear that the operators Θ'_i preserve the groups of invertible elements $(\Pi')^\times$, $(\Pi)^\times$, and \mathfrak{t}^\times . The following result is obtained by a direct computation.

Proposition 3.18. *The operators Θ'_i , $i \in I$, preserve the subgroup G of $(\Pi')^\times$ generated by $(\Pi)^\times$ and the $\Psi_{j,a}$'s. On this subgroup, we have $(\Theta'_i)^2 = \text{Id}$ on $(\Pi)^\times$ and $(\Theta'_i)^2(\Psi_{j,a})$ is equal to $\Psi_{j,a}$ up to a factor in $\pm \mathfrak{t}^\times$. Hence we have a well-defined action of the Weyl group W on $G/(\pm \mathfrak{t}^\times)$.*

4. WEYL GROUP ACTION AND SOLUTIONS OF THE EXTENDED TQ -RELATIONS

In this section we use the Weyl group action from Section 3.5 to prove Conjectures 3.9 and 3.10 when $w \in W$ is a simple reflection. We then relate these conjectures for arbitrary elements $w \in W$ to a conjectural formula for the q -character of $L(\Psi_{w(\omega_i), aq_i^{-1}})$, up to non-zero factor from $\mathcal{E} \subset \mathcal{E}_\ell$ (Conjecture 4.8). This formula generalizes the formulas in Theorem 3.11 in the case when w is a simple reflection. In the process, we define generating functions $Q_{w(\omega_i), a} \in \Pi$ (Theorem 4.5) and show that they satisfy the extended TQ -relations (Theorem 4.10). In the next section, we will show that they also satisfy the extended QQ -system (Theorem 5.6).

4.1. Relations obtained from the Weyl group action. For each element w of the Weyl group W , we have the extended TQ -relations from Conjecture 3.9 and their equivalent q -character versions from Conjecture 3.10. In the special case $w = s_j, j \in I$, Theorem 3.11 gives us an explicit q -character formula for $L(\Psi_{s_j(\omega_j), a})$, and therefore we obtain the following equivalent reformulation of Conjecture 3.10 in the case $w = s_j, j \in I$.

Corollary 4.1. *Let V be a finite-dimensional simple representation of $U_q(\widehat{\mathfrak{g}})$. Its q -character $\chi_q(V)$, viewed as an element of the completion $\widetilde{\mathcal{Y}}^e$ of \mathcal{Y} , is invariant under the substitution*

$$Y_{i,a} \mapsto E_e(\Theta_j(Y_{i,a})), \quad \forall i \in I, a \in \mathbb{C}^\times.$$

We will now state and prove a generalization of Conjecture 4.1 for an arbitrary element $w \in W$ (instead of $w = s_i$). In fact, we will formulate two versions: for $\widetilde{\mathcal{Y}}^e$ and for $\Pi = \bigoplus_{w \in W} \widetilde{\mathcal{Y}}^w$.

Theorem 4.2. *Let V be a finite-dimensional representation of $U_q(\widehat{\mathfrak{g}})$ and $w \in W$.*

(1) *The q -character $\chi_q(V)$, viewed as an element of the completion $\widetilde{\mathcal{Y}}^e$ of \mathcal{Y} , is invariant under the substitution*

$$Y_{i,a} \mapsto E_e(\Theta_w(Y_{i,a})), \quad \forall i \in I, a \in \mathbb{C}^\times.$$

(2) *The q -character $\chi_q(V)$, viewed as an element of Π under the diagonal embedding $\mathcal{Y} \hookrightarrow \Pi$, is invariant under the substitution*

$$Y_{i,a} \mapsto \Theta_w(Y_{i,a}), \quad \forall i \in I, a \in \mathbb{C}^\times.$$

Proof. Theorem 3.13 implies that the q -character $\chi_q(V)$, viewed as an element of Π under the diagonal embedding $\mathcal{Y} \hookrightarrow \Pi$, is invariant under all operators $\Theta_w, w \in W$. This proves part (2). Applying the projection $E_e : \Pi \rightarrow \widetilde{\mathcal{Y}}^e$, we obtain part (1). \square

In the case $w = e$, the statement of this theorem is trivial. In the case $w = s_j$, because we know the q -character of $L(\Psi_{s_i(\omega_i), a})$ from Theorem 3.11, we obtain the statement of Corollary 4.1 which in turn implies Conjectures 3.9 and 3.10 when w is a simple reflection (note that for $w = e$ we have proved these conjectures in Theorems 3.1 and 3.2). Thus, we obtain the following.

Theorem 4.3. *Conjectures 3.9 and 3.10 hold if $w = s_j, j \in I$.*

In light of this discussion, it is natural to conjecture the following generalization of Theorem 3.11 and formula (3.16) for $\Theta_i(Y_{i,a})$.

Conjecture 4.4. *For any $w \in W$, we have the following identity in $\widetilde{\mathcal{Y}}^e$:*

$$(4.1) \quad E_e(\Theta_w(Y_{i,a})) = [w(\omega_i)] \frac{\chi_q(L(\Psi_{w(\omega_i), aq_i^{-1}}))}{\chi_q(L(\Psi_{w(\omega_i), aq_i}))}$$

Theorem 3.13 and Conjecture 4.4 imply Conjectures 3.9 and 3.10 for an arbitrary $w \in W$.

At the moment, we can only prove Conjecture 4.4 for $w = e$ and $w = s_j, j \in I$ (as explained above) and for an arbitrary $w \in W$ if \mathfrak{g} has rank 2 (see Section 7). However, in the next subsection we will show that this conjecture pins down the q -character of $L(\Psi_{w(\omega_i), aq_i^{-1}})$ up to multiplication by a non-zero factor from a completion of $\mathcal{E} \subset \mathcal{E}_\ell$ (i.e. a possibly infinite

linear combination of constant ℓ -weights, see Section 2.3). This in turn implies Conjectures 3.9 and 3.10 (see Corollary 4.11 below).

4.2. Computation of $\Theta_w(Y_{i,a})$. We will say that a family of ℓ -weights Ψ_a parametrized by $a \in \mathbb{C}^\times$ is \mathbb{C}^\times -equivariant if $\Psi_a = \tau_a(\Psi_1)$ for any $a \in \mathbb{C}^\times$. We have an analogous definition of \mathbb{C}^\times -equivariant families in Π . We also have a notion of \mathbb{C}^\times -invariant element, that is fixed by τ_a , $a \in \mathbb{C}^\times$.

Note that by Theorem 3.13, $\Theta_w(Y_{i,a})$ only depends on $a \in \mathbb{C}^\times$ and $w(\omega_i) \in \mathcal{P}_{\omega_i}$. Recall from Lemma 2.2,(3) that the subsets \mathcal{P}_{ω_i} , $i \in I$, of P are mutually disjoint.

Theorem 4.5. *Let $i \in I$ and $w \in W$.*

(1) *There is a family of ℓ -weights $\Psi_{w(\omega_i),a}$ and a family of invertible elements $Q_{w(\omega_i),a} \in \Pi$, both \mathbb{C}^\times -equivariant and unique, such that*

$$(4.2) \quad \Theta_w(Y_{i,a}) = [w(\omega_i)] \frac{\Psi_{w(\omega_i),aq_i^{-1}} Q_{w(\omega_i),aq_i^{-1}}}{\Psi_{w(\omega_i),aq_i} Q_{w(\omega_i),aq_i}},$$

with $\Lambda(Q_{w(\omega_i),a}) = 1 = \varpi(Q_{w(\omega_i),a})$ and $\Psi_{w(\omega_i),a}(0) = 1$.

(2) *The ℓ -weight $\Psi_{w(\omega_i),a}$ defined in part (1) coincides with the ℓ -weight defined $\Psi_{w(\omega_i),a}$ explicitly in Definition 3.5.*

Example 4.6. *For example, we have*

$$(4.3) \quad Q_{\omega_i,a} = 1, \quad Q_{s_i(\omega_i),a} = \Sigma_{i,aq_i^{-2}}(1 - [-\alpha_i]).$$

Proof of Theorem 4.5. To prove uniqueness, consider $\Psi_{w(\omega_i),a}$ and $Q_{w(\omega_i),a}$ as in the proposition. Then

$$(4.4) \quad \Lambda(\Theta_w(Y_{i,a})) = [w(\omega_i)] \frac{\Psi_{w(\omega_i),aq_i^{-1}}}{\Psi_{w(\omega_i),aq_i}}.$$

If $\Psi'_{w(\omega_i),a}$ and $Q'_{w(\omega_i),a}$ give rise to another solution, then we setting $\Psi := \Psi'_{w(\omega_i),1} \Psi_{w(\omega_i),1}^{-1}$, we obtain that $\Psi(z) = \Psi(zq_i^2)$. This implies that Ψ is a constant ℓ -weight, and so by the conditions of the theorem, $\Psi = \Psi(0) = 1$. Hence $\Psi_{w(\omega_i),a}$ is unique. Now $Q_{w(\omega_i),a}/Q'_{w(\omega_i),a}$ is a constant H that does not depend on a . Hence we have $H = \varpi(H) = 1$. Thus, uniqueness is proved.

Now let $i \in I$, $a \in \mathbb{C}^\times$ and $w \in W$. Recall that $\Theta_w(Y_{i,a})$ depends only on a and $w(\omega_i)$. The existence of $\Psi_{w(\omega_i),a}$ and $Q_{w(\omega_i),a}$ is proved by induction on the length $l(w)$ of w . It is clear if $w = e$. Then consider ws_j so that $l(ws_j) = l(w) + 1$. For $j \neq i$, the result is clear for ws_j as $\Theta_{ws_j}(Y_{i,a}) = \Theta_w(Y_{i,a})$. For ws_i , we have

$$\Theta_{ws_i}(Y_{i,a}) = \Theta_w(Y_{i,a} A_{i,aq_i^{-1}}^{-1}) \times \frac{\Theta_w(\Sigma_{i,aq_i^{-3}})}{\Theta_w(\Sigma_{i,aq_i^{-1}})}.$$

For the second factor, we have

$$\frac{\Theta_w(\Sigma_{i,aq_i^{-3}})}{\Theta_w(\Sigma_{i,aq_i^{-1}})} = \frac{\Lambda(\Theta_w(\Sigma_{i,aq_i^{-3}})) \Theta_w(\Sigma_{i,aq_i^{-3}}) \Lambda(\Theta_w(\Sigma_{i,aq_i^{-3}}^{-1}))}{\Lambda(\Theta_w(\Sigma_{i,aq_i^{-1}})) \Theta_w(\Sigma_{i,aq_i^{-1}}) \Lambda(\Theta_w(\Sigma_{i,aq_i^{-1}}^{-1}))},$$

which is of the correct form as $\Lambda(\Theta_w(\Sigma_{i,aq_i^{-3}}))$ is in \mathcal{M} (and so is an ℓ -weight).

For the first factor, the induction hypothesis on the length of w gives that for each $j \in I$, $b \in \mathbb{C}^\times$, $\Theta_w(Y_{j,b})$ is of the form (4.2). Note that the spectral parameter shift in the expression for j is q_j , which is not necessarily equal to q_i . However we explain it leads to the desired expression for $\Theta_w(Y_{i,a}A_{i,aq_i^{-1}}^{-1})$. Indeed, the factor $\Theta_w(Y_{i,aq_i^{-2}}^{-1})$ is already in the correct form. Let j so that $C_{j,i} \neq 0$.

If $C_{j,i} = -1$, the factor $\Theta_w(Y_{j,b})$ occurs with $b = aq_i^{-1}$. If $d_j = d_i$, it is in the correct form and if $d_j = 2 > d_i = 1$, we have

$$\Theta_w(Y_{j,b}) = [w(\omega_j)] \frac{\Psi_{w(\omega_j),bq^{-2}} Q_{w(\omega_i),bq^{-2}}}{\Psi_{w(\omega_j),bq^2} Q_{w(\omega_j),bq^2}} = [w(\omega_j)] \frac{\Psi_{w(\omega_j),bq^{-2}} Q_{w(\omega_i),bq^{-2}} \Psi_{w(\omega_j),b} Q_{w(\omega_j),b}}{\Psi_{w(\omega_j),b} Q_{w(\omega_j),b} \Psi_{w(\omega_j),ab^2} Q_{w(\omega_j),bq^2}}.$$

(it is analogous if $d_j = 3 > d_i = 1$).

If $C_{j,i} = -2$, we have $d_j = 1$, $d_i = 2$ and the factor $\Theta_w(Y_{j,aq^{-1}}Y_{j,aq})$ occurs. It is of the correct form

$$[w(2\omega_j)] \frac{\Psi_{w(\omega_j),aq^{-2}} Q_{w(\omega_i),aq^{-2}} \Psi_{w(\omega_j),a} Q_{w(\omega_i),a}}{\Psi_{w(\omega_j),a} Q_{w(\omega_i),a} \Psi_{w(\omega_j),aq^2} Q_{w(\omega_j),aq^2}} = [w(2\omega_j)] \frac{\Psi_{w(\omega_j),aq_i^{-1}} Q_{w(\omega_i),aq_i^{-1}}}{\Psi_{w(\omega_j),aq_i} Q_{w(\omega_j),aq_i^{-1}}}.$$

It is analogous if $C_{j,i} = -3$. This completes the proof of part (1) of the theorem.

To prove part (2) of the theorem, we will use the following generalization of (3.23).

Proposition 4.7. (1) For any $w \in W$,

(i) we have

$$(4.5) \quad \Lambda \circ \Theta_w = \sigma \circ T_w \circ \sigma;$$

(ii) for any $i \in I$ and $a \in \mathbb{C}^\times$, the ℓ -weight $\Psi_{w(\omega_i),a}$ in the statement of Theorem 4.5 satisfies

$$(4.6) \quad [w(\omega_i)] \frac{\Psi_{w(\omega_i),aq_i^{-1}}}{\Psi_{w(\omega_i),aq_i}} = \sigma(Y_{w(\omega_i),a^{-1}}).$$

Proof. Since $T_w = \Lambda_w \circ \Theta_w$ is an automorphism of \mathcal{M} , we can consider its inverse T_w^{-1} . For example, we have for $i, j \in I$ and $a \in \mathbb{C}^\times$,

$$T_{s_i}^{-1}(Y_{j,a}) = Y_{j,a} A_{i,aq_i^{-1}}^{-\delta_{i,j}} = (\sigma T_{s_i} \sigma)(Y_{j,a}).$$

This implies that for a general $w \in W$ we have

$$T_{w^{-1}}^{-1} = \sigma \circ T_w \circ \sigma.$$

Now consider

$$T_{w^{-1}}(m) = m_{w^{-1}}.$$

As $\chi_q(L(m))$ is fixed by Θ_w , m is the monomial of highest weight in

$$\chi_q(L(m)) = E_e(\Theta_w(\chi_q(L(m)))).$$

It follows from the first part (existence) in Theorem 4.5 that for a monomial m' of weight ω , the monomial in $E_e(\Theta_w(m'))$ of highest weight has weight $w(\omega)$. Let ω be the weight of m . Then $m_{w^{-1}}$ is the only monomial of weight $w^{-1}(\omega)$ and for any other monomial m' in $\chi_q(L(m))$ of weight $\omega' \neq w^{-1}(\omega_m)$, the weight $w(\omega')$ is strictly lower than ω . We obtain

that only $\Theta_w(m_{w^{-1}})$ can contribute in $E_e(\Theta_w(\chi_q(L(m))))$ to the terms of weight ω . We have established

$$\Lambda(\Theta_w(m_{w^{-1}})) = m.$$

Since this is true for any dominant monomial m , we have

$$\Lambda \circ \Theta_w \circ T_{w^{-1}} = \text{Id},$$

that is

$$\Lambda \circ \Theta_w = T_{w^{-1}}^{-1} = \sigma \circ T_w \circ \sigma.$$

This proves part (i). To prove part (ii), observe that $\Lambda \circ \Theta_w(Y_{i,a})$ is equal the LHS of equation (4.6), by the definition of $\Psi_{w(\omega_i),a}$ given in the statement of Theorem 4.5, and $\sigma \circ T_w \circ \sigma(Y_{i,a})$ is equal to the RHS. \square

Now we can conclude the proof of part (2) of Theorem 4.5. Using equation (4.6), we obtain an expression of $\Psi_{w(\omega_i),1}$ in terms of $\Psi_{j,b}^{\pm 1}$ using the factorization of $Y_{w(\omega_i),1}$ in terms of $Y_{j,c}^{\pm 1}$. This expression coincides with the one given in Definition 3.5 because we have

$$\begin{aligned} [\omega_k] \frac{\Psi_{k,aq_i^{-1}}}{\Psi_{k,aq_i}} &= Y_{k,a} \text{ if } d_i = d_k, \\ [\omega_k] \frac{\Psi_{k,aq^{-2}} \Psi_{k,a}}{\Psi_{k,a} \Psi_{k,aq^2}} &= Y_{k,a} \text{ if } C_{i,k} = -2, \\ [\omega_k] \frac{\Psi_{k,aq^{-3}} \Psi_{k,aq^{-1}} \Psi_{k,aq}}{\Psi_{k,aq^{-1}} \Psi_{k,aq} \Psi_{k,aq^3}} &= Y_{k,a} \text{ if } C_{i,k} = -3, \\ [2\omega_k] \frac{\Psi_{k,aq^{-2}}}{\Psi_{k,q^2}} &= Y_{k,aq^{-1}} Y_{k,aq} \text{ if } C_{k,i} = -2, \\ [3\omega_k] \frac{\Psi_{k,aq^{-3}}}{\Psi_{k,aq^3}} &= Y_{k,aq^{-2}} Y_{k,a} Y_{k,aq^2} \text{ if } C_{k,i} = -3. \end{aligned} \quad \square$$

Proof of Proposition 3.6. From formula 4.6, we obtain that

$$\Psi_{w(\omega_i),aq_i^{-1}} \Psi_{w(\omega_i),aq_i}^{-1} = \sigma(T'_w(\Psi_{i,a^{-1}q_i^{-1}} \Psi_{i,a^{-1}q_i}^{-1}))$$

and so

$$\Psi_{w(\omega_i),aq_i^{-1}} \sigma(T'_w(\Psi_{i,a^{-1}q_i})) = \Psi_{w(\omega_i),aq_i} \sigma(T'_w(\Psi_{i,a^{-1}q_i}^{-1}))$$

which implies that

$$\Psi_{w(\omega_i),aq_i^{-1}} (\sigma(T'_w(\sigma(\Psi_{i,aq_i}^{-1}))))^{-1} = \Psi_{w(\omega_i),aq_i} (\sigma(T'_w(\sigma(\Psi_{i,aq_i}))))^{-1}.$$

Hence $\Psi_{w(\omega_i),aq_i^{2m}} (\sigma \circ T'_w \circ \sigma(\Psi_{i,aq_i^{2m}}))^{-1}$, $m \in \mathbb{Z}$, does not depend on m , and since it is a Laurent monomial in the $\Psi_{j,aq^r}^{\pm 1}$, $j \in I$, $r \in \mathbb{Z}$, it must be equal to 1. Hence the result. \square

4.3. Conjectural q -character formula for $L(\Psi_{w(\omega_i),a})$. Theorem 4.5 implies that Conjecture 4.4 is equivalent to the following conjecture (which holds for $w = e$ by formula (3.5); for $w = s_j$, $j \in I$, by Theorem 3.11; and in the rank 2 case, as we show in Section 7).

Conjecture 4.8. *We have the q -character formula*

$$(4.7) \quad \chi_q(L(\Psi_{w(\omega_i),a})) = \Psi_{w(\omega_i),a} E_e(Q_{w(\omega_i),a}) \chi(L(\Psi_{w(\omega_i),a})).$$

Remark 4.9. *Instead of $E_e(Q_{w(\omega_i),a})$, let us consider $E_y(Q_{w(\omega_i),a}) \in \widetilde{\mathcal{Y}}^y$ for some $y \in W$. We expect that this is also equal to the q -character of an irreducible $U_q(\widehat{\mathfrak{b}})$ -module (up to a non-zero renormalization factor from $\mathcal{E} \subset \mathcal{E}_\ell$). But this module belongs to a “twist” of the category \mathcal{O} by $y \in W$, which is analogous to the twist of the category \mathcal{O} of representations of \mathfrak{g} by $v \in W$ – it consists of modules that are locally finite with the respect to the twisted Borel subalgebra $\mathfrak{y}\mathfrak{b}y^{-1}$. These categories have been defined and studied by Keyu Wang [W2].*

We now establish the extended TQ -relations for the right hand sides of formula (4.7).

Theorem 4.10. *Let V be a finite dimensional representation of $U_q(\widehat{\mathfrak{g}})$ and $w \in W$.*

(1) *The q -character $\chi_q(V)$, viewed as an element of the completion $\widetilde{\mathcal{Y}}^e$ of \mathcal{Y} , is invariant under the substitution*

$$Y_{i,a} \mapsto [w(\omega_i)] \frac{[\Psi_{w(\omega_i),aq_i^{-1}}] E_e(Q_{w(\omega_i),aq_i^{-1}})}{[\Psi_{w(\omega_i),aq_i}] E_e(Q_{w(\omega_i),aq_i})}, \quad \forall i \in I, a \in \mathbb{C}^\times.$$

(2) *The q -character $\chi_q(V)$, viewed as an element of Π under the diagonal embedding $\mathcal{Y} \hookrightarrow \Pi$, is invariant under the substitution*

$$(4.8) \quad Y_{i,a} \mapsto [w(\omega_i)] \frac{[\Psi_{w(\omega_i),aq_i^{-1}}] Q_{w(\omega_i),aq_i^{-1}}}{[\Psi_{w(\omega_i),aq_i}] Q_{w(\omega_i),aq_i}}, \quad \forall i \in I, a \in \mathbb{C}^\times.$$

Proof. We follow the same argument as in the proof of Theorem 4.2: Theorem 3.13 implies that the q -character $\chi_q(V)$, viewed as an element of Π under the diagonal embedding $\mathcal{Y} \hookrightarrow \Pi$, is invariant under all operators $\Theta_w, w \in W$. But the right hand side of formula (4.8) is equal to $\Theta_w(Y_{i,a})$. This proves part (2). Applying the projection $E_e : \Pi \rightarrow \widetilde{\mathcal{Y}}^e$, we obtain part (1). \square

This theorem has the following immediate corollary.

Corollary 4.11. *Conjecture 4.8 implies Conjectures 3.9 and 3.10.*

5. EXTENDED QQ-SYSTEMS

We start by recalling in Theorem 5.1 the QQ -system in the Grothendieck ring $K_0(\mathcal{O})$ of the category \mathcal{O} established in [FH2]. We then formulate the extended QQ -system in $K_0(\mathcal{O})$ (Conjecture 5.9).

Note that the analogous QQ -systems for Yangians were studied in [T, BFLMS] for type A, in [FFK, EV] for type D, and in [MV, ESV] for types ADE.

5.1. The QQ -system. The following theorem was proved in [FH2].

Theorem 5.1. *For all $i \in I$ and $a \in \mathbb{C}^\times$, we have the following system of relations in $K_0(\mathcal{O})$ called the QQ -system:*

$$(5.1) \quad \begin{aligned} & \widetilde{Q}_{i,aq_i} Q_{i,aq_i^{-1}} - [-\alpha_i] \widetilde{Q}_{i,aq_i^{-1}} Q_{i,aq_i} \\ &= \prod_{j, C_{i,j} = -1} Q_{j,a} \prod_{j, C_{i,j} = -2} Q_{j,aq} Q_{j,aq^{-1}} \prod_{j, C_{i,j} = -3} Q_{j,aq^2} Q_{j,a} Q_{j,aq^{-2}}, \end{aligned}$$

where

$$\mathbf{Q}_{i,a} = [L(\Psi_{i,a})]/\chi(L(\Psi_{i,a})), \quad \tilde{\mathbf{Q}}_{i,a} = [L(\tilde{\Psi}_{i,aq_i^{-2}})]\chi_i/(\chi(L(\tilde{\Psi}_{i,aq_i^{-2}}))), \quad \chi_i = \sum_{n \geq 0} [-n\alpha_i]$$

and $\tilde{\Psi}_{i,aq_i^{-2}}$ is defined by formula (3.7).

Remark 5.2. Here we use the normalization we used in [FHR], which is slightly different from the normalization we used in [FH2]. The relation between the two is explained in [FHR, Section 5.8]. The system (5.1) was originally introduced in [MRV1, MRV2] in the context of affine opers.

5.2. Generalization. We are going to formulate a generalization of the system (5.1), called the extended QQ -system, which is a system of relations on the variables $\mathbf{Q}_{w(\omega_i),a}$, $w \in W$, $i \in I$, $a \in \mathbb{C}^\times$. It includes equations (5.1) in which we set $\mathbf{Q}_{i,a} = \mathbf{Q}_{\omega_i,a}$ and $\tilde{\mathbf{Q}}_{i,a} = \mathbf{Q}_{s_i(\omega_i),a}$.

We will need the following result.

Lemma 5.3. *There is a unique family*

$$\chi_{w(\omega_i)} \in \mathbb{Z}[(1 - [-\alpha])^{-1}]_{\alpha \in \Delta} \text{ for } w \in W, i \in I$$

such that $\chi_{\omega_i} = 1$ and

$$(5.2) \quad \chi_{ws_i(\omega_i)}\chi_{w(\omega_i)} = (1 - [-w(\alpha_i)])^{-1} \prod_{j \neq i} \chi_{w(\omega_j)}^{-C_{i,j}} \text{ if } l(ws_i) = l(w) + 1.$$

Recall the topological completion $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^v$, $v \in W$, of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ defined in Section 3.6. We have a natural embedding

$$(5.3) \quad \mathbf{e}_v : \mathbb{Z}[(1 - [-\alpha])^{-1}]_{\alpha \in \Delta} \hookrightarrow (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^v$$

obtained by expanding each $(1 - [-\alpha])^{-1}$, $\alpha \in \Delta$, in positive powers of $[-\alpha]$ if $\alpha \in v(\Delta_+)$ and in positive powers of $[\alpha]$ if $\alpha \in v(\Delta_-)$.

In what follows, we will consider $\chi_{w(\omega_i)}$ as an element the ring

$$(5.4) \quad \bar{\Pi}' = \bigoplus_{v \in W} \bar{\mathcal{Y}}'^v$$

introduced in Section 3.6, whose component in $\bar{\mathcal{Y}}'^v$ is $\mathbf{e}_v(\chi_{w(\omega_i)})$ (note that according to the definitions given in Section 3.6, we have natural inclusion $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^v \subset \bar{\mathcal{Y}}'^v$).

Remark 5.4. *With this interpretation of $\chi_{w(\omega_i)}$, the relations (5.2) coincide with the image of the extended QQ -system (5.1) under the map ϖ (i.e. we forget the spectral parameters in (5.2)).*

Proof of Lemma 5.3. By induction on the length of w , we prove that there exists a unique solution of this system that *a priori* depends on both $i \in I$ and $w \in W$. Moreover, it follows by induction that the highest weight term of

$$\mathbf{e}_e(\chi_{w(\omega_i)}) \in (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^e = \mathbb{Z}[[y_i^{-1}]]_{i \in I}$$

is equal to 1. It remains to show that this solution only depends on $w(\omega_i)$.

To do that, recall the operators Θ'_w defined in Proposition 3.17. Since $\Sigma_{i,a} = \tilde{\Psi}_{i,a}^{-1} \Theta'_i(\Psi_{i,aq_i^2})$ and $A_{i,a} = [\alpha_i] \tilde{\Psi}_{i,a} \Psi_{i,a} \tilde{\Psi}_{i,aq_i^{-2}}^{-1} \Psi_{i,aq_i^2}^{-1}$, applying $\varpi \circ \Theta'_w$ to equation (3.18), we obtain that $\chi_{w(\omega_j)} = \varpi(\Theta'_w(\Psi_{i,a}))$, $w \in W$, is a solution of the system (5.2).

Now, by Proposition 3.18 and Lemma 2.2, $\varpi(\Theta'_w(\Psi_{i,a}))$ depends on $w(\omega_i)$ up to a factor in $\pm \mathfrak{t}^\times$. But the leading term of $\mathbf{e}_e(\varpi(\Theta'_w(\Psi_{i,a}))) = \varpi(E_e(\Theta'_w(\Psi_{i,a})))$ must be equal to 1 because it is a solution of our system, as we have shown above. Therefore, $\varpi(\Theta'_w(\Psi_{i,a}))$ depends only on $w(\omega_i)$. \square

Example 5.5. For $i \in I$ and $w = s_i$, we have

$$\chi_{s_i(\omega_i)} = (1 - [-\alpha_i])^{-1}.$$

Therefore

$$\mathbf{e}_e(\chi_{s_i(\omega_i)}) = \sum_{n \geq 0} [-n\alpha_i]$$

which is the series χ_i from Theorem 5.1.

For $\alpha \in \Delta$, we set $[\alpha]_+ = 1$ if $\alpha \in \Delta_-$ and $[\alpha]_+ = -[\alpha]$ if $\alpha \in \Delta_+$.

Recall the ring $\overline{\Pi}'$ defined in Section 3.7. We introduce the following elements of $\overline{\Pi}'$ for $w \in W$, $i \in I$, and $a \in \mathbb{C}^\times$:

$$(5.5) \quad \mathcal{Q}'_{w(\omega_i),a} = \Psi_{w(\omega_i),a} \mathcal{Q}_{w(\omega_i),a}, \quad \mathcal{Q}_{w(\omega_i),a} = \chi_{w(\omega_i)} \mathcal{Q}'_{w(\omega_i),a}.$$

According to Conjecture 4.8, we have

$$(5.6) \quad E_e(\mathcal{Q}_{w(\omega_i),a}) = \chi_q(L(\Psi_{w(\omega_i),a})) \cdot \mathbf{e}_e(\chi_{w(\omega_i)} / \chi(L(\Psi_{w(\omega_i),a}))).$$

Theorem 5.6. For all $i \in I$, $w \in W$, $a \in \mathbb{C}^\times$, we have the extended QQ-system in $\overline{\Pi}'$

$$\begin{aligned} & \mathcal{Q}_{ws_i(\omega_i),aq_i} \mathcal{Q}_{w(\omega_i),aq_i^{-1}} - [-w(\alpha_i)] \mathcal{Q}_{ws_i(\omega_i),aq_i^{-1}} \mathcal{Q}_{w(\omega_i),aq_i} \\ &= [-w(\alpha_i)]_+ \prod_{j, C_{i,j} = -1} \mathcal{Q}_{w(\omega_j),a} \prod_{j, C_{i,j} = -2} \mathcal{Q}_{w(\omega_j),aq} \mathcal{Q}_{w(\omega_j),aq^{-1}} \prod_{j, C_{i,j} = -3} \mathcal{Q}_{w(\omega_j),aq^2} \mathcal{Q}_{w(\omega_j),a} \mathcal{Q}_{w(\omega_j),aq^{-2}}, \end{aligned}$$

where $[-w(\alpha_i)]_+ = 1$ if $w(\alpha_i) \in \Delta_+$ and $[-w(\alpha_i)]_+ = -[-w(\alpha_i)]$ if $w(\alpha_i) \in \Delta_-$.

Remark 5.7. For each $w' \in W$, by applying $E_{w'}$ to the extended QQ-system in Theorem 5.6, we obtain a relation between the expansions $E_{w'}(\mathcal{Q}_{w(\omega_j),a})$ in the ring $\overline{\mathcal{Y}}^{w'}$ introduced in Section 3.7. Our proof below is given in the ring $\overline{\Pi}'$ which is the direct sum of $\overline{\mathcal{Y}}^{w'}$ over all $w' \in W$. Hence we also obtain the proof of the corresponding relations in $\overline{\mathcal{Y}}^{w'}$ for each w' .

Proof of Theorem 5.6. First let us consider

$$(5.7) \quad \Sigma_{i,w,a} = \Theta_w(\Sigma_{i,a}) \mathcal{Q}_{ws_i(\omega),aq_i^{-1}}^{-1}.$$

We can express $\Sigma_{i,w,a}$ in terms of the $\mathcal{Q}_{w(\omega_j),b}$ in the following way.

Lemma 5.8. *For any $i \in I$, $w \in W$, $a \in \mathbb{C}^\times$, we have*

$$\frac{\Lambda(\Sigma_{i,w,a}^{-1})\Sigma_{i,w,a}}{\varpi(\Lambda(\Sigma_{i,w,a}^{-1})\Sigma_{i,w,a})} =$$

$$Q_{w(\omega_i),a} \prod_{j,C_{i,j}=-1} Q_{w(\omega_j),aq_i}^{-1} \prod_{j,C_{i,j}=-2} Q_{w(\omega_j),aq^2}^{-1} Q_{w(\omega_j),a}^{-1} \prod_{j,C_{i,j}=-3} Q_{w(\omega_j),aq^3}^{-1} Q_{w(\omega_j),aq}^{-1} Q_{w(\omega_j),aq^{-1}}^{-1}.$$

Proof. We have

$$\Theta_{ws_i}(Y_{i,a}) = \Theta_w(Y_{i,a}A_{i,aq_i^{-1}}^{-1}) \frac{\Theta_w(\Sigma_{i,aq_i^{-3}})}{\Theta_w(\Sigma_{i,aq_i^{-1}})} = \Lambda(\Theta_{ws_i}(Y_{i,a})) \frac{Q_{ws_i(\omega_i),aq_i^{-1}}}{Q_{ws_i(\omega_i),aq_i}},$$

$$\Theta_w(Y_{i,a}A_{i,aq_i^{-1}}^{-1}) = \Lambda(\Theta_{ws_i}(Y_{i,a})) \frac{\Sigma_{i,w,aq_i^{-1}}}{\Sigma_{i,w,aq_i^{-3}}},$$

$$(5.8) \quad \Theta_w(A_{i,a}^{-1}) = \Lambda(\Theta_{ws_i}(Y_{i,a})\Theta_w(Y_{i,a}^{-1})) \frac{\Sigma_{i,w,a}Q_{w(\omega_i),aq_i^2}}{\Sigma_{i,w,aq_i^{-2}}Q_{w(\omega_i),a}}.$$

But from (4.2) and the definition of $A_{i,a}$, we obtain

$$\Theta_w(A_{i,a})\Lambda(\Theta_w(A_{i,a}^{-1})) =$$

$$\frac{Q_{w(\omega_i),aq_i^{-2}}}{Q_{w(\omega_i),aq_i^2}} \prod_{j,C_{j,i}=-1} \frac{Q_{w(\omega_j),aq_j}}{Q_{w(\omega_j),aq_j^{-1}}} \prod_{j,C_{j,i}=-2} \frac{Q_{w(\omega_j),aq^2}}{Q_{w(\omega_j),aq^{-2}}} \prod_{j,C_{j,i}=-3} \frac{Q_{w(\omega_j),aq^3}}{Q_{w(\omega_j),aq^{-3}}},$$

which gives the formula in the lemma. \square

Now we can finish the proof of Theorem 5.6. By (3.18), we have

$$\Sigma_{i,a} = 1 + A_{i,a}^{-1}\Sigma_{i,aq_i^{-2}} \text{ and so } \Theta_w(\Sigma_{i,a}) = 1 + \Theta_w(A_{i,a}^{-1})\Theta_w(\Sigma_{i,aq_i^{-2}}),$$

$$Q_{ws_i(\omega_i),aq_i^2}\Sigma_{i,w,a} = 1 + \Lambda(\Theta_{ws_i}(Y_{i,a})\Theta_w(Y_{i,a}^{-1})) \frac{\Sigma_{i,w,a}Q_{w(\omega_i),aq_i^2}}{\Sigma_{i,w,aq_i^{-2}}Q_{w(\omega_i),a}} Q_{ws_i(\omega_i),a}\Sigma_{i,w,aq_i^{-2}}.$$

Then we obtain

$$Q_{ws_i(\omega_i),aq_i^2}Q_{w(\omega_i),a} = \frac{Q_{w(\omega_i),a}}{\Sigma_{i,w,a}} + \Lambda(\Theta_{ws_i}(Y_{i,a})\Theta_w(Y_{i,a}^{-1}))Q_{ws_i(\omega_i),a}Q_{i,w,aq_i^2}.$$

After a shift of spectral parameters by q_i^{-1} , using Lemma 5.8, we obtain that

$$\Lambda(\Sigma_{i,w,aq_i^{-1}})Q_{ws_i(\omega_i),aq_i}Q_{w(\omega_i),aq_i^{-1}} - \Lambda(\Sigma_{i,w,aq_i^{-3}})\Lambda(\Theta_w(A_{i,aq_i^{-1}}^{-1}))Q_{ws_i(\omega_i),aq_i^{-1}}Q_{w(\omega_i),aq_i}$$

$$= \varpi(\Lambda(\Sigma_{i,w,a})\Sigma_{i,w,a}^{-1}) \prod_{j,C_{i,j}=-1} Q_{w(\omega_j),a} \prod_{j,C_{i,j}=-2} Q_{w(\omega_j),aq}Q_{w(\omega_j),aq^{-1}} \times$$

$$\times \prod_{j,C_{i,j}=-3} Q_{w(\omega_j),aq^2}Q_{w(\omega_j),a}Q_{w(\omega_j),aq^{-2}}.$$

We have $\Lambda(\Sigma_{i,w,a}) = \Lambda(\Theta_w(\Sigma_{i,a}))$ as $\Lambda(Q_{ws_i(\omega_i),a}) = 1$. Formula (5.8) implies that

$$(5.9) \quad \Lambda(\Theta_w(A_{i,aq_i^{-1}}^{-1})) = [-w(\alpha_i)] \frac{\Psi_{ws_i(\omega_i),aq_i^{-1}} \Psi_{w(\omega_i),aq_i} \Lambda(\Sigma_{i,w,aq_i^{-1}})}{\Psi_{ws_i(\omega_i),aq_i} \Psi_{w(\omega_i),aq_i^{-1}} \Lambda(\Sigma_{i,w,aq_i^{-3}})}.$$

Besides, from the explicit expression of $A_{i,a}$, the same computation as above gives

$$(5.10) \quad \Lambda(\Theta_w(A_{i,a})) = [w(\alpha_i)] \frac{\tilde{\Psi}_{w(\omega_i),aq_i^{-2}}^{-1} \Psi_{w(\omega_i),a}}{\tilde{\Psi}_{w(\omega_i),a}^{-1} \Psi_{w(\omega_i),aq_i^2}}.$$

where

$$\begin{aligned} & \tilde{\Psi}_{w(\omega_i),aq_i^{-1}} = \Psi_{w(\omega_i),aq_i^{-1}}^{-1} \\ & \times \prod_{j,C_{i,j}=-1} \Psi_{w(\omega_j),a} \prod_{j,C_{i,j}=-2} \Psi_{w(\omega_j),aq} \Psi_{w(\omega_j),aq^{-1}} \prod_{j,C_{i,j}=-3} \Psi_{w(\omega_j),aq^2} \Psi_{w(\omega_j),a} \Psi_{w(\omega_j),aq^{-2}}. \end{aligned}$$

Note that $\tilde{\Psi}_{\omega_i,a}$ is equal to $\tilde{\Psi}_{i,a}$ from Example 3.8,(ii). Therefore from (5.9) and (5.10) we obtain that

$$\Lambda(\Sigma_{i,w,a}) = \varpi(\Lambda(\Sigma_{i,w,a})) \tilde{\Psi}_{w(\omega_i),a}^{-1} \Psi_{ws_i(\omega_i),aq_i^2}.$$

This gives

$$\begin{aligned} & \Psi_{ws_i(\omega_i),aq_i} \Psi_{w(\omega_i),aq_i^{-1}} Q_{ws_i(\omega_i),aq_i} Q_{w(\omega_i),aq_i^{-1}} - \\ & [-w(\alpha_i)] \Psi_{ws_i(\omega_i),aq_i^{-1}} \Psi_{w(\omega_i),aq_i} Q_{ws_i(\omega_i),aq_i^{-1}} Q_{w(\omega_i),aq_i} \\ & = \Psi_{ws_i(\omega_i),aq_i} \Psi_{w(\omega_i),aq_i^{-1}} \tilde{\Psi}_{w(\omega_i),aq_i^{-1}} \Psi_{(ws_i)(\omega_i),aq_i}^{-1} \varpi(\Sigma_{i,w,a}^{-1}) \times \\ & \times \prod_{j,C_{i,j}=-1} Q_{w(\omega_j),a} \prod_{j,C_{i,j}=-2} Q_{w(\omega_j),aq} Q_{w(\omega_j),aq^{-1}} \prod_{j,C_{i,j}=-3} Q_{w(\omega_j),aq^2} Q_{w(\omega_j),a} Q_{w(\omega_j),aq^{-2}}. \end{aligned}$$

Note that

$$\varpi(\Sigma_{i,w,a}^{-1}) = \varpi(\Theta_w(\Sigma_{i,a}^{-1})) = w(\varpi(\Sigma_{i,a}^{-1})) = (1 - [-w(\alpha_i)]).$$

Therefore, we have

$$\begin{aligned} & \mathcal{Q}'_{ws_i(\omega_i),aq_i} \mathcal{Q}'_{w(\omega_i),aq_i^{-1}} - [-w(\alpha_i)] \mathcal{Q}'_{ws_i(\omega_i),aq_i^{-1}} \mathcal{Q}'_{w(\omega_i),aq_i} \\ & = (1 - [-w(\alpha_i)]) \prod_{j,C_{i,j}=-1} \mathcal{Q}'_{w(\omega_j),a} \prod_{j,C_{i,j}=-2} \mathcal{Q}'_{w(\omega_j),aq} \mathcal{Q}'_{w(\omega_j),aq^{-1}} \times \\ & \quad \times \prod_{j,C_{i,j}=-3} \mathcal{Q}'_{w(\omega_j),aq^2} \mathcal{Q}'_{w(\omega_j),a} \mathcal{Q}'_{w(\omega_j),aq^{-2}}, \end{aligned}$$

which is the extended QQ-system. Theorem 5.6 is proved. \square

5.3. Extended QQ -system Conjecture. Conjecture 4.8, injectivity of the q -character homomorphism, and Theorem 5.6 (in which we apply the projection E_e to the $Q_{w(\omega_i),a}$), imply the following conjecture, which we call the Extended QQ -system Conjecture. It describes a system of relations in $K_0(\mathcal{O})$ generalizing the QQ -system of Theorem 5.6. In Theorem 7.1, we will establish it together with Conjecture 4.8 for all \mathfrak{g} of rank 2.

In what follows we will use the notation

$$(5.11) \quad \chi_{w(\omega_i)} := \mathbf{e}_e(\chi_{w(\omega_i)}).$$

Conjecture 5.9. *Let $w \in W$. For any $i \in I$ and $a \in \mathbb{C}^\times$, we have in $K_0(\mathcal{O})$ the relations*

$$(5.12) \quad \mathbf{Q}_{(ws_i)(\omega_i),aq_i} \mathbf{Q}_{w(\omega_i),aq_i^{-1}} - [-w(\alpha_i)] \mathbf{Q}_{(ws_i)(\omega_i),aq_i^{-1}} \mathbf{Q}_{w(\omega_i),aq_i} = [-w(\alpha_i)]_+ \cdot \prod_{j, C_{i,j}=-1} \mathbf{Q}_{w(\omega_j),a} \prod_{j, C_{i,j}=-2} \mathbf{Q}_{w(\omega_j),aq} \mathbf{Q}_{w(\omega_j),aq^{-1}} \prod_{j, C_{i,j}=-3} \mathbf{Q}_{w(\omega_j),aq^2} \mathbf{Q}_{w(\omega_j),a} \mathbf{Q}_{w(\omega_j),aq^{-2}},$$

where

$$(5.13) \quad \mathbf{Q}_{w(\omega_i),a} = [L(\Psi_{w(\omega_i),a})] \cdot \chi_{w(\omega_i)} / \chi(L(\Psi_{w(\omega_i),a})).$$

6. SHIFTED QUANTUM AFFINE ALGEBRAS

A drawback of the conjectural formula (4.7) for the q -character $\chi_q(L(\Psi_{w(\omega_i),a}))$ is that for a general Lie algebra \mathfrak{g} we don't have an explicit expression for the third factor on the right hand side of (4.7); namely, the ordinary character $\chi(L(\Psi_{w(\omega_i),a}))$ (not even for $w = e$ or $s_j, \in I$). This is not important for the extended TQ -relations because $\chi_q(L(\Psi_{w(\omega_i),a}))$ enters in them through the ratios

$$\frac{\chi_q(L(\Psi_{w(\omega_i),aq_i^{-1}}))}{\chi_q(L(\Psi_{w(\omega_i),aq_i}))}$$

in which the factors $\chi(L(\Psi_{w(\omega_i),a}))$ cancel out. But the variables $\mathbf{Q}_{w(\omega_i),a}$ appearing in the QQ -system of Conjecture 5.9 correspond to a particular normalization of $\chi_q(L(\Psi_{w(\omega_i),a}))$ (or equivalently, $[L(\Psi_{w(\omega_i),a})]$) given by formulas (5.6) and (5.13). This suggests that perhaps the factor $\chi_{w(\omega_i)} / \chi(L(\Psi_{w(\omega_i),a}))$ appearing in these formulas is equal to 1. However, this is not the case, even for $w = e$, when $\chi_{\omega_i} = 1$ but $\chi(L(\Psi_{\omega_i,a}))$ is known to be an infinite series.

Nonetheless, it turns out that there are certain modifications of $U_q(\widehat{\mathfrak{g}})$ called *shifted quantum affine algebras* for which there are analogous representations $L^{\text{sh}}(\Psi_{w(\omega_i),a})$ satisfying (conjecturally, in general) the extended TQ - and QQ -systems and for which we expect that

$$(6.1) \quad \chi(L^{\text{sh}}(\Psi_{w(\omega_i),a})) = \chi_{w(\omega_i)}, \quad \forall w \in W.$$

In this section we recall the definition of the shifted quantum affine algebras and their categories of representations analogous to the category \mathcal{O} , and formulate the analogues of the extended TQ - and QQ -systems for representations from these categories. We also conjecture q -character formulas for these representations in Conjecture 6.8. In the next section, we will establish these conjectures for all Lie algebras of rank 2.

6.1. Representations of shifted quantum affine algebras. We collect some information on the shifted quantum affine algebras of [FT] and of their representations (see [He] for more details and references).

Fix an integral coweight μ of \mathfrak{g} . The following definition is due to [FT].

Definition 6.1. *The shifted quantum affine algebra $U_q^\mu(\widehat{\mathfrak{g}})$ is the algebra with the same Drinfeld generators (2.2) as the algebra $U_q(\widehat{\mathfrak{g}})$ and the same relations except that the definition of the series $\phi_i^-(z)$ given in (2.3) is modified as follows:*

$$(6.2) \quad \phi_i^-(z) = \sum_{m \in \mathbb{Z}} \phi_{i,m}^- z^m = \phi_{i,\alpha_i(\mu)}^- z^{\alpha_i(\mu)} \exp \left(-(q_i - q_i^{-1}) \sum_{r>0} h_{i,-r} z^{-r} \right)$$

(the definition of the series $\phi_i^+(z)$ stays the same). In addition, we impose the condition that $\phi_{i,0}^+$ and $\phi_{i,\alpha_i(\mu)}^-$ are invertible (but they are not necessarily inverse to each other).

Note that in particular, for $i \in I$, we have set $\phi_{i,m}^+ = 0$ for $m < 0$ and $\phi_{i,m}^- = 0$ for $m > \alpha_i(\mu)$.

We define the category \mathcal{O}_μ of $U_q^\mu(\widehat{\mathfrak{g}})$ modules similarly to the category \mathcal{O} of $U_q(\widehat{\mathfrak{b}})$ -modules in Definition 2.10. The notions of Cartan-diagonalizable representation and weight spaces V_ω are the same as before.

Definition 6.2. [He] *A $U_q^\mu(\widehat{\mathfrak{g}})$ -module V is said to be in category \mathcal{O}_μ if*

- i) V is Cartan-diagonalizable;
- ii) for all $\omega \in \mathfrak{t}^\times$ we have $\dim(V_\omega) < \infty$;
- iii) there exist a finite number of elements $\lambda_1, \dots, \lambda_s \in \mathfrak{t}^\times$ such that the weights of V are in $\bigcup_{j=1, \dots, s} D(\lambda_j)$.

Theorem 6.3. [He] *The simple representations $L^{\text{sh}}(\Psi)$ in \mathcal{O}_μ are parametrized by rational ℓ -weights $\Psi = (\Psi_i(z))_{i \in I}$ in \mathfrak{r} such that*

$$\mu = \sum_{i \in I} \deg(\Psi_i(z)) \omega_i^\vee \in P^\vee.$$

Each representation V in the category \mathcal{O}_μ has a q -character

$$\chi_q(V) = \sum_{\Psi \in \mathfrak{r}_\mu} \dim(V_\Psi) [\Psi]$$

where V_Ψ is the ℓ -weight space of ℓ -weight Ψ as above and \mathfrak{r}_μ is the group of rational ℓ -weights $\Psi = (\Psi_i(z))_{i \in I}$ such that $\deg(\Psi_i(z)) = \alpha_i(\mu)$ for each $i \in I$.

Theorem 6.4. [He] *The ℓ -weights of a simple representations in \mathcal{O}_μ belong to \mathfrak{r}_μ .*

As in the case of $U_q(\widehat{\mathfrak{b}})$, the q -character map extends to an injective group homomorphism

$$(6.3) \quad \chi_q : K_0(\mathcal{O}_\mu) \rightarrow \mathcal{E}_\ell.$$

Using the fusion product of simple representations constructed from the deformed Drinfeld coproduct, the following result was established in [He]. Note that this fusion procedure is known to work for simple representations but not for arbitrary representations in \mathcal{O}_μ .

Theorem 6.5. *The direct sum $\mathbf{K} := \bigoplus_{\mu \in P^\vee} K_0(\mathcal{O}_\mu)$ has a ring structure such that the product of elements from $K_0(\mathcal{O}_\mu)$ and $K_0(\mathcal{O}_\nu)$ belongs to $K_0(\mathcal{O}_{\mu+\nu})$ and the maps χ_q given by (6.3) combine into an injective ring homomorphism from \mathbf{K} to \mathcal{E}_ℓ . Moreover, the structure constants on simple representations are positive integers.*

Remark 6.6. *The ring \mathbf{K} will be studied from the point of view of cluster algebras in [GHL].*

6.2. Langlands dual Lie algebra. We have already seen the occurrence of the Langlands dual Lie algebra ${}^L\mathfrak{g}$ in the context of the ℓ -weights $\Psi_{w(\omega_i),a}$ (see Remark 3.7). In the context of representations of shifted quantum affine algebras recalled in the preceding subsection, we also assign the coweight ω_k^\vee to $\Psi_{k,a}$ that suggests a connection to ${}^L\mathfrak{g}$.

Now for $i \in I$, define $\phi_i : P_{\mathbb{Q}} \rightarrow P_{\mathbb{Q}}^\vee$ as a homomorphism of W -modules given by the formula

$$\phi_i(\omega_k) = \frac{d_k}{d_i} \omega_k^\vee, \quad k \in I.$$

Then we have

$$\phi_i(\alpha_k) = \sum_{j \in I} C_{j,k} \frac{d_j}{d_i} \omega_j^\vee = \frac{d_k}{d_i} \sum_{j \in I} C_{k,j} \omega_j^\vee = \frac{d_k}{d_i} \alpha_k^\vee.$$

Indeed,

$$\phi_i(s_j(\omega_k)) = \phi_i(\omega_k - \delta_{j,k} \alpha_k) = \frac{d_k}{d_i} (\omega_k^\vee - \delta_{j,k} \alpha_k^\vee) = \frac{d_k}{d_i} (s_j(\omega_k^\vee)) = s_j(\phi_i(\omega_k)).$$

In particular for $w \in W$, we have

$$\phi_i(w(\omega_i)) = w(\omega_i^\vee)$$

and so

$$\phi_i(W \cdot \omega_i) = W \cdot \omega_i^\vee \subset P^\vee.$$

We obtain the following.

Proposition 6.7. *Let $i \in I$, $w \in W$ and $a \in \mathbb{C}^\times$.*

(1) *The coweight associated to $\Psi_{w(\omega_i),a}$ is*

$$w(\omega_i^\vee) = \phi_i(w(\omega_i)),$$

that is the image of the weight of $Y_{w(\omega_i)}(z)$ by ϕ_i .

(2) *The coweight μ associated to $L^{\text{sh}}(\Psi_{w(\omega_i),a})$ is*

$$\mu = w(\omega_i^\vee).$$

Thus, $L^{\text{sh}}(\Psi_{w(\omega_i),a})$ is a simple object of the category $\mathcal{O}^{w(\omega_i^\vee)}$.

Part (2) follows from part (1) and the definition of coweights of simple representations of shifted quantum affine algebras.

6.3. Conjectural q -character formula. We now formulate a more explicit version of Conjecture 4.8 for shifted quantum affine algebras.

Conjecture 6.8. *The q -character of the simple representation $L^{\text{sh}}(\Psi_{w(\omega_i),a})$ is*

$$\chi_q(L^{\text{sh}}(\Psi_{w(\omega_i),a})) = [\Psi_{w(\omega_i),a}] E_e(Q_{w(\omega_i),a}) \chi_{w(\omega_i)}.$$

In particular, its character is equal to $\chi_{w(\omega_i)}$:

$$\chi(L^{\text{sh}}(\Psi_{w(\omega_i),a})) = \chi_{w(\omega_i)}.$$

Example 6.9. (i) *The conjecture is true for $w = e$ because it is known that $L^{\text{sh}}(\Psi_{i,a})$ is a one-dimensional representation by [He, Example 4.13].*

(ii) *The conjecture is true for a simple reflexion $w = s_i$ because by [He, Example 5.2], we have*

$$\chi_q(L(\tilde{\Psi}_{i,a})) = \tilde{\Psi}_{i,a} E_e(\Sigma_{i,a}).$$

Remark 6.10. *Note that Conjecture 6.8 implies positivity of $\chi_{w(\omega_i)}$ for any $w \in W$, $i \in I$, which is a purely combinatorial statement.*

Theorem 5.6 and Conjecture 6.8 imply the following analogue of Conjecture 5.9 for shifted quantum affine algebras (here we use the multiplicative structure of Theorem 6.5).

Conjecture 6.11. *The extended QQ-system (5.12) is satisfied if we set*

$$(6.4) \quad \mathbf{Q}_{w(\omega_i),a} = [L^{\text{sh}}(\Psi_{w(\omega_i),a})].$$

The proof of Conjecture 6.8 (which implies Conjecture 6.11) for rank 2 Lie algebras will be given in the next section.

7. RANK 2 LIE ALGEBRAS

We prove Conjecture 4.8 (hence also 3.9, 3.10, and 5.9) and Conjecture 6.8 (hence also 6.11) for all rank 2 Lie algebras.

Theorem 7.1. *Conjectures 4.8 and 6.8 hold for rank 2 Lie algebras.*

This is clear for type $A_1 \times A_1$ from the case of \mathfrak{sl}_2 discussed above.

For $i \in I$, $a \in \mathbb{C}^\times$ and $\alpha \geq 0$, define the element of \mathcal{M} ,

$$V_{i,a}^{(\alpha)} := (A_{i,a} A_{i,aq_i^{-2}} \cdots A_{i,aq_i^{2-2\alpha}})^{-1}, \quad \alpha > 0, \quad V_{i,a}^{(0)} := 1.$$

We will also use the notation $a_i = [\alpha_i]$.

Type A_2

It suffices to consider the case $i = 1$ from the symmetry of the Dynkin quiver. We have :

$$\chi_{\omega_1} = 1, \quad \chi_{\omega_2 - \omega_1} = \frac{1}{1 - a_1^{-1}}, \quad \chi_{-\omega_2} = \frac{1}{(1 - a_1^{-1} a_2^{-1})(1 - a_2^{-1})},$$

$$\Psi_{\omega_1,a} = \Psi_{1,a}, \quad \Psi_{\omega_2 - \omega_1,a} = \Psi_{1,aq^{-2}}^{-1} \Psi_{2,a}, \quad \Psi_{-\omega_2,a} = \Psi_{2,aq^{-3}}^{-1}.$$

Now the cases $w = e, s_2, s_1, s_1 s_2$ follow from the discussion above. It suffices to treat the case $w = s_2 s_1$. We have by [FH3, Section 4.6]

$$E_e(Q_{w(\omega_1), a}) = \sum_{0 \leq \beta \leq \alpha} V_{2, aq^{-3}}^{(\alpha)} V_{1, aq^{-2}}^{(\beta)},$$

$$\chi_{w(\omega_1)} = \varpi(E_e(Q_{w(\omega_1), a})) = \mathbf{e}_e \left(\frac{1}{(1 - a_2^{-1})(1 - a_1^{-1} a_2^{-1})} \right).$$

This can be compared with [FH3, Example 6.1]. Now, this corresponds to a negative prefundamental representations, and by [HJ, Section 4.1] we have

$$\chi_q(L(\Psi_{2, aq^{-3}}^{-1})) = \Psi_{2, aq^{-3}}^{-1} E_e(Q_{w(\omega_1), a}) = \Psi_{2, aq^{-3}}^{-1} \chi(L(\Psi_{2, aq^{-3}}^{-1})) E_e(Q_{w(\omega_1), a}) (\chi_{w(\omega_1)})^{-1}.$$

But by [He, Corollary 4.11] this is also $\chi_q(L^{\text{sh}}(\Psi_{2, aq^{-3}}^{-1}))$.

Type B_2

We denote $d_1 = 2, d_2 = 1$. We have

$$\chi_{\omega_1} = 1, \chi_{2\omega_2 - \omega_1} = \frac{1}{1 - a_1^{-1}}, \chi_{\omega_1 - 2\omega_2} = \frac{1}{(1 - a_1^{-1} a_2^{-2})(1 - a_2^{-1})},$$

$$\chi_{-\omega_2} = \chi_{1, s_1 s_2 s_1 s_2} = \frac{1}{(1 - a_1^{-1} a_2^{-2})(1 - a_1^{-1})(1 - a_1^{-1} a_2^{-1})},$$

$$\chi_{\omega_2} = 1, \chi_{\omega_1 - \omega_2} = \frac{1}{1 - a_2^{-1}}, \chi_{\omega_2 - \omega_1} = \frac{1}{(1 - a_1^{-1} a_2^{-1})(1 - a_1^{-1})^2},$$

$$\chi_{-\omega_2} = \frac{1}{(1 - a_1^{-1} a_2^{-1})(1 - a_2^{-1})(1 - a_1^{-1} a_2^{-2})^2}.$$

The orbit of ℓ -weights are :

$$\Psi_{\omega_1, a} = \Psi_{1, a}, \Psi_{2\omega_2 - \omega_1} = \Psi_{1, aq^{-4}}^{-1} \Psi_{2, aq^{-2}}, \Psi_{\omega_1 - 2\omega_2} = \Psi_{2, aq^{-4}}^{-1} \Psi_{1, aq^{-2}}, \Psi_{-\omega_1, a} = \Psi_{1, aq^{-6}}^{-1},$$

$$\Psi_{\omega_2, a} = \Psi_{2, a}, \Psi_{\omega_1 - \omega_2} = \Psi_{2, aq^{-2}}^{-1} \Psi_{1, aq^{-2}} \Psi_{1, a}, \Psi_{\omega_2 - \omega_1} = \Psi_{2, aq^{-4}} \Psi_{1, aq^{-6}}^{-1} \Psi_{1, aq^{-4}}^{-1}, \Psi_{-\omega_2, a} = \Psi_{2, aq^{-6}}^{-1}.$$

The cases $w = e, s_2, s_1$ follow from the discussion above. It suffices to treat the cases :

(A) $s_2 s_1(\omega_1) = \omega_1 - 2\omega_2$, (B) $(s_1 s_2 s_1)(\omega_1) = -\omega_1$, (C) $(s_1 s_2)(\omega_2) = \omega_2 - \omega_1$, (D) $(s_2 s_1 s_2)(\omega_2) = -\omega_2$.

The cases (B) and (D) correspond to negative prefundamental representations and can be treated as in type A_2 above. More precisely, let $i = 1$ or 2 , we have $\Psi_{-\omega_i, a} = \Psi_{i, aq^{-6}}^{-1}$. By [FH3, Section 4.7] (formulas (4.34) and (4.47)) :

$$E_e(Q_{-\omega_1, a}) = \sum_{0 \leq \gamma \leq \beta/2 \leq \alpha} V_{1, aq^{-6}}^{(\alpha)} V_{2, aq^{-4}}^{(\beta)} V_{1, aq^{-4}}^{(\gamma)}.$$

$$E_e(Q_{-\omega_2, a}) = \sum_{0 \leq \beta \leq \alpha/2, 0 \leq \beta' \leq (\alpha+1)/2, 0 \leq \gamma \leq \text{Min}(1+2\beta, 2\beta'-1)} V_{2, aq^{-6}}^{(\alpha)} V_{1, aq^{-6}}^{(\beta)} V_{1, aq^{-4}}^{(\beta')} V_{2, aq^{-2}}^{(\gamma)}.$$

We have $\chi_{-\omega_i} = \varpi(E_e(Q_{-\omega_i, a}))$. Now by [HJ], $\chi_q(L(\Psi_{i, aq^{-6}}^{-1}))$ can be obtained as a limit of q -characters of Kirillov-Reshetikhin modules whose q -character can be obtained algorithmically. Comparing these results, we obtain

$$\chi_q(L(\Psi_{i, aq^{-6}}^{-1})) = \Psi_{i, aq^{-6}}^{-1} E_e(Q_{-\omega_i, a}) = \Psi_{i, aq^{-6}}^{-1} \chi(L(\Psi_{i, aq^{-6}}^{-1})) E_e(Q_{-\omega_i, a}) (\chi_{-\omega_i})^{-1}.$$

But by [He, Corollary 4.11], this is also $\chi_q(L^{\text{sh}}(\Psi_{i, aq^{-6}}^{-1}))$.

Case (A): by formula (4.45) of [FH3, Section 4.7], we have

$$E_e(Q_{\omega_1 - 2\omega_2, a}) = \sum_{0 \leq 2\beta \leq \alpha} V_{2, aq^{-4}}^{(\alpha)} V_{1, aq^{-4}}^{(\beta)}.$$

An explicit construction of the representation $L^{\text{sh}}(\Psi_{1, aq^{-2}} \Psi_{2, aq^{-4}}^{-1})$ (similar to [He, Example 5.2]) implies that its q -character is equal to $\Psi_{1, aq^{-2}} \Psi_{2, aq^{-4}}^{-1} E_e(Q_{\omega_1 - 2\omega_2, a})$. We have again $\chi_{s_2 s_1(\omega_1)} = \varpi(E_e(Q_{\omega_1 - 2\omega_2, a}))$. Also, a computation as in [FH1] shows that

$$\chi_q(L(\Psi_{\omega_1 - 2\omega_2, a})) = \Psi_{\omega_1 - 2\omega_2, a} \chi(L(\Psi_{\omega_1 - 2\omega_2, a})) E_e(Q_{\omega_1 - 2\omega_2, a}) (\chi_{s_2 s_1(\omega_1)})^{-1}.$$

Case (C) : by [FH3, Section 4.7] Formula (4.41), we have

$$E_e(Q_{\omega_2 - \omega_1, a}) = \sum_{0 \leq \beta \leq \text{Min}(2\alpha, 2\alpha' + 1)} V_{1, aq^{-4}}^{(\alpha)} V_{1, aq^{-6}}^{(\alpha')} V_{2, aq^{-2}}^{(\beta)}.$$

An explicit construction of the representation $L^{\text{sh}}(\Psi_{2, aq^{-4}} \Psi_{1, aq^{-4}}^{-1} \Psi_{1, aq^{-6}}^{-1})$ (similar to [He, Example 5.2]) gives that its q -character is equal to $\Psi_{2, aq^{-4}} \Psi_{1, aq^{-4}}^{-1} \Psi_{1, aq^{-6}}^{-1} E_e(Q_{\omega_2 - \omega_1, a})$. Also, a computation as in [FH1] shows that

$$\chi_q(L(\Psi_{\omega_2 - \omega_1, a})) = \Psi_{\omega_2 - \omega_1, a} \chi(L(\Psi_{\omega_2 - \omega_1, a})) E_e(Q_{\omega_2 - \omega_1, a}) (\chi_{s_1 s_2(\omega_2)})^{-1}.$$

Type G_2

We denote $d_1 = 3, d_2 = 1$. We have

$$\begin{aligned} \chi_{\omega_1} &= 1, \chi_{3\omega_2 - \omega_1} = \frac{1}{1 - a_1^{-1}}, \chi_{2\omega_1 - 3\omega_2} = \frac{1}{(1 - a_1^{-1} a_2^{-3})(1 - a_2^{-1})}, \\ \chi_{-2\omega_1 + 3\omega_2} &= \frac{1}{(1 - a_1^{-1} a_2^{-3})(1 - a_1^{-1})^2 (1 - a_1^{-1} a_2^{-1})}, \\ \chi_{\omega_1 - 3\omega_2} &= \frac{1}{(1 - a_1^{-2} a_2^{-3})(1 - a_1^{-1} a_2^{-2})(1 - a_1^{-1} a_2^{-3})^2 (1 - a_2)}, \\ \chi_{-\omega_1} &= \frac{1}{(1 - a_1^{-2} a_2^{-3})^2 (1 - a_1^{-1} a_2^{-1})(1 - a_1)(1 - a_1^{-1} a_2^{-2})(1 - a_1^{-1} a_2^{-3})}, \\ \chi_{\omega_2} &= 1, \chi_{\omega_1 - \omega_2} = \frac{1}{1 - a_2^{-1}}, \chi_{2\omega_2 - \omega_1} = \frac{1}{(1 - a_1^{-1} a_2^{-1})(1 - a_1^{-1})^3}, \\ \chi_{\omega_1 - 2\omega_2} &= \frac{1}{(1 - a_1^{-1} a_2^{-1})(1 - a_2^{-1})^2 (1 - a_1^{-1} a_2^{-2})}, \\ \chi_{\omega_2 - \omega_1} &= \frac{1}{(1 - a_1^{-1} a_2^{-1})^2 (1 - a_1^{-1})^3 (1 - a_1^{-1} a_2^{-2})(1 - a_1^{-2} a_2^{-3})^3}, \end{aligned}$$

$$\chi^{-\omega_2} = \frac{1}{(1 - a_1^{-1}a_2^{-1})(1 - a_2^{-1})(1 - a_1^{-1}a_2^{-2})^3(1 - a_1^{-2}a_2^{-3})^2(1 - a_1^{-1}a_2^{-3})^3}.$$

The orbit of ℓ -weights are :

$$\begin{aligned} \Psi_{\omega_1, a} &= \Psi_{1, a}, \quad \Psi_{3\omega_2 - \omega_1} = \Psi_{1, aq^{-6}}^{-1} \Psi_{2, aq^{-3}}, \quad \Psi_{2\omega_1 - 3\omega_2} = \Psi_{2, aq^{-5}}^{-1} \Psi_{1, aq^{-4}} \Psi_{1, aq^{-2}}, \\ \Psi_{-2\omega_1 + 3\omega_2} &= \Psi_{2, aq^{-7}} \Psi_{1, aq^{-10}}^{-1} \Psi_{1, aq^{-8}}^{-1}, \quad \Psi_{\omega_1 - 3\omega_2} = \Psi_{2, aq^{-9}}^{-1} \Psi_{1, aq^{-6}}, \quad \Psi_{-\omega_1, a} = \Psi_{1, aq^{-12}}^{-1}, \\ \Psi_{\omega_2, a} &= \Psi_{2, a}, \quad \Psi_{\omega_1 - \omega_2} = \Psi_{2, aq^{-2}}^{-1} \Psi_{1, aq} \Psi_{1, aq^{-1}} \Psi_{1, aq^{-3}}, \\ \Psi_{2\omega_2 - \omega_1} &= \Psi_{2, aq^{-4}} \Psi_{2, aq^{-6}} \Psi_{1, aq^{-5}}^{-1} \Psi_{1, aq^{-7}}^{-1} \Psi_{1, aq^{-9}}^{-1}, \quad \Psi_{\omega_1 - 2\omega_2} = \Psi_{2, aq^{-6}}^{-1} \Psi_{2, aq^{-8}}^{-1} \Psi_{1, aq^{-3}} \Psi_{1, aq^{-5}} \Psi_{1, aq^{-7}}, \\ \Psi_{\omega_2 - \omega_1} &= \Psi_{2, aq^{-10}} \Psi_{1, aq^{-9}}^{-1} \Psi_{1, aq^{-11}}^{-1} \Psi_{1, aq^{-13}}^{-1}, \quad \Psi_{-\omega_2, a} = \Psi_{2, aq^{-12}}^{-1}. \end{aligned}$$

The results for the weight ω_i , $s_i(\omega_i)$, $i = 1, 2$, have been treated above. The weights $-\omega_i$, $i = 1, 2$, correspond to negative prefundamental representations and can be treated as in type A_2 above. The explicit formula for $Q_{i, a}$ can be found in [FH3, Section 4.8].

The other intermediate weights are treated as for type B_2 above. For example, we have

$$E_e(Q_{2\omega_1 - 3\omega_2, a}) = \sum_{0 \leq 3\beta \leq \alpha} V_{2, aq^{-5}}^{(\alpha)} V_{1, aq^{-6}}^{(\beta)},$$

and

$$\chi_q(L(\Psi_{2\omega_1 - 3\omega_2, a})) = \Psi_{2\omega_1 - 3\omega_2, a} \chi(L(\Psi_{2\omega_1 - 3\omega_2, a})) E_e(Q_{2\omega_1 - 3\omega_2, a}) (\chi_{s_2 s_1(\omega_1)})^{-1}.$$

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