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# Hydrodynamics and relaxation limit for multilane exclusion process and related hyperbolic systems

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## Abstract

We investigate the hydrodynamic behavior and local equilibrium of the multilane exclusion process, whose invariant measures were studied in our previous paper [1]. The dynamics on each lane follows a hyperbolic time scaling, whereas the interlane dynamics has an arbitrary time scaling. We prove the following: (i) the hydrodynamic behavior of the global density (i.e. summed over all lanes) is governed by a scalar conservation law; (ii) the latter, as well as the limit of individual lanes, is the relaxation limit of a weakly coupled hyperbolic system of balance laws that approximates the particle system. For the hydrodynamic limit, to highlight new phenomena arising in our model, a precise computation of the flux function, with the transitions between different possible shapes (and a physical interpretation thereof), is given for the two-lane model.

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# 1 Introduction

The one-dimensional totally asymmetric simple exclusion process (abbreviated as TASEP) is a popular simplified microscopic model of traffic-flow on a one lane highway ([11]). Its hydrodynamic limit under hyperbolic time scaling was first established in [26] under  $1-0$  step initial condition, and obtained under general initial conditions in [25]. It is given by a scalar conservation law known in traffic-flow modeling as the *car traffic equation* or *Lighthill-Witham model*:

$$\partial_t \rho(t, x) + \partial_x [f(\rho(t, x))] = 0, \quad (1)$$

where  $\rho(t, x) \in [0; 1]$  is the density of cars at time  $t \geq 0$  and spatial location  $x \in \mathbb{R}$ . The function  $f$ , given here by

$$f(\rho) = \rho(1 - \rho), \quad (2)$$

is called the *flux function*, or *current-density relation* in traffic-flow modeling, and yields the local flux as a closed function of the sole local density. Equation (1) is meant in the sense of *entropy conditions*, that select the unique physical solution ([28]) among many possible ones. The strict concavity of (2) implies spontaneous creation of increasing shocks, which may be viewed as a simplified mechanism for the formation of traffic jams, that is regions with a sharp transition from low density to high density.

Multilane asymmetric exclusion processes are natural generalizations of TASEP where particles perform an asymmetric exclusion process on each lane, with lane-dependent parameters, and additionally change lanes according to a certain transverse jump kernel. For instance, this may be relevant to incorporate the effect of overtaking. Alternatively, a lane can be viewed as a species of particles and such models as multi-species exclusion processes. Particles on different lanes may have different speeds (and move in different directions), in which case the lane (or species) can be interpreted as a kinetic parameter.

From a mathematical standpoint, the asymmetric multilane exclusion is an interesting intermediate model between one and two dimensional asymmetric exclusion process, especially when it comes to studying the structure of invariant measures, that is well understood in the former case ([20, 7, 9]) but still widely open in the latter ([8]). Recently, a fairly complete characterization of invariant measures was obtained for a wide class of multilane asymmetric models ([1]) as well as for their symmetric counterpart ([24]).

Related models have been studied in the physics or mathematics literature, whether in a multilane or multi-species perspective, like for instance two-lane cellular automata for traffic-flow ([6]) or two-species interacting exclusions with spin flip ([4]).

We are concerned here with the hydrodynamic behaviour under hyperbolic scaling of the class of multilane asymmetric exclusion processes considered in [1]. In

the case of *symmetric* multilane exclusion, under diffusive time scaling on each lane and no time rescaling for transverse jumps, the hydrodynamic limit was addressed in [27] and found to be given by a system of linear diffusions with linear balance terms. In our setting, the problem was recently studied heuristically through mean-field approximation and Monte-Carlo simulations ([12, 13]). Rather than viewing the model as a unique exclusion process, we privilege the point of view of decoupling the dynamics on each lane (longitudinal dynamics) and the interlane (or transverse) dynamics. We shall therefore decompose the generator of the process as

$$\mathcal{L}^N = NL_h + \theta(N)L_v = N \left( L_h + \frac{\theta(N)}{N} L_v \right) \quad (3)$$

where  $N \rightarrow +\infty$  is the space scaling parameter,  $\theta(N) \rightarrow +\infty$ ,  $L_h$  and  $L_v$  respectively denote generators of the longitudinal and transverse jumps. Therefore the longitudinal and transverse time scales are decoupled, although this includes the usual situation  $\theta(N) = N$  where  $\mathcal{L}^N = NL$  for the fixed generator  $L = L_h + L_v$  speeded up in time. In particular, (3) includes the regime  $\theta(N) \ll N$  of *weak coupling* between lanes. This regime is natural to model overtaking, whose rate is expected to be negligible with respect to longitudinal motion. It will bring out a new type of scaling condition requiring ad hoc analysis.

For the study of hydrodynamics, we make a reversibility assumption on the transverse dynamics. This implies existence of a family of product invariant measures which are homogeneous on each lane but with transverse inhomogeneity.

Our first main result (Theorem 3.1) is the hydrodynamic limit and conservation of local equilibrium in the weak sense. We prove that the hydrodynamic limit of the *global* density field (i.e. summed over all lanes) is given by entropy solutions of a scalar conservation law of the form (1). We also show that when the number of lanes  $n \rightarrow +\infty$ , under suitable rescaling, it is possible to obtain a singular type of limit that is not governed by a scalar conservation law (Theorem 4.2).

Besides the global density, we derive the density fields of each lane; however, these densities cannot be described by a set of evolution equations, but are functions of the total density, as we now explain.

The limiting flux function and lane densities can be understood as *relaxation limits*. To this end, we approximate the microscopic model with a system of conservation laws with stiff relaxation term, whose solution  $\rho^\varepsilon$  depends on  $\varepsilon$ ; namely,

$$\partial_t \rho_i^\varepsilon(t, x) + \partial_x f_i[\rho_i^\varepsilon(t, x)] = \varepsilon^{-1} c_i[\rho^\varepsilon(t, x)], \quad i = 0, \dots, n-1 \quad (4)$$

where  $\varepsilon = \theta(N)^{-1} \rightarrow 0$ . Here  $n$  is the number of lanes and  $\rho = (\rho_0, \dots, \rho_{n-1})$ , where  $\rho_i$  denotes the density field on lane  $i$ ,

$$f_i(\rho_i) := \gamma_i \rho_i (1 - \rho_i) \quad (5)$$

is the flux on lane  $i$ , where  $\gamma_i$  is the mean drift of a particle, and

$$c_i(\rho) = \sum_{j=0}^{n-1} [q(j, i) \rho_j (1 - \rho_i) - q(i, j) \rho_i (1 - \rho_j)] \quad (6)$$

where  $q(i, j)$  denotes the transverse jump rate from lane  $i$  to lane  $j$ . In fact, (4) is obtained by replacing the longitudinal dynamics by the hydrodynamic limit of single-lane ASEP and the transverse dynamics by their mean-field approximation. The system (4) is known as a hyperbolic relaxation system with *weak coupling*, see [10, 15, 22], i.e. independent scalar conservation laws coupled only through their source terms.

Our second object of study is the relaxation limit  $\varepsilon \rightarrow 0$  for (4). Relaxation limits were obtained in some particular cases of weak coupling such as the Jin-Xin model and a more general class of discrete kinetic systems ([22]). The relaxation limit, like the hydrodynamic limit, involves a local equilibrium closure of an initially non-closed conservation law. Indeed, by summing the equations in (4) we obtain a conservation law

$$\partial_t R^\varepsilon + \partial_x \sum_{i=0}^{n-1} f_i[\rho_i^\varepsilon] = 0 \quad (7)$$

for the total density

$$R^\varepsilon(t, x) := \sum_{i=0}^{n-1} \rho_i^\varepsilon(t, x). \quad (8)$$

The flux function in (7) is a function of  $\rho$  but *not a closed* function of  $R^\varepsilon$ . In order to close equation (7), one must prove that in the limit  $\varepsilon \rightarrow 0$ ,  $(\rho_0, \dots, \rho_{n-1})$  lies in an *equilibrium manifold*  $\mathcal{F}$  parametrized by the total density, say  $R$ , so that the flux depends only on  $R$ . This manifold is given by

$$\mathcal{F} := \{\rho \in [0; 1]^n : c_i(\rho) = 0, \quad \forall i = 0, \dots, n-1\}. \quad (9)$$

Under reversibility assumptions on  $q(i, j)$ , we can show (in Proposition 2.1) that this manifold is given by

$$\mathcal{F} := \{\rho \in [0; 1]^n : \forall i, j \in \{0, \dots, n-1\}, q(i, j) \rho_i (1 - \rho_j) = q(j, i) \rho_j (1 - \rho_i)\} \quad (10)$$

and can indeed be parametrized by the total density, i.e., the function

$$\psi : \rho \in [0; 1]^n \mapsto \sum_{i=0}^{n-1} \rho_i \in [0; n] \quad (11)$$

is a bijection. Besides,  $\mathcal{F}$  coincides with the set of vectors  $\rho$  for which there exists a product invariant measure for the multilane exclusion process with mean density  $\rho_i$  on lane  $i = 0, \dots, n-1$  (Theorem 2.1). Remark that if the lane number  $i$  is viewed as a kinetic parameter through the mean particle velocity  $\gamma_i$  in (5), we can understand the function  $i \mapsto \psi^{-1}(R)_i$  as the velocity distribution when the total density is  $R$ , that is, the analogue in this context of a Maxwellian.

Our second main result (Theorem 3.2) states that, as  $\varepsilon \rightarrow 0$ , the global density (8) converges locally in  $L^1$  to a limiting field  $R(t, x)$  that is, the entropy solution to a conservation law of the form (1), and that the lane densities vector is given by the ‘‘Maxwellian’’  $\rho = \psi^{-1}(R)$ . The flux function arising both in the relaxation and hydrodynamic limit is the projection  $f := F \circ \psi^{-1}$  of the total flux function

$$F(\rho) := \sum_{i=0}^{n-1} f_i(\rho_i) \quad (12)$$

on the equilibrium manifold  $\mathcal{F}$ . The relaxation limits obtained in this setting are indeed similar to those obtained in Theorem 3.1 for the particle system.

For the hydrodynamic limit of Theorem 3.1, to highlight new phenomena arising in our model, a precise computation of the flux function, with the transitions between different possible shapes (and a physical interpretation thereof), is given for the two-lane model. These include, as in the KLS or AS models ([17, 23, 3]), fluxes that are bell-shaped or with two maxima and one minimum; but additionally, when the two drifts have opposite signs, fluxes with one positive maximum and one negative minimum. However, this analysis reveals (see Theorem 4.1, and Appendix A for graphical illustrations) an unexpected non-monotone transition of the flux shape with respect to the strength of interlane coupling. When one of the lanes is symmetric and the other one asymmetric, we can obtain a singular non-differentiable type of flux similar to the one obtained recently in [14] for the facilitated ASEP. Altogether, the different possible shapes and transitions are found to reflect the interplay and competition between lanes.

Theorem 3.1 is valid in the weak coupling regime  $\theta(N) \ll N$ , as well as  $\theta(N) \sim N$  and  $\theta(N) \gg N$ . In the former case, the transverse dynamics slows down the local equilibrium mechanism, and a minimal growth condition is required on  $\theta(N)$ , cf. (52). This condition is related to the diameter of the transverse random walk graph, and disappears when this diameter is 1, i.e., for two lanes or more generally mean-field walks. Hence the interest of an analysis going beyond the two-lane model.

We note that hydrodynamic and relaxation results of the same nature were established in [29, 30] for the particle and relaxation system when the left-hand side of (4) is replaced by linear transport equations, which corresponds to interaction-free longitudinal particle dynamics (that is, particles are independent as long as they stay on the same lane). The relaxation terms considered

there have a different structure, and the proof in this non-interacting situation was based on the moment method; this is not feasible in our setting, which requires the full hydrodynamic limit machinery, though we believe our approach would also be valid in the linear case.

We end up with an outline of the methods of proof.

For the hydrodynamic limit, we rely on the general scheme developed in [5] for hyperbolic type limits, that reduces the hydrodynamic limit for the Cauchy problem to the case of step initial functions (the so-called *Riemann problem*), for which we have an explicit variational representation that we reproduce at microscopic level. An essential property for this reduction is the so-called *macroscopic stability* property, for which a significant refinement of [9, Lemma 3.1] is required when  $\theta(N) \ll N$ . This accounts for condition (52), which arises from short-time analysis of transverse couplings, see Lemma 7.2.

Condition (52) is also involved in the derivation of Riemann hydrodynamics, as well as the derivation of individual lane profiles from the global one. Indeed for these purposes, we need one and two-block estimates that can be obtained here by revisiting an argument from [25]. The latter was based on a so-called *interface property* for single-lane ASEP ([20]), whereby the number of sign changes between two coupled systems cannot increase. However, this property fails for multilane models. Here, thanks to condition (52), we can show that the interface property remains true up to macroscopically negligible errors, which we call the *quasi-interface* property.

For the relaxation limit, we write entropy conditions for the system with a suitable family of dissipative entropies with respect to the relaxation terms in (4). These are combinations of Kruřkov entropies ([19, 28]) relative to equilibrium states. The entropy dissipation enables us to prove relaxation to the equilibrium manifold, and then close the entropy conditions. The special reversible and monotone structure of the relaxation terms plays here an important role for these entropies to be dissipative.

The paper is organized as follows. In Section 2, we introduce the model and assumptions, then the equilibrium manifold, and we describe the relevant invariant measures for the hydrodynamic limit. In Section 3, we state our main results for the hydrodynamic limit and relaxation limit. In Section 4, general properties of the flux functions are stated as well as various examples and the more precise treatment of the two-lane model. The corresponding proofs and graphical illustrations for the two-lane model are given respectively in Section 6 and in Appendix A, where phase transitions are discussed. In Section 5, we establish the structure of the equilibrium manifold under reversibility assumptions and general properties of the flux. The results stated in Section 3 are then established in Sections 7 (for the hydrodynamic limit) and 8 (for the relaxation limit), and the singular limit is proved in Section 9.

## 2 Multilane exclusion processes and their invariant measures

### 2.1 Multilane exclusion process

**State space.** Let  $V = \mathbb{Z} \times W$ , where

$$W := \{0, \dots, n-1\}. \quad (13)$$

We call  $V$  a *ladder*. An element  $x$  of  $V$  will be generically written in the form  $x = (x(0), x(1))$ , with  $x(0) \in \mathbb{Z}$  and  $x(1) \in W$ . The set of particle configurations on  $V$  is denoted by  $\mathcal{X} := \{0; 1\}^V$ , that is, a compact polish space with respect to product topology. For  $\eta \in \mathcal{X}$  and  $x \in V$ ,  $\eta(x)$  denotes the number of particles at  $x$ . In traffic-flow modeling, we may think of  $V$  as a highway, where for  $i \in W$ ,

$$\mathbb{L}_i := \{x \in V : x(0) \in \mathbb{Z}, x(1) = i\} \quad (14)$$

denotes the  $i$ 'th lane. Then  $x \in V$  is interpreted as spatial location  $x(0)$  on lane  $x(1)$ . For  $i \in W$ , we denote by  $\eta^i$  the particle configuration on  $\mathbb{Z}$  defined by

$$\eta^i(z) = \eta(z, i), \quad (15)$$

that is, the configuration on lane  $i$ . Another interpretation is that  $i \in W$  represents a particle species; then  $\eta(z, i) = \eta^i(z)$  is the number of particles of species  $i$  at site  $z$ . We also denote the total number of particles at  $z \in \mathbb{Z}$  by

$$\bar{\eta}(z) = \sum_{i \in W} \eta^i(z). \quad (16)$$

**Dynamics.** For  $\eta \in \mathcal{X}$  and  $x, y \in V$ , denote by  $\eta^{x,y}$  the new configuration after a particle has jumped, if possible at all, from  $x$  to  $y$ : that is,

$$\eta^{x,y}(w) = \begin{cases} \eta(w) & w \neq x, y \\ \eta(x) - 1 & w = x \\ \eta(y) + 1 & w = y \end{cases},$$

**Definition 2.1.** We call kernel on a nonempty countable set  $S$  a mapping  $\pi : S \times S \rightarrow [0; +\infty)$  such that

$$\sup_{x \in S} \sum_{y \in S} [\pi(x, y) + \pi(y, x)] < +\infty. \quad (17)$$

Let  $q(\cdot, \cdot)$  be a kernel on  $W$ , and for each  $i \in W$ , let  $q_i(\cdot, \cdot)$  be a translation invariant kernel on  $\mathbb{Z}$ . By translation invariant we mean that there exists a summable function  $Q_i : \mathbb{Z} \rightarrow [0; +\infty)$  such that

$$\forall u, v \in \mathbb{Z}, \quad q_i(u, v) = Q_i(v - u). \quad (18)$$



We define kernels  $p_h(\cdot, \cdot)$  and  $p_v(\cdot, \cdot)$  on  $V$  by

$$p_h(x, y) := \sum_{i \in W} q_i[x(0), y(0)] \mathbf{1}_{\{x(1)=y(1)=i\}} \quad (19)$$

$$p_v(x, y) := q[x(1), y(1)] \mathbf{1}_{\{x(0)=y(0)\}}. \quad (20)$$

where  $h$  stands for “horizontal” and  $v$  for “vertical” (in accordance with the interpretation of the model given later on): For  $\theta \geq 0$ , we define the kernel  $p^\theta$  on  $V$  by

$$p^\theta(x, y) := p_h(x, y) + \theta p_v(x, y). \quad (21)$$

The ladder process with kernel  $p^\theta(\cdot, \cdot)$  is the Markov process  $(\eta_t)_{t \geq 0}$  on  $\mathcal{X}$  ([21]) with generator

$$\begin{aligned} L^\theta f(\eta) &:= \sum_{x, y \in V} p^\theta(x, y) \eta(x) (1 - \eta(y)) (f(\eta^{x, y}) - f(\eta)) \\ &= (L_h + \theta L_v) f(\eta) \end{aligned} \quad (22)$$

where  $L_h$  and  $L_v$  are defined respectively by replacing  $p^\theta(\cdot, \cdot)$  by  $p_h(\cdot, \cdot)$  and  $p_v(\cdot, \cdot)$ . In (22),  $f$  is a local function on  $V$ , i.e., a function depending on finitely many coordinates. Thus  $L^\theta$  is the generator of a simple exclusion process (SEP) on  $V$  with jump kernel  $p^\theta$ , where  $L_h$  corresponds to jumps along a lane and  $L_v$  corresponds to interlane jumps. The latter occur with a time scaling parameter  $\theta$ .

The ladder process can be constructed through the so-called *Harris graphical representation* ([16]). Suppose  $(\Omega, \mathfrak{F}, \mathbb{P})$  is a probability space that supports a family  $\mathcal{N} = \{\mathcal{N}_{(x, y)} : (x, y) \in V\}$  (called a Harris system) of independent Poisson processes  $\mathcal{N}_{(x, y)}$  with respective intensities  $p^\theta(x, y)$ . For a given  $\omega \in \Omega$ , we let the process evolve according to the following rule: if there is a particle at site  $x \in V$  at time  $t^-$  where  $t \in \mathcal{N}_{(x, y)}$ , it shall attempt to jump to site  $y$ . The attempt is suppressed if at time  $t^-$  site  $y$  is occupied.

**Remark 2.1.** For  $\theta = 0$ , i.e.  $L^\theta = L_h$ , the process generated by  $L^\theta$  is a collection of independent SEP’s on different lanes with jump kernel  $q_i(\cdot, \cdot)$  on lane  $i$ , whereas for  $\theta > 0$ ,  $\theta$  encodes the strength of interlane coupling.

## 2.2 Assumptions

In the sequel, a family of real numbers  $\rho_i$  indexed by  $i \in W$  will be denoted by  $(\rho_i)_{i=0, \dots, n-1}$ , or  $(\rho_0, \dots, \rho_{n-1})$ .

Recall Definition 2.1. For  $x, y \in S$  such that  $x \neq y$ , and  $\ell \in \mathbb{N}$ , we write  $x \xrightarrow{\ell} \pi y$  if there exists a path  $(x = x_0, \dots, x_\ell = y)$  of length  $\ell$  such that  $\pi(x_k, x_{k+1}) > 0$  for  $k = 0, \dots, \ell - 1$ . We write  $x \rightarrow \pi y$  if there exists  $\ell \in \mathbb{N}$  such that  $x \xrightarrow{\ell} \pi y$ . We omit mention of  $\pi$  whenever there is no ambiguity on the kernel. We say  $x$  and  $y$  are  $\pi$ -connected if  $x \rightarrow \pi y$  or  $y \rightarrow \pi x$ . We set the following definition.

**Definition 2.2.** A kernel  $\pi$  will be called weakly irreducible if, for every  $x, y \in S$  such that  $x \neq y$ ,  $x \rightarrow_\pi y$  or  $y \rightarrow_\pi x$ .

We refer to the above property as *weak* irreducibility as opposed to irreducibility for which one requires  $x \rightarrow_\pi y$  and  $y \rightarrow_\pi x$ . In the context of interacting particle systems, this is a more natural assumption, since it includes for instance totally asymmetric jumps which do not satisfy full irreducibility (see examples in Section 4).

We make the following assumptions on the transition kernel  $p^\theta$  (see (19)–(21)) of the ladder process.

**Assumption 2.1.**

(i) For every  $i = 0, \dots, n-1$ , there exists  $d_i \geq 0$  and  $l_i \geq 0$  such that  $d_i + l_i > 0$  and

$$Q_i(z) = d_i \mathbf{1}_{\{z=1\}} + l_i \mathbf{1}_{\{z=-1\}}. \quad (23)$$

(ii) The kernel  $q(\cdot, \cdot)$  is weakly irreducible.

On lane  $i \in W$ ,  $\gamma_i$  denotes the mean drift, that is,

$$\gamma_i := \sum_{z \in \mathbb{Z}} z Q_i(z) = d_i - l_i. \quad (24)$$

Note that the drift may have different signs (including 0) on different lanes. For instance, we may have a TASEP on each lane, but going left to right on some lanes and right to left on others. If one thinks of the multilane TASEP as a highway connecting two cities A and B, then cars can travel from city A to city B on some lanes and from city B to city A on others.

**Assumption 2.2.** For every irreducibility class  $\mathcal{C}$  of  $q(\cdot, \cdot)$ , there exists a nonzero reversible measure  $\lambda = (\lambda_i)_{i \in \mathcal{C}}$  for the chain restricted to  $\mathcal{C}$ . That is, for all  $i, j \in \mathcal{C}$ ,

$$\lambda_i q(i, j) = \lambda_j q(j, i). \quad (25)$$

**Remark 2.2.** Regarding Assumption 2.2:

1. If  $\mathcal{C}$  is a singleton, the condition is void. In this case, any measure on  $\mathcal{C}$  (that is any constant value  $\lambda_i$  assigned to the unique element  $i$  of  $\mathcal{C}$ ) is reversible.
2. Since the restriction of the kernel to  $\mathcal{C}$  is irreducible, the reversible measure  $\lambda$  is positive on  $\mathcal{C}$  and unique up to a multiplicative factor.

Several examples of models satisfying the above assumptions will be given in Section 4.

### 2.3 Invariant measures

In the sequel, we denote by  $(\tau_k)_{k \in \mathbb{Z}}$  the group of space shifts on  $\mathbb{Z}$ , defined as follows. The shift operator  $\tau_k$  acts on a particle configuration  $\eta \in \mathcal{X}$  through

$$(\tau_k \eta)(z, w) := \eta(z + k, w), \quad \forall (z, w) \in \mathbb{Z} \times W. \quad (26)$$

We write  $\tau$  instead of  $\tau_1$ . We denote by  $\mathcal{S}$  the set of all probability measures on  $\mathcal{X}$  that are invariant under  $\tau$ , and by  $\mathcal{I}$  the set of probability measures on  $\mathcal{I}$  that are invariant for the generator (22). We are interested here in the (compact convex) set  $\mathcal{I} \cap \mathcal{S}$ . By Choquet-Deny theorem, every one of its elements is a mixture of its extremal elements, the set of which is denoted by  $(\mathcal{I} \cap \mathcal{S})_e$ .

It is well known that product Bernoulli measures are invariant for translation-invariant exclusion processes on  $\mathbb{Z}$  ([21]). It follows from Remark 2.1 that the family of measures  $\nu^{\rho_0, \dots, \rho_{n-1}}$  defined for  $(\rho_0, \dots, \rho_{n-1}) \in [0, 1]^n$  by

$$\nu^{\rho_0, \dots, \rho_{n-1}} \{\eta(z, i) = 1\} = \rho_i, \quad (z, i) \in \mathbb{Z} \times W \quad (27)$$

is invariant for the generator (22) when  $\theta = 0$ . When  $\theta > 0$ , the vertical kernel is present, and we expect the coupling of lanes to select a family of invariant measures where  $(\rho_0, \dots, \rho_{n-1})$  is restricted to a one-dimensional relaxation (or equilibrium) manifold defined by the relations

$$\rho_i(1 - \rho_j)q(i, j) = \rho_j(1 - \rho_i)q(j, i) \quad (28)$$

for every  $(i, j) \in W^2$  such that  $i \neq j$ . We denote this manifold by

$$\mathcal{F} := \{(\rho_0, \dots, \rho_{n-1}) \in [0, 1]^n \text{ satisfying relation (28)}\}. \quad (29)$$

The following proposition shows that we can parametrize the set  $\mathcal{F}$  by a global density parameter  $\rho \in [0, n]$ . See Section 5 for its proof.

**Proposition 2.1.**

(i) The mapping  $\psi : \mathcal{F} \rightarrow [0, n]$  defined by

$$\psi(\rho_0, \dots, \rho_{n-1}) := \sum_{i=0}^{n-1} \rho_i \quad (30)$$

is a bijection.

(ii) For every  $i \in W$ , the mapping

$$\tilde{\rho}_i : \rho \mapsto \tilde{\rho}_i(\rho) := (\psi^{-1}(\rho))_i \quad (31)$$

is continuous and nondecreasing.

It follows from Proposition 2.1 that if we set

$$\nu_\rho := \nu^{\tilde{\rho}_0(\rho), \dots, \tilde{\rho}_{n-1}(\rho)} \quad (32)$$

for every  $\rho \in [0, n]$ , then

$$\{\nu^{\rho_0, \dots, \rho_{n-1}} : (\rho_0, \dots, \rho_{n-1}) \in \mathcal{F}\} = \{\nu_\rho, \rho \in [0, n]\}. \quad (33)$$

The following result is an extension of [1, Theorem 2.1] and can be established similarly. Therefore we omit its proof.

**Theorem 2.1.**

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^{\rho_0, \dots, \rho_{n-1}}, (\rho_0, \dots, \rho_{n-1}) \in \mathcal{F}\} = \{\nu_\rho, \rho \in [0, n]\}. \quad (34)$$

**Remark 2.3.** In [1, Theorems 2.2 and 2.3], we studied more generally the structure of the set  $\mathcal{I}_e$  of extremal invariant measures for two-lane exclusion processes and explained in [1, Appendix A] how these results could be partially extended to multilane processes. Although the class of multilane processes mentioned in [1] is less general than the one considered in this paper, the same approach could also apply here. See also item 3. of Remark 4.1 about the role played by Proposition 4.1 below in this extension. However the characterization of  $\mathcal{I}_e$  is not required for hydrodynamic limit, that is the main purpose of this paper.

### 3 Hydrodynamics, convergence and relaxation

Before stating our results in Subsections 3.2 and 3.3, we introduce the necessary definitions in Subsection 3.1.

#### 3.1 Definitions

*Empirical measures and density profiles.* Let  $\mathcal{M}$  be the set of measures on  $\mathbb{R}$  equipped with the vague topology, and  $N \in \mathbb{N}^* := \{1, 2, \dots\}$  represent the scaling parameter, that is the inverse of the macroscopic distance between two consecutive sites on a lane. For a particle configuration  $\xi \in \{0, \dots, \ell\}^{\mathbb{Z}}$ , we define its associated empirical measure at scale  $N$  by

$$\alpha^N(\xi, dx) := N^{-1} \sum_{z \in \mathbb{Z}} \xi(z) \delta_{\frac{z}{N}}(dx). \quad (35)$$

In particular, if  $\eta$  is a configuration for the multilane asymmetric simple exclusion process (ASEP), recall definitions (15) and (16). Then for  $i \in W$ ,  $\alpha^N(\bar{\eta}, dx)$ ,  $\alpha^N(\eta^i, dx)$  respectively represent the global empirical measure and the empirical measures on lane  $i$ .

Let  $u(\cdot)$  be a  $[0, n]$ -valued Borel function on  $\mathbb{R}$ . We say a sequence  $(\xi^N)_{N \in \mathbb{N}^*}$

of random  $\mathcal{X}$ -valued configurations has *density profile*  $u(\cdot)$  if  $\alpha^N(\xi^N, dx)$  converges to  $u(x) dx$  in probability as  $N \rightarrow \infty$ ; that is, for every  $\epsilon > 0$  and every continuous function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  with compact support,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \left| N^{-1} \sum_{z \in \mathbb{Z}} \phi \left( \frac{z}{N} \right) \xi^N(z) - \int \phi(y) u(y) dy \right| > \epsilon \right) = 0. \quad (36)$$

*Local Gibbs states.* In the following,  $\mathcal{B}_\rho$  denotes the Bernoulli distribution with parameter  $\rho \in [0, 1]$ . Let  $(\xi^N)_{N \in \mathbb{N}^*}$  be a sequence of random configurations on  $\mathcal{X}$ . Recall the mappings  $\tilde{\rho}_i$  defined in Proposition 2.1. We say  $(\xi^N)_{N \in \mathbb{N}^*}$  is a local Gibbs state (l.g.s.) with global profile  $u(\cdot)$ , or with lane profiles  $u^0(\cdot), \dots, u^{n-1}(\cdot)$ , where

$$u^i(\cdot) := \tilde{\rho}_i[u(\cdot)], \quad i \in W, \quad (37)$$

if the law  $\mu^N$  of  $\xi^N$  writes

$$\mu^N(d\xi) := \bigotimes_{x \in \mathbb{Z}, i \in W} \mathcal{B}_{u_x^{N,i}}[d\xi(x, i)] \quad (38)$$

where

$$u_x^{N,i} = \tilde{\rho}_i[u_x^N], \quad i \in W, x \in \mathbb{Z} \quad (39)$$

and  $(u_x^N)_{N \in \mathbb{N}^*, x \in \mathbb{Z}}$  is a  $[0, n]$ -valued family such that, for every  $a < b$  in  $\mathbb{R}$ ,

$$\lim_{N \rightarrow +\infty} \int_a^b \left| u_{[Nx]}^N - u(x) \right| dx = 0. \quad (40)$$

In the special case of so-called *Riemann* profiles, that is

$$u(x) = \alpha \mathbf{1}_{\{x \leq 0\}} + \beta \mathbf{1}_{\{x > 0\}} =: R_{\alpha, \beta}(x) \quad (41)$$

where  $\alpha, \beta \in [0, n]$ , a natural choice is

$$u_x^N := \alpha \mathbf{1}_{\{x \leq 0\}} + \beta \mathbf{1}_{\{x > 0\}}, \quad x \in \mathbb{Z}. \quad (42)$$

The measure  $\mu^N$  in (38) (which no longer depends on  $N$ ) is then denoted by  $\mu_{\alpha, \beta}$ .

*Local equilibrium.* Let  $f$  be a local function of  $\mathcal{X}$ . For  $(\rho_0, \dots, \rho_{n-1}) \in [0, 1]^n$  and  $\rho \in [0, n]$ , we define

$$\langle f \rangle(\rho_0, \dots, \rho_{n-1}) := \int_{\mathcal{X}} f(\eta) d\nu^{\rho_0, \dots, \rho_{n-1}}(\eta), \quad (43)$$

$$\bar{f}(\rho) := \int_{\mathcal{X}} f(\eta) d\nu_\rho(\eta) = \langle f \rangle(\tilde{\rho}_0(\rho), \dots, \tilde{\rho}_{n-1}(\rho)). \quad (44)$$

Remark that  $\bar{f}$  is continuous, and that it is increasing if  $f$  is nondecreasing. We say  $(\xi^N)_{N \in \mathbb{N}^*}$  satisfies the *weak local equilibrium* property with profile  $u(\cdot)$ ,

or with lane profiles  $u^0(\cdot), \dots, u^{n-1}(\cdot)$  if for every  $\varphi \in C_K^0(\mathbb{R})$ , and every local function  $f$  on  $\mathcal{X}$ , the following limit holds in probability:

$$\begin{aligned} N^{-1} \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{N}\right) \tau_x f(\xi^N) &\xrightarrow{N \rightarrow +\infty} \int \varphi(x) \langle f \rangle(u^0(x), \dots, u^{n-1}(x)) dx \\ &= \int \varphi(x) \bar{f}[u(x)] dx. \end{aligned} \quad (45)$$

We say  $(\xi^N)_{N \in \mathbb{N}^*}$  satisfies the *strong local equilibrium* property with profile  $u(\cdot)$ , or lane profiles  $u^0(\cdot), \dots, u^{n-1}(\cdot)$  if, for every point of continuity  $x \in \mathbb{R}$  of  $u(\cdot)$ ,

$$\lim_{N \rightarrow +\infty} \mathbb{E} \{ \tau_{\lfloor Nx \rfloor} f(\xi^N) \} = \langle f \rangle(u^0(x), \dots, u^{n-1}(x)) = \bar{f}[u(x)]. \quad (46)$$

By the law of large numbers, a local Gibbs state  $(\xi^N)_{N \in \mathbb{N}^*}$  satisfies the weak local equilibrium property with the same profiles. The weak local equilibrium property implies that  $(\xi^{N,i})_{N \in \mathbb{N}^*}$  has density profile  $u^i(\cdot)$ , hence that  $(\bar{\xi}^N)_{N \in \mathbb{N}^*}$  has density profile  $u(\cdot)$ , as can be seen by choosing

$$f(\eta) = \eta^i(0), \quad \langle f \rangle(\rho^0, \dots, \rho^{n-1}) = \rho^i, \quad \bar{f}(\rho) = \tilde{\rho}_i(\rho). \quad (47)$$

### 3.2 Hydrodynamic limit and convergence results

Let  $(\theta(N))_{N \in \mathbb{N}^*}$  be a positive integer-valued sequence such that

$$\lim_{N \rightarrow +\infty} \theta(N) = +\infty. \quad (48)$$

When  $n \geq 3$ , depending on  $n$  and  $q(\cdot, \cdot)$ , we need a stronger growth assumption on  $\theta(N)$ , still allowing  $\theta(N) \ll N$  with some restriction. Namely, let

$$n^* = n^*(n, q(\cdot, \cdot)) := \max\{d_q(i, j) : (i, j) \in W, i < j\}, \quad \text{where} \quad (49)$$

$$d_q(i, j) := \inf\{\ell \in \mathbb{N} : i \xrightarrow{\ell}_q j \text{ or } j \xrightarrow{\ell}_q i\}. \quad (50)$$

In words,  $n^*$  is the maximal value over all pairs  $i \neq j \in W$  of the minimum length of a path connecting  $i$  to  $j$  by  $q(\cdot, \cdot)$  or its reverse  $\check{q}(i, j) := q(j, i)$ . This is also the diameter of  $W$  for the unoriented graph distance induced by the kernel  $q(\cdot, \cdot)$ . Let

$$m^* := m^*(n, q(\cdot, \cdot)) = \left\lfloor \frac{n^*}{2} \right\rfloor \left( n^* - 1 - \left\lfloor \frac{n^*}{2} \right\rfloor \right) + n^*. \quad (51)$$

Then we make the assumption that

$$\lim_{N \rightarrow +\infty} \frac{\theta(N)}{N^{1-\frac{1}{m^*}}} = +\infty. \quad (52)$$

**Remark 3.1.** When  $n = 2$ , or more generally when

$$q^* := \min\{q(i, j) + q(j, i) : (i, j) \in \{0, \dots, n-1\}^2, i < j\} > 0 \quad (53)$$

we have  $n^* = 1$ , hence  $m^* = 1$  (see Example 7.1), hence (52) reduces to (48).

We consider a sequence of processes  $\eta_t^N = (\eta_t^N)_{t \geq 0}$  such that, for each  $N \in \mathbb{N}^*$ ,  $\eta_t^N$  has generator  $L^{\theta(N)/N}$  (cf. (22)). We are interested in the speeded up process under hyperbolic scaling, i.e.,  $(\eta_{Nt}^N)_{t \geq 0}$ , which has generator

$$\mathcal{L}^N := NL^{\theta(N)/N} = NL_h + \theta(N)L_v. \quad (54)$$

To describe the hydrodynamic behaviour of these processes, we need to define the so-called macroscopic flux function  $G$  that appears in the hydrodynamic limit. As usual in this context, the *macroscopic* flux function is expressed as an equilibrium average of a *microscopic* flux function, that is a function of the microscopic configuration  $\eta \in \mathcal{X}$ . For our model, the latter writes

$$j := \sum_{k \in W} j_k \quad (55)$$

where, for  $k \in W$  (recall (23)),

$$j_k(\eta) := d_k \eta^k(0)[1 - \eta^k(1)] - l_k \eta^k(1)[1 - \eta^k(0)] \quad (56)$$

represents the microscopic current on lane  $k \in W$ . Note that, for every  $x \in \mathbb{Z}$ ,

$$\mathcal{L}^N[\bar{\eta}(x)] = NL_h[\bar{\eta}(x)] = N(\tau_{x-1}j(\eta) - \tau_x j(\eta)). \quad (57)$$

Indeed, since the vertical component  $L_v$  of the generator contains only vertical jumps, it leaves  $\bar{\eta}(z)$  unchanged, and thus does not contribute to (57). The macroscopic flux is then given by, for  $\rho \in [0, n]$ ,

$$G(\rho) = \int j(\eta) \nu_\rho(d\eta). \quad (58)$$

Using (32) and (33), this yields

$$G(\rho) := \sum_{i=0}^{n-1} \gamma_i G_0[\tilde{\rho}_i(\rho)] \quad (59)$$

where the mean drift  $\gamma_i$  was defined in (24), and  $G_0$  is the flux function of the single-lane TASEP, given by

$$G_0(\alpha) = \alpha(1 - \alpha) \quad \forall \alpha \in [0, 1]. \quad (60)$$

Properties and examples of the flux function defined by (58)–(60) are studied in Section 4. We can now state hydrodynamic limit and local equilibrium results for the multilane ASEP. We consider the conservation law

$$\begin{aligned} \partial_t u + \partial_x G(u) &= 0 \\ u(x, 0) &= u_0(x). \end{aligned} \quad (61)$$

with  $G$  given by (59). An important particular case is the *Riemann* initial conditions, that is (cf. (41))

$$u_0(x) = R_{\alpha, \beta}(x), \quad x \in \mathbb{R}. \quad (62)$$

**Theorem 3.1.** Let  $u_0(\cdot)$  be a  $[0, n]$ -valued Borel function on  $\mathbb{R}$ , and  $(\eta_t^N)_{t \geq 0, N \in \mathbb{N}^*}$  be a sequence of processes with generator  $L^{\theta(N)/N}$ . Then the following statements hold under assumption (52):

(i) Assume  $(\bar{\eta}_0^N)_{N \in \mathbb{N}^*}$  has density profile  $u_0(\cdot)$ . Then  $(\bar{\eta}_{Nt}^N)_{N \in \mathbb{N}^*}$  has density profile  $u(\cdot, t)$ , the entropy solution at time  $t$  to the conservation law (61).

(ii) Assume  $(\eta_0^N)_{N \in \mathbb{N}^*}$  is a local Gibbs state with lane profiles  $u_0^0(\cdot), \dots, u_0^{n-1}(\cdot)$ . Then  $(\eta_{Nt}^N)_{N \in \mathbb{N}^*}$  satisfies weak local equilibrium with lane profiles  $u^0(\cdot, t), \dots, u^{n-1}(\cdot, t)$  defined from  $u(\cdot, t)$  as in (37).

(iii) Assume  $\eta_0^N \sim \mu_{\alpha, \beta}$  for some  $0 \leq \alpha, \beta \leq 1$ . Then  $(\eta_{Nt}^N)_{N \in \mathbb{N}^*}$  satisfies strong local equilibrium with lane profiles  $u^0(\cdot), \dots, u^{n-1}(\cdot)$  defined by (37).

The hydrodynamic profiles  $u^i(\cdot, t)$  of individual lanes in statements (ii)–(iii) of Theorem 3.1 do not obey an autonomous equation. We next show that they can be interpreted as *relaxation limits* for a hyperbolic system of scalar conservation laws with so-called *weak coupling*.

### 3.3 Relaxation limit

When the lanes are not coupled by a vertical kernel, by Remark 2.1, the hydrodynamic limit of the system is given by a *system of uncoupled* conservation laws:

$$\begin{cases} \partial_t \rho_0 + \partial_x [\gamma_0 \rho_0 (1 - \rho_0)] & = 0 \\ & \vdots \\ \partial_t \rho_{n-1} + \partial_x [\gamma_{n-1} \rho_{n-1} (1 - \rho_{n-1})] & = 0 \end{cases} \quad (63)$$

where the entropy solution is picked for each of these equations. Indeed the  $i$ -th equation in (63) was shown ([25]) to be the hydrodynamic limit for a simple exclusion process on  $\mathbb{Z}$  with mean drift  $\gamma_i$ .

We can view the vertical dynamics as adding creation/annihilation terms on each lane. This can be formally understood as a relaxation system obtained from (63) by adding fast balance terms on the right-hand side, whose equilibrium manifold is exactly  $\mathcal{F}$  defined by the relations (28).

Indeed, using the generator (54) defined from (22)–(23), we see that the time evolutions of the expected densities on each lane are given by, for  $i \in W$ ,

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[\eta_{Nt}^i(x)] &= N \mathbb{E} [d_i \eta_{Nt}^i(x-1)(1 - \eta_{Nt}^i(x)) - l_i \eta_{Nt}^i(x)(1 - \eta_{Nt}^i(x-1))] \\ &\quad - N \mathbb{E} [d_i \eta_{Nt}^i(x)(1 - \eta_{Nt}^i(x+1)) - l_i \eta_{Nt}^i(x+1)(1 - \eta_{Nt}^i(x))] \\ &\quad + \theta(N) \mathbb{E} \sum_{j \in W} [q(j, i) \eta_{Nt}^j(x)(1 - \eta_{Nt}^i(x)) - q(i, j) \eta_{Nt}^i(x)(1 - \eta_{Nt}^j(x))]. \end{aligned} \quad (64)$$



In (64), the first two lines are produced by the horizontal part of the dynamics, that is ASEP on each lane, and the third line by the vertical part.

Following the usual heuristic in hydrodynamic limits, if we believe *a priori* that the measures (27) describe local equilibrium states, we replace the above expectations with expectations computed under these measures, and approximate the lattice gradients with continuous space gradients. This yields formally the following relaxation system for  $i = 0, \dots, n-1$ :

$$\partial_t \rho_i + \gamma_i \partial_x [\rho_i(1 - \rho_i)] = \theta(N) \sum_{j \in W} [q(j, i) \rho_j(1 - \rho_i) - q(i, j) \rho_i(1 - \rho_j)] \quad (65)$$

where (forgetting about the particle system)  $\theta(N)$  now plays the role of the relaxation parameter. It is important to note that the interlane jumps do not produce a lattice gradient (it should be viewed as a creation rather than a transport term), hence the time scaling of order  $\theta(N)$  does not disappear from the right-hand side of (65). Adding the equations in (65) shows that

$$\rho := \sum_{i \in W} \rho_i \quad (66)$$

is a conserved quantity, which satisfies the conservation law

$$\partial_t \rho + \partial_x \sum_{i \in W} \gamma_i \rho_i(1 - \rho_i) = 0. \quad (67)$$

Note that for finite  $N$ , (67) is not a closed equation, since the flux

$$\sum_{i \in W} \gamma_i \rho_i(1 - \rho_i) = \sum_{i \in W} \gamma_i G_0(\rho_i) \quad (68)$$

in (67) cannot be expressed as a function of  $\rho$ . However, it is expected that in the limit  $N \rightarrow +\infty$ ,  $(\rho_0, \dots, \rho_{n-1})$  converges to the so-called *equilibrium manifold* defined by declaring that each right-hand side should vanish. This is justified heuristically by the fact that otherwise the left-hand side would become infinite in the limit. On this manifold, that will turn out to be exactly  $\mathcal{F}$  defined by the relations (28), the densities  $\rho_0, \dots, \rho_{n-1}$  are (see Lemma 2.1) functions  $\rho_i = \tilde{\rho}_i(\rho)$  of the total density. Thus the flux (68) is exactly equal to (59).

In the next results, we investigate the relaxation limit for (65) and show that the limiting global density is governed by the same conservation law as the hydrodynamic limit in Theorem 3.1. Moreover, we prove that in the limit the  $\rho_i$ 's coincide with the individual lane densities in Theorem 3.1.

We begin by recalling from [15] the definition and properties of entropy solutions for a general hyperbolic system with weak coupling, namely

$$\partial_t \rho_i + \partial_x [f_i(\rho_i)] = c_i(\rho_0, \dots, \rho_{n-1}) \quad (69)$$

where the flux functions  $f_i : [0, 1] \rightarrow \mathbb{R}$ , and the reaction terms  $c_i : [0, 1]^n \rightarrow \mathbb{R}$ ,  $i = 0, \dots, n-1$ , are functions of class  $C^1$ . We call *entropy* a convex function

$h \in C^1([0, 1], \mathbb{R})$ . The  $i$ -th entropy flux associated to  $h$  (defined modulo a constant) is  $g_i(\rho) = \int h'(\rho) f'_i(\rho) d\rho$ . We then say  $(h, g_i)$  is an entropy-flux pair for the  $i$ -th equation.

**Definition 3.1.** Let be given  $\rho_i^0 \in L^\infty([0, +\infty) \times \mathbb{R}; [0, 1])$  for  $i = 0, \dots, n-1$ . The family of functions  $\rho^i(\cdot, \cdot) \in L^\infty([0, +\infty) \times \mathbb{R}; [0, 1])$ , where  $i = 0, \dots, n-1$ , is an entropy solution to (69) if and only if the following conditions are satisfied for every  $i = 0, \dots, n-1$ :

1. (Entropy conditions) For every entropy  $h : [0, 1] \rightarrow \mathbb{R}$  with associated entropy flux  $g_i$ ,

$$\partial_t h[\rho_i(t, x)] + \partial_x g_i[\rho_i(t, x)] \leq h'[\rho_i(t, x)] c_i[\rho_0(t, x), \dots, \rho_{n-1}(t, x)] \quad (70)$$

in the sense of distributions on  $[0, +\infty) \times \mathbb{R}$ .

2. (Initial conditions) For every  $a < b$  in  $\mathbb{R}$ ,

$$\lim_{t \rightarrow 0} \int_a^b |\rho_i(t, x) - \rho_i^0(x)| = 0. \quad (71)$$

**Remark 3.2.** As in the case of a single scalar conservation law, it is equivalent to impose the entropy conditions (70) for  $C^1$  entropies  $h$ , or to impose it for one of the following families of Kruřkov entropy pairs ([19, 28]):

$$h_c(\rho) = |\rho - c|, \quad g_{i,c}(\rho) = \text{sgn}(\rho - c)[f_i(\rho) - f_i(c)], \quad c \in [0, 1], \quad (72)$$

$$h_{c+}(\rho) = (\rho - c)^+, \quad g_{i,c+}(\rho) = \mathbf{1}_{\{\rho > c\}}[f_i(\rho) - f_i(c)], \quad c \in [0, 1], \quad (73)$$

$$h_{c-}(\rho) = (\rho - c)^-, \quad g_{i,c-}(\rho) = -\mathbf{1}_{\{\rho < c\}}[f_i(\rho) - f_i(c)], \quad c \in [0, 1]. \quad (74)$$

Indeed, each of these families can be obtained by regularization of smooth convex entropies, and conversely generates by mixture all convex entropies (up to linear functions).

**Remark 3.3.** Specializing (70)–(71) to the case  $n = 1$ ,  $c_0 = 0$ ,  $f_0 = f$ ,  $\rho_0(\cdot, \cdot) = \rho(\cdot, \cdot)$ ,  $\rho_0^0(\cdot) = \rho^0(\cdot)$ , we recover the usual definition of entropy solutions to the scalar conservation law

$$\partial_t \rho(t, x) + \partial_x [f(\rho(t, x))] = 0. \quad (75)$$

We now state the relaxation and equilibrium limit for (65) (in fact, a slightly more general version thereof based on (69)). Recall Assumptions 2.1 and 2.2.

**Theorem 3.2.** Let  $\rho_\varepsilon(\cdot, \cdot) = [\rho_{0,\varepsilon}(\cdot, \cdot), \dots, \rho_{n-1,\varepsilon}(\cdot, \cdot)]$  denote the entropy solution to

$$\partial_t \rho_i + \partial_x [f_i(\rho_i)] = \varepsilon^{-1} c_i(\rho_0, \dots, \rho_{n-1}), \quad i = 0, \dots, n-1 \quad (76)$$

with initial data  $\rho^0(\cdot) = [\rho_0^0(\cdot), \dots, \rho_{n-1}^0(\cdot)]$ , where  $c_i$  is of the form

$$c_i(\rho) = \sum_{j=0}^{n-1} [c_{ji}(\rho) - c_{ij}(\rho)] \quad (77)$$

with

$$c_{ij}(\rho) = q(i, j)\rho_i(1 - \rho_j) \quad (78)$$

where we assume that the kernel  $q(\cdot, \cdot)$  on  $\{0, \dots, n-1\}$  satisfies assumption 2.1, (ii) and assumption 2.2. Define the relaxation flux function  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(\rho) := \sum_{i=0}^{n-1} f_i[\tilde{\rho}_i(\rho)] \quad (79)$$

where  $\tilde{\rho}_i(\rho)$  is defined in Proposition 2.1. Then, as  $\varepsilon \rightarrow 0$ :

(i) (Relaxation limit) The global density field

$$R_\varepsilon(t, x) := \sum_{i=0}^{n-1} \rho_{i,\varepsilon}(t, x) \quad (80)$$

converges in  $L^1_{\text{loc}}((0, +\infty) \times \mathbb{R})$  to  $\rho(\cdot, \cdot)$  defined as the entropy solution to (75) with initial data

$$R^0(x) := \sum_{i=0}^{n-1} \rho_i^0(x). \quad (81)$$

(ii) (Equilibrium limit). For each  $i = 0, \dots, n-1$ ,  $\rho_{i,\varepsilon}(\cdot, \cdot)$  converges in  $L^1_{\text{loc}}((0, +\infty) \times \mathbb{R})$  to  $\rho_i(\cdot, \cdot)$  defined by

$$\rho_i(t, x) = \tilde{\rho}_i[\rho(t, x)]. \quad (82)$$

## 4 Examples and a singular limit

In order to state examples and some properties, it is convenient to reformulate Definition 2.2 of weak irreducibility in an equivalent way. The following Lemma is proved in Section 5.

**Lemma 4.1.** *Let  $\pi(\cdot, \cdot)$  be the transition kernel for a Markov jump process on a finite set  $S$ . Then the following statements are equivalent:*

(1) *The kernel  $\pi(\cdot, \cdot)$  is weakly irreducible.*

(2) *There is a unique labeling  $(\Gamma_\alpha)_{\alpha=0, \dots, m-1}$  of the irreducibility classes of  $\pi(\cdot, \cdot)$  that satisfies the following property: for every  $\alpha \in \{0, \dots, m-2\}$ , there exists  $i \in \Gamma_\alpha$  and  $j \in \Gamma_{\alpha+1}$  such that  $\pi(i, j) > 0$ .*

*Then,  $\Gamma_{m-1}$  is the unique recurrent class.*

Denote by  $(\mathcal{C}_\alpha)_{\alpha=0, \dots, m-1}$ , the labeling of the irreducibility classes of  $q(\cdot, \cdot)$  induced by condition (ii) of Assumption 2.1 and Lemma 4.1. We arbitrarily select

a nonzero reversible measure on  $\mathcal{C}_\alpha$  and denote it by  $\lambda^\alpha$ . For  $\alpha = 0, \dots, m-1$ , let us define

$$n_\alpha := |\mathcal{C}_\alpha|, \quad N_\alpha := \sum_{\beta=\alpha+1}^{m-1} n_\beta \quad (83)$$

to be, respectively, the number of lanes in class  $\mathcal{C}_\alpha$ , and the number of lanes ahead of this class. In particular,  $N_{m-1} = 0$ ; by extension we set  $N_{-1} = n$ . Lane  $i \in \{0, \dots, n-1\}$  belongs to class  $\mathcal{C}_\alpha$  for  $\alpha \in \{0, \dots, m-1\}$  if and only if  $i \in \{n - N_{\alpha-1} + 1, \dots, n - N_\alpha\}$ .

We next illustrate condition (ii) of Assumption 2.1 and Assumption 2.2 for the ‘‘vertical’’ (that is, interlane) jump kernel.

**Example 4.1.** Consider the two-lane SEP, that is  $W = \{0, 1\}$  and transition probability

$$q(0; 1) = p, \quad q(1; 0) = q, \quad (84)$$

with  $p, q \geq 0$  and  $p + q > 0$ .

(i) If  $pq > 0$ , then  $m = 1$ ,  $\mathcal{C}_0 = \{0, 1\}$ ,  $n_0 = 2$ ,  $N_0 = 0$ ,  $N_{-1} = 2$ . A reversible measure is given by  $\lambda_0^\alpha = 1$ ,  $\lambda_1^\alpha = \frac{p}{q}$ .

(ii) If  $q = 0 < p$ , then  $m = 2$ ,  $\mathcal{C}_0 = \{0\}$ ,  $\mathcal{C}_1 = \{1\}$ ,  $n_0 = n_1 = 1$ ,  $N_1 = 0$ ,  $N_0 = 1$ ,  $N_{-1} = 2$ . For  $\alpha \in \{0, 1\}$  and the unique element  $i$  of  $\mathcal{C}_\alpha$ , any value  $\lambda_i^\alpha > 0$  yields a nonzero reversible measure on  $\mathcal{C}_\alpha$ .

For three lanes, we have the following possibilities (up to a permutation of lanes).

**Example 4.2.** Assume  $W = \{0, 1, 2\}$ ,  $q(0, 1) = p > 0$ ,  $q(1, 0) = q \geq 0$ ,  $q(1, 2) = r > 0$ ,  $q(2, 1) = s \geq 0$ ,  $q(0, 2) = t \geq 0$ ,  $q(2, 0) = u \geq 0$ .

(i) If  $q = u = s = 0$ , then  $m = 3$ ,  $\mathcal{C}_\alpha = \{\alpha\}$  for  $\alpha \in \{0, 1, 2\}$ ,  $n_0 = n_1 = n_2 = 1$ ,  $N_2 = 0$ ,  $N_1 = 1$ ,  $N_0 = 2$ ,  $N_{-1} = 3$ . Any constant  $\lambda^\alpha > 0$  yields a nonzero reversible measure on  $\mathcal{C}_\alpha$ .

(ii) If  $u = s = 0 < q$ , then  $m = 2$ ,  $\mathcal{C}_0 = \{0, 1\}$ ,  $\mathcal{C}_1 = \{2\}$ ,  $n_0 = 2$ ,  $n_1 = 1$ ,  $N_1 = 0$ ,  $N_0 = 1$ ,  $N_{-1} = 3$ . Assumption 2.2 holds for instance with  $\lambda_0^0 = 1$ ,  $\lambda_1^0 = \frac{p}{q}$  and  $\lambda_2^1 = 1$ .

(iii) If  $q > 0$  and  $s > 0$ , then  $m = 1$ ,  $\mathcal{C}_\alpha = \{0, 1, 2\}$ ,  $N_0 = 0$ ,  $N_{-1} = 3$ . Assumption 2.2 holds if and only if

$$t = u = 0, \quad \text{or} \quad u \neq 0 \quad \text{and} \quad \frac{t}{u} = \frac{p r}{q s}. \quad (85)$$

In this case, a reversible measure is given by

$$\lambda_0^0 = 1, \quad \lambda_1^0 = \frac{p}{q}, \quad \lambda_2^0 = \frac{p r}{q s} = \frac{t}{u}. \quad (86)$$

**Example 4.3.** We can generalize Example 4.1 to  $n$  lanes, with  $W = \{0, \dots, n-1\}$ , and the kernel  $q(\cdot, \cdot)$  given by

$$q(i, j) = \begin{cases} 0 & \text{if } |i - j| \neq 1 \\ p & \text{if } j = i + 1 < n \\ q & \text{if } j = i - 1 > 0 \end{cases} \quad (87)$$

with  $p, q \geq 0$  and  $p + q > 0$ . In other words, considering  $\mathbb{L}_0$  as the top lane and  $\mathbb{L}_{n-1}$  as the bottom lane,  $p$  is the rate at which particles go down and  $q$  the rate of going up. Without loss of generality, let us consider  $p > 0$  and  $q \geq 0$ . Then there are two cases:

Case 1. If  $q > 0$ , then  $m = 0$  and  $\mathcal{C}_0 = W$ . A reversible measure on  $\mathcal{C}_0$  is

$$\lambda_i^0 = \left(\frac{p}{q}\right)^i, \quad i \in \mathcal{C}_0. \quad (88)$$

Case 2. If  $q = 0$ , then  $m = n$  and  $\mathcal{C}_i = \{i\}$  for  $i \in W$ . A reversible measure on  $\mathcal{C}_i$  is any positive constant value associated with  $i$ .

In the following example the kernel  $q(\cdot, \cdot)$  is irreducible and Assumption 2.2 is satisfied. In the traffic interpretation, it could model a highway interchange.

**Example 4.4.** Let  $W$  be the set of vertices of a tree. If  $i, j \in W$ , we write  $i \sim j$  if and only if  $i$  and  $j$  are connected by an edge of the tree. We assume that the kernel  $q(\cdot, \cdot)$  in (21) satisfies

$$q(i, j) > 0 \text{ if and only if } i \sim j. \quad (89)$$

*Proof of Assumption 2.2.* For  $i, j \in W$  and  $i \sim j$ , we define

$$r(i, j) := \frac{q(i, j)}{q(j, i)}, \quad i, j \in W. \quad (90)$$

We extend this definition to an arbitrary  $(i, j) \in W^2$  by setting

$$r(i, j) = \prod_{k=0}^{l-1} r(i_k, i_{k+1}) \quad (91)$$

where  $(i = i_0, \dots, i_l = j)$  is the unique path from  $i$  to  $j$  on the tree. Fix an arbitrary vertex  $i^* \in W$  (for convenience the reader may think of  $i^*$  as the root of the tree, but in fact we do not need the tree to be rooted). For every  $\lambda_{i^*} \geq 0$ , define a family  $(\lambda_i)_{i \in W}$  by the relation

$$\lambda_i = \lambda_{i^*} r(i^*, i) \quad (92)$$

where  $(i_0 = i^*, \dots, i_l = i)$  is the unique path connecting  $i^*$  to  $i$ . Then  $\lambda$  satisfies (25) for all  $i, j \in W$ . Conversely, any  $\lambda$  satisfying (25) for all  $i, j \in \{0, \dots, n-1\}$  is of the form (92).  $\square$

In relation to (28)–(29), we now define a mapping  $\phi_r$  from  $[0, 1]$  to  $[0, 1]$  by

$$\phi_r(\rho) := \frac{r\rho}{1 - \rho + r\rho}, \quad \forall \rho \in [0, 1]. \quad (93)$$

Note that  $\phi_r$  is an increasing continuous bijection, and

$$\phi_{r'} \circ \phi_r = \phi_{rr'}, \quad \phi_{1/r} = \phi_r^{-1}. \quad (94)$$

Defining again  $r(i, j)$  by (90) whenever  $q(j, i) > 0$ , the relation (28) becomes

$$\rho_j = \phi_{r(i, j)}(\rho_i). \quad (95)$$

By (94) and (91),

$$\phi_{r(i, j)} = \phi_{r(i_{m-1}, i_m)} \circ \cdots \circ \phi_{r(i_0, i_1)}. \quad (96)$$

Then, we have

$$\mathcal{F} := \{(\rho_i)_{i \in W} \in [0, 1]^W : \forall i \in W, \rho_i = \phi_{r(i^*, i)}(\rho_{i^*})\}. \quad (97)$$

Equation (97) gives an explicit construction of  $\mathcal{F}$  as indexed by the density at some “reference” vertex  $i^*$ . Properties (94) and (96) imply that the set  $\mathcal{F}$  given by (97) does not depend on the choice of  $i^*$ .

#### 4.1 The shape of the flux function and a phase transition

The following proposition, proved in Section 5, states some properties of the flux function. Recall the notation introduced in (83).

**Proposition 4.1.**

(i) For each  $\alpha \in \{0, \dots, m-1\}$ , the restriction of  $G$  to  $[N_\alpha, N_{\alpha-1}]$  is continuously differentiable and depends only on the restriction of  $q(\cdot, \cdot)$  to  $\mathcal{C}^\alpha \times \mathcal{C}^\alpha$ .

(ii) The flux function  $G$  has the following one-sided derivatives at  $N_\alpha$ : for  $-1 \leq \alpha < m-1$ , resp.  $-1 < \alpha \leq m-1$ , it holds that

$$G'(N_\alpha-) = - \sum_{i \in \mathcal{C}_{\alpha+1}} \gamma_i f_i^{\alpha+1}, \quad \text{resp.} \quad G'(N_\alpha+) = \sum_{i \in \mathcal{C}_\alpha} \gamma_i e_i^\alpha \quad (98)$$

where

$$e_i^\alpha := \frac{\lambda_i^\alpha}{\lambda_j^\alpha}, \quad f_i^\alpha := \frac{\lambda_j^\alpha}{\lambda_i^\alpha}. \quad (99)$$

(iii) Let  $\alpha \in \{0, \dots, m-1\}$ . If there exists  $i \in \mathcal{C}_\alpha$  such that

$$\sum_{j \in \mathcal{C}_\alpha: \lambda_j^\alpha = \lambda_i^\alpha} \gamma_j \neq 0, \quad (100)$$

then the derivative of  $G$  vanishes finitely many times on  $[N_\alpha, N_{\alpha-1}]$ . Otherwise,  $G$  is identically 0 on  $[N_\alpha, N_{\alpha-1}]$ .

**Remark 4.1.**

1. It follows from (ii) of Proposition 4.1 that if lanes in classes  $\alpha$  and  $\alpha+1$  have either all nonnegative or all nonpositive drifts (with the two classes on the same side), and at least one lane in at least one of these classes has nonzero drift, then  $G$  is not differentiable at  $N_\alpha$ . This generalizes the non-differentiability of the flux in Example 4.5 below.

2. By condition (25), level sets of  $\lambda^\alpha$  are singletons or maximal subsets of  $\mathcal{C}_\alpha$  on which  $q(\cdot, \cdot)$  is symmetric. Condition (100) can be rephrased saying that the total drift on at least one such subset is nonzero.

3. Proposition 4.1, (iii) implies that for a given integer  $k \in \{1, \dots, n\}$ , the equation  $G(\rho + k) - G(\rho) = 0$  (with unknown  $\rho \in [0, n]$ ) has finitely many solutions. This implies that there are only finitely many possible shocks of integer amplitude for the hydrodynamic equation. This property plays an important role to extend the arguments and results of [1, Theorem 2.2], and prove that up to horizontal translations, there are only finitely many measures in  $\mathcal{I}_e \setminus (\mathcal{I} \cap \mathcal{S})_e$ .

We illustrate Theorem 3.1 and Proposition 4.1 by computing flux functions for our examples. We begin with the two-lane model of Example 4.1 that we develop later in this section. The following computation is taken from [1, Section 5.2], where  $r := q/p \in [0; +\infty]$  (if  $q = 0$  the expressions below must be taken in the sense of their limit as  $r \rightarrow +\infty$ ).

$$G(\rho) = (\gamma_0 + \gamma_1) \frac{\rho}{2} \left(1 - \frac{\rho}{2}\right) + (\gamma_0 - \gamma_1)(1 - \rho)\varphi(\rho) - (\gamma_0 + \gamma_1)\varphi(\rho)^2 \quad (101)$$

with

$$\begin{aligned} \varphi(\rho) &= \frac{1}{2} \left( \frac{r+1}{r-1} \right) \left( 1 - \sqrt{\psi(\rho)} \right) \text{ if } r \neq 1 \\ &= 0 \text{ if } r = 1 \end{aligned} \quad (102)$$

$$\psi(\rho) = 1 + \left( \frac{r-1}{r+1} \right)^2 \rho(\rho-2). \quad (103)$$

In particular,  $G$  depends on  $p, q$  only through  $r$ . To emphasize dependence of  $G$  on the parameters of the model, whenever necessary, we write  $G = G_{\gamma_0, \gamma_1, r}$ . The following properties, taken from [1, Proposition 4.8 and Section 5.2], are useful to reduce the number and range of parameters of the model.

**Proposition 4.2.** *The function  $G_{\gamma_0, \gamma_1, r}$  satisfies the following:*

1. *Symmetry:*

$$G_{\gamma_0, \gamma_1, r}(2 - \rho) = G_{\gamma_1, \gamma_0, r}(\rho) = G_{\gamma_0, \gamma_1, r^{-1}}(\rho). \quad (104)$$

2. *Homogeneity: if  $\gamma_0 + \gamma_1 \neq 0$ ,*

$$G_{\gamma_0, \gamma_1, r}(\rho) = (\gamma_0 + \gamma_1) G_{\frac{\gamma_0}{\gamma_0 + \gamma_1}, \frac{\gamma_1}{\gamma_0 + \gamma_1}, r}(\rho). \quad (105)$$

3. Sign: if  $\gamma_0\gamma_1 < 0$ ,  $G$  vanishes on  $(0; 2)$  if and only if

$$\min(q\gamma_0 + p\gamma_1; p\gamma_0 + q\gamma_1) < 0. \quad (106)$$

In this case, the zero is unique,  $G$  has a single minimum and a single maximum, of opposite signs, on  $(0; 2)$ .

In the following special cases, the function  $G$  has a simple explicit expression.

**Example 4.5.** Assume  $p = 0 < q$ . Then

$$G(\rho) = \begin{cases} \gamma_0\rho(1 - \rho) & \text{if } \rho \in [0, 1] \\ \gamma_1(\rho - 1)(2 - \rho) & \text{if } \rho \in (1, 2]. \end{cases} \quad (107)$$

In particular, when  $\gamma_0 = \gamma_1$ , the flux is a function of period 1 whose restriction to  $[0, 1]$  is the TASEP flux. Note that there is a point of non-differentiability ( $\rho = 1$ ), a property not seen in usual single-lane models with product invariant measures. Moreover, if  $\gamma_0\gamma_1 = 0$ , the flux function is strictly concave on  $[0; 1]$  or  $[1; 2]$  and identically zero on its complement: similar unusual behaviour has been established recently for the asymmetric facilitated exclusion process ([14]).

**Example 4.6.** Assume  $p = q > 0$ . Then

$$G(\rho) = \frac{\gamma_0 + \gamma_1}{4}\rho(2 - \rho). \quad (108)$$

Here, unless  $\gamma_0 + \gamma_1 = 0$ , the flux has the same shape as the single-lane TASEP flux (from which it is obtained by a scale change). It is in particular strictly concave.

**Remark 4.2.** In cases  $\gamma_0 = \gamma_1 = 0$  of Example 4.5, and  $\gamma_0 + \gamma_1 = 0$  of Example 4.6, the flux function is identically 0, and the entropy solution to (61) does not evolve in time. In these situations, the hyperbolic time scale is irrelevant, and one expects a nontrivial diffusive hydrodynamic limit to arise under diffusive time scaling.

Following are examples beyond the two-lane model.

**Example 4.7.** Consider case (ii) in Example 4.2. Then

$$G(\rho) = \begin{cases} \gamma_2\rho(1 - \rho) & \text{if } \rho \in [0, 1] \\ G_{\gamma_0, \gamma_1, q/p}(\rho - 1) & \text{if } \rho \in [1, 3] \end{cases} \quad (109)$$

where the second line refers to the two-lane flux. For  $\alpha = 0$ , unless  $p = q$  and  $\gamma_1 = -\gamma_0$ , condition (100) holds for the flux function on  $[N_\alpha, N_{\alpha-1}] = [1, 3]$ .

**Example 4.8.** Consider Example 4.3 with  $p = 1$ ,  $q = 0$  and a TASEP from left to right on each lane, that is  $0 = l_i < d_i$ . In this case we have

$$\tilde{\rho}_i(\rho) = (\rho - i)\mathbf{1}_{[i, i+1]}(\rho) + \mathbf{1}_{\{i+1 < \rho\}} \quad (110)$$



so that we obtain the flux function

$$G^n(\rho) = \sum_{i=0}^{n-1} d_i(\rho - i)(i + 1 - \rho)\mathbf{1}_{(i,i+1)}(\rho) \quad (111)$$

that consists of a succession of hills of height  $d_i$  separated by valleys.

For the model of Example 4.3, in view of (110), part (ii) of Theorem 3.1 says that the density on lane  $i = \lfloor u(x, t) \rfloor$  is  $u(x, t) - i$ , lanes with lower index are full and lanes with higher index are empty. Thus all the current is carried out by the lane of index  $\lfloor u(x, t) \rfloor$ .

We now come back to the two-lane model. The following theorem yields precise information on the shape of the flux function for the two-lane model (84) and shows in particular that phase transitions occur according to the parameters of the model. We prove that the flux function  $G$  may have either zero, one or two inflexion points, and possibly change sign at critical values that can be expressed explicitly.

**Theorem 4.1.** *We assume that  $d \in [1/2; +\infty)$  and  $r \in [1; +\infty)$ . Then the number of inflexion points of  $G = G_{d,1-d,r}$  is given by the following phase diagram.*

Let

$$\frac{1}{2} < \tilde{d}_1 := \frac{1}{2} + \frac{1}{6}\sqrt{3 + 2\sqrt{3}} < \tilde{d}_0 := \frac{1}{2} + \frac{\sqrt{3}}{4} < \bar{d}_1 := \frac{1}{2} + \frac{1}{4}\sqrt{2\sqrt{3}}. \quad (112)$$

Then there exist functions of  $d$

$$\bar{r}_1 : [1/2; \bar{d}_1] \rightarrow [1; +\infty), r_3 : [\tilde{d}_1; +\infty) \rightarrow [1; +\infty), r_4 : [\tilde{d}_1; 1) \rightarrow [1; +\infty) \quad (113)$$

with the following properties:

1, (i):  $r_4$  is increasing in  $d$ ,  $\bar{r}_1$  and  $r_3$  are decreasing in  $d$ , and all these three functions are continuous; besides,

$$\lim_{d \rightarrow 1^-} r_4(d) = +\infty. \quad (114)$$

1, (ii): It holds that

$$\begin{aligned} d = \tilde{d}_0 &\Rightarrow \bar{r}_1(d) = r_3(d) \\ d \neq \tilde{d}_0 &\Rightarrow \bar{r}_1(d) < r_3(d). \end{aligned}$$

2, (i): For  $d \in [1/2; \tilde{d}_1]$ : if  $r \leq \bar{r}_1(d)$ ,  $G$  is strictly concave; if  $r > \bar{r}_1(d)$ ,  $G$  has two inflexion points;

2, (ii): For  $d \in [\tilde{d}_1; \tilde{d}_0]$ : if  $r \leq \bar{r}_1(d)$ ,  $G$  is strictly concave; if  $\bar{r}_1(d) < r < r_3(d)$ ,

or  $r > r_4(d)$ ,  $G$  has two inflexion points; if  $r_3(d) \leq r \leq r_4(d)$ ,  $G$  has a single inflexion point;

2, (iii): For  $d \in [\tilde{d}_0; 1)$ : if  $r \leq r_3(d)$ ,  $G$  is strictly concave; if  $r_3(d) < r < r_4(d)$ ,  $G$  has a single inflexion point; if  $r > r_4(d)$ ,  $G$  has two inflexion points.

2, (iv): For  $d = 1$ : if  $r \leq r_3(1) = 1 + \sqrt{2}$ ,  $G$  is strictly concave; if  $r > r_3(d)$ ,  $G$  has a single inflexion point.

2, (v): For  $d \in (1; +\infty)$ : if  $r \leq r_3(d)$ ,  $G$  is positive on  $(0; 2)$  and strictly concave; if  $r_3(d) < d \leq \frac{d}{d-1}$ ,  $G$  is positive on  $(0; 2)$  and has a single inflexion point; if  $r > \frac{d}{d-1}$ ,  $G$  has a single inflexion point, a single zero on  $(0; 2)$ , is positive with a single local maximum on the left of this zero, negative with a single local minimum on the right.

Graphical illustrations of the above theorem, and interpretations thereof, are given in Appendix A.

**Remark 4.3.**

(i) By the symmetry and homogeneity properties in Proposition 4.2, the above theorem yields the number of inflexion points of  $G_{\gamma_0, \gamma_1, r}$  for all values of  $\gamma_0 \in \mathbb{R}$ ,  $\gamma_1 \in \mathbb{R}$  and  $r > 0$  such that  $(\gamma_0, \gamma_1) \neq (0, 0)$ . The case  $\gamma_0 + \gamma_1 = 0$  is attained by the limit

$$\lim_{d \rightarrow +\infty} \frac{1}{d} G_{d, 1-d, r} = G_{1, -1, r}. \quad (115)$$

(ii) The value of  $\bar{r}_1(d)$  is explicit: cf. (163)–(165). The values of  $r_3(d)$  and  $r_4(d)$  could be written explicitly as roots of a cubic equation, but we omit the complicated formulas which would shed no additional light on our results.

(iii) Cases 2, (iv)–(v) are (as regards inflexion points) a natural extension of 2, (iii) and could be included in the latter using the extension  $r_4(d) = +\infty$  for  $d \in [1; +\infty)$ .

## 4.2 Many-lane limit

In this subsection, we investigate a singular behaviour of this model as the number of lanes  $n \rightarrow +\infty$  and the density is properly renormalized. We consider the  $n$ -lane model of Example 4.3. For each  $n$ , we consider a sequence  $(\eta^{n, N})_N$  of such processes. Since the density range for the total density is  $[0, n]$ , to have a fixed range  $[0, 1]$ , we consider for (16) the empirical measure renormalized by the number of lanes:

$$\hat{\alpha}^N (\bar{\eta}^{n, N}, dx) := n^{-1} \alpha^N (\bar{\eta}^{n, N}, dx). \quad (116)$$

For each  $n$ , for a given Cauchy datum  $u_0^n$ , by Theorem 3.1, the unnormalized empirical measure  $\alpha_{Nt}^{n, N}$  converges as  $N \rightarrow +\infty$  to  $u^n(\cdot, t)$ , the entropy solution

at time  $t$  of (61) with a flux function  $G^n$  defined on  $[0, n]$  by (111). The normalized measure (116) converges the normalized profile  $\hat{u}^n(\cdot, t) := n^{-1}u^n(\cdot, t)$ . The latter is the entropy solution to the normalized conservation law

$$\partial_t u + \partial_x \widehat{G}^n(u) = 0 \quad (117)$$

where the normalized flux function

$$\widehat{G}^n(u) := n^{-1}G^n(nu) \quad (118)$$

is now defined on  $[0, 1]$ . Let  $F \in C^1[0, 1]$  be positive on  $(0, 1)$  with  $F(0) = F(1) = 0$ . In Example 4.3, we consider

$$d_i = 4nF(i/n). \quad (119)$$

The resulting normalized flux

$$\widehat{G}^n(\rho) = 4 \sum_{i=0}^{n-1} \mathbf{1}_{(i/n, (i+1)/n)}(\rho) F(i/n) (n\rho - i)(i + 1 - n\rho) \quad (120)$$

is a highly oscillating function (as the hills now have width  $1/n$ ) and the height of the  $i$ -th local maxima is  $F(i/n)$ . Thus in the maximal current regime  $\widehat{G}^n$  approximates  $F$ . We are interested in the behavior of  $\hat{u}^n$  as  $n \rightarrow +\infty$  when the normalized initial condition  $\hat{u}_0^n$  is fixed. In particular, it is natural to ask if there is any connection with the conservation law

$$\partial_t u + \partial_x F(u) = 0. \quad (121)$$

The following theorem yields an answer for Riemann conditions  $\hat{u}_0$ .

**Theorem 4.2.** *Assume  $\hat{u}_0$  is the Riemann condition (62). Define  $\hat{u}(\cdot, 1)$  as follows: if  $\alpha > \beta$ ,  $\hat{u}(\cdot, 1)$  is the entropy solution at time 1 of the Riemann problem for (121). If  $\alpha \leq \beta$ ,  $\hat{u}(\cdot, 1) = \hat{u}_0(\cdot)$ . For  $v \in \mathbb{R}$ , let  $\widehat{U}(v, 1)$  denote the closed interval with bounds  $\hat{u}(v \pm, 1)$ . Then*

$$\lim_{n \rightarrow +\infty} d[\hat{u}^n(v, 1); \widehat{U}(v, 1)] = 0 \quad (122)$$

*locally uniformly in  $v$ . In particular,  $\hat{u}^n(\cdot, 1)$  converges to  $\hat{u}(\cdot, 1)$  at every point of continuity of the latter.*

Since the Riemann problem is expected to characterize the Cauchy problem, this result suggests the following. If we consider the  $n$ -lane model (87) with  $N \rightarrow +\infty$  and  $n = n(N) \rightarrow +\infty$  with  $n/N \rightarrow 0$ , starting from a Cauchy datum  $\hat{u}_0$  for the normalized measure (116), the latter should converge at time  $Nt$  to the solution  $\hat{u}(\cdot, t)$  of a singular evolution problem. Solutions of this problem should coincide with entropy solutions of (121) for nondecreasing initial data, remain fixed for nonincreasing data, and involve some interaction between these behaviors for general data.

## 5 Proofs of Lemma 4.1, Proposition 2.1 and Proposition 4.1

*Proof of Lemma 4.1.* Let  $(\gamma_\alpha)_{\alpha=0,\dots,m-1}$  denote the irreducibility classes of  $\pi(\cdot, \cdot)$  labeled in an arbitrary way. We write  $\alpha \preceq \beta$  if there is a path from  $\gamma_\alpha$  to  $\gamma_\beta$ ; that is, if there exists a path  $(\alpha = u_0, \dots, u_k = \beta)$  such that for every  $l = 0 \dots, k-1$ , there is a possible transition from class  $\gamma_{u_l}$  to class  $\gamma_{u_{l+1}}$ . When  $\alpha = \beta$ , we consider by convention that both  $\alpha \preceq \beta$  and  $\beta \preceq \alpha$  hold (we consider by extension that there is a path of length one leading from  $\mathcal{C}_\alpha$  to itself). We write  $\alpha \prec \beta$  if  $\alpha \preceq \beta$  and  $\alpha \neq \beta$ . The relation  $\preceq$  is a total order on  $\{0, \dots, m-1\}$ . It follows that the finite set  $\{0, \dots, m-1\}$  is linearly ordered by  $\preceq$ . That is, there exists a (unique) sequence  $(\alpha_0, \dots, \alpha_{m-1})$  such that

$$\alpha_0 \prec \dots \prec \alpha_{m-1} = \alpha^*. \quad (123)$$

Then, for  $k = 0, \dots, m-1$ , we set  $\Gamma_k := \gamma_{\alpha_k}$ .  $\square$

The proof of Proposition 2.1 relies on Lemmas 5.1 and 5.2 below.

### Lemma 5.1.

Let  $\alpha = 0, \dots, m-1$ , and  $\rho = (\rho_i)_{i \in \mathcal{C}_\alpha}$  be a  $[0, 1]$ -valued family. Then  $\rho$  satisfies (28) for all  $i, j \in \mathcal{C}_\alpha$ , if and only if, either  $\rho$  is the constant function with value 1 on  $\mathcal{C}_\alpha$ , henceforth denoted by  $\rho_c^{\alpha, \infty}$ ; or  $\rho$  is of the form

$$\rho_i^{\alpha, c} := \frac{c\lambda_i^\alpha}{1 + c\lambda_i^\alpha} \quad (124)$$

for  $c \geq 0$ .

*Proof.* Assume  $\rho$  satisfies (28) and there exists  $i \in \mathcal{C}_\alpha$  such that  $\rho_i = 1$ . For every  $j \in \mathcal{C}_\alpha$ , by irreducibility, there is a path  $(i_0 = i, \dots, i_{k-1} = j)$  from  $i$  to  $j$ . The relations

$$\rho_{i_k} (1 - \rho_{i_{k+1}}) q(i_k, i_{k+1}) = \rho_{i_{k+1}} (1 - \rho_{i_k}) q(i_{k+1}, i_k)$$

with  $q(i_k, i_{k+1}) > 0$  and  $q(i_{k+1}, i_k) > 0$  imply  $\rho_j = 1$ . Thus  $\rho$  uniformly equals 1. Assume now that  $\rho$  never takes value 1. Then, defining  $\lambda_i$  by

$$\lambda_i := \frac{\rho_i}{1 - \rho_i},$$

(28) is equivalent to (25). Since  $q(\cdot, \cdot)$  restricted to  $\mathcal{C}_\alpha$  is irreducible, any reversible measure on  $\mathcal{C}_\alpha$  is a multiple of  $\lambda^\alpha$ ; thus so is  $\lambda$ .  $\square$

### Lemma 5.2. The following are equivalent:

(i) The vector  $(\rho_0, \dots, \rho_{n-1})$  satisfies (28) for every  $i = 0, \dots, n-1$ ;

(ii) There exists  $\alpha \in \{0, \dots, m-1\}$  such that, for every  $\beta < \alpha$ ,  $\rho_\cdot$  is identically 0 on  $\mathcal{C}_\beta$ , and for every  $\beta > \alpha$ , it is identically 1 on  $\mathcal{C}_\beta$ . For this  $\alpha$ , there exists a unique  $c \in [0, +\infty]$  such that  $\rho_\cdot = \rho_\cdot^{\alpha, c}$  on  $\mathcal{C}_\alpha$ .

Besides, if  $c \in (0, +\infty)$ ,  $\alpha$  is uniquely determined.

*Proof.* Let  $\alpha \in \{0, \dots, m-1\}$  and assume  $\rho_\cdot$  is neither of the constants 0 or 1 on  $\mathcal{C}_\alpha$ . By Lemma 5.1, we have  $\rho_i \in (0, 1)$  for all  $i \in \mathcal{C}_\alpha$ . Assume  $\alpha < \beta$ ,  $i \in \mathcal{C}_\alpha$  and  $j \in \mathcal{C}_\beta$ . Then there is a path  $(i_0 = i, \dots, i_{k-1} = j)$  such that  $q(i_l, i_{l+1}) > 0$  for every  $l = 0, \dots, k-1$ , and  $q(i_{l+1}, i_l) = 0$  whenever  $i_l$  and  $i_{l+1}$  do not belong to the same class.

$$\rho_{i_l} (1 - \rho_{i_{l+1}}) q(i_l, i_{l+1}) = \rho_{i_{l+1}} (1 - \rho_{i_l}) q(i_{l+1}, i_l).$$

It follows that  $\rho_l = 1$  for  $l \geq L$ , where

$$L = \min\{l = 0, \dots, k\} : i_l \notin \mathcal{C}_\alpha\}.$$

Thus  $\rho_\cdot$  is identically 1 on  $\mathcal{C}_\beta$ . A similar argument shows that it is identically 0 on all classes  $\mathcal{C}_\beta$  with  $\beta < \alpha$ . The relation  $\rho_\cdot = \rho_\cdot^{\alpha, c}$  on  $\mathcal{C}_\alpha$  follows from Lemma 5.1. If  $\rho_\cdot$  is neither of the constants 0 or 1 on  $\mathcal{C}_\alpha$ ,  $\alpha$  is uniquely determined by the fact that  $\rho_\cdot$  is identically 0 or 1 on all other classes.  $\square$

*Proof of Proposition 2.1.*

*Proof of (i).* Let  $\rho \in [0, n]$  and assume  $\rho \in [N_{\alpha^*}, N_{\alpha^*-1}]$  for some  $\alpha \in \{0, \dots, m-1\}$ . Note that  $\rho_i^{\alpha^*, c}$  defined by (124) increases from 0 to 1 as  $c$  increases from 0 to  $+\infty$ . Thus there is a unique  $c \in [0, +\infty]$  such that

$$\sum_{i \in \mathcal{C}_{\alpha^*}} \rho_i^{\alpha^*, c} + N_{\alpha^*} = \rho. \quad (125)$$

We then define  $(\rho_0, \dots, \rho_{n-1})$  as follows:

$$\rho_i = \begin{cases} 1 & \text{if } i \in \mathcal{C}_\beta \text{ for } \beta < \alpha^* \\ 0 & \text{if } i \in \mathcal{C}_\beta \text{ for } \alpha^* < \beta \\ \rho_i^{\alpha^*, c} & \text{if } i \in \mathcal{C}_{\alpha^*}. \end{cases} \quad (126)$$

By Lemma 5.2, we have  $(\rho_0, \dots, \rho_{n-1}) \in \mathcal{F}$  (defined in (29)), and (125) implies

$$\sum_{i=0}^{n-1} \rho_i = \rho. \quad (127)$$

We now prove uniqueness of  $(\rho_0, \dots, \rho_{n-1}) \in \mathcal{F}$  satisfying (127). The value  $\alpha$  in Lemma 5.2 is such that

$$\rho \in [N_\alpha, N_{\alpha-1}]. \quad (128)$$

(a) If  $\rho \notin \{N_{\alpha^*-1}, N_{\alpha^*}\}$ , then  $\alpha = \alpha^*$  is uniquely determined. Thus the lemma implies that  $(\rho_0, \dots, \rho_{n-1})$  is given by (126).

(b) If  $\rho = N_{\alpha^*}$ , by (128), we must have  $\alpha = \alpha^*$  or  $\alpha = \alpha^* + 1$ . In the former case, by Lemma 5.2, we recover (126). In the latter case, the lemma gives  $\rho_i = 0$  if  $i \in \mathcal{C}_\beta$  for  $\beta < \alpha^* + 1$  and  $\rho_i = 1$  if  $i \in \mathcal{C}_\beta$  for  $\alpha^* + 1 < \beta$ . Thus

$$\rho = N_{\alpha^*} = \sum_{i \in \mathcal{C}_{\alpha^*+1}} \rho_i + N_{\alpha^*+1}. \quad (129)$$

Hence, the sum in (129) must be equal to  $n_{\alpha^*+1}$ , so  $\rho_i = 1$  for every  $i \in \mathcal{C}_{\alpha^*+1}$ . This gives the same  $(\rho_0, \dots, \rho_{n-1})$  as in (126).

(c) If  $\rho = N_{\alpha^*-1}$ , a similar argument than in (b) shows that we recover (126).  $\square$

We now turn to the proof of Proposition 4.1. We shall need the following lemma, which completes the proof of Proposition 2.1.

**Lemma 5.3.** *For  $i = 0, \dots, n-1$ , the function  $\tilde{\rho}_i(\cdot)$  defined in Proposition 2.1 is increasing and continuously differentiable on  $[N_\alpha, N_{\alpha-1}]$ , identically 0 on  $[0, N_\alpha]$ , and identically 1 on  $[N_{\alpha-1}, n]$ , where  $\alpha \in \{0, \dots, m-1\}$  is such that  $i$  belongs to  $\mathcal{C}_\alpha$ . Its restriction to  $[N_\alpha, N_{\alpha-1}]$  depends only on the restriction of  $q(\cdot, \cdot)$  to  $\mathcal{C}_\alpha \times \mathcal{C}_\alpha$ .*

*Proof of lemma 5.3.* The constant values 0 and 1 outside  $[N_\alpha, N_{\alpha-1}]$  follow from the first two lines of (126). The regularity on  $[N_\alpha, N_{\alpha-1}]$  follows from the third line. Indeed,  $c \mapsto \rho_i^{\alpha, c}$  lies in  $C^1([0, +\infty))$ , and so does

$$c \mapsto \rho^\alpha(c) := \sum_{i \in \mathcal{C}_\alpha} \rho_i^{\alpha, c} + N_\alpha. \quad (130)$$

Notice that  $\rho_i^{\alpha, \cdot}$  is an increasing function such that

$$\rho_i^{\alpha, 0} = 0, \quad \lim_{c \rightarrow +\infty} \rho_i^{\alpha, c} = 1. \quad (131)$$

Hence  $\rho^\alpha$  is an increasing function such that

$$\rho^\alpha(0) = N_\alpha \quad , \quad \lim_{c \rightarrow +\infty} \rho^\alpha(c) = N_{\alpha-1} \quad (132)$$

$$\lim_{\rho \rightarrow N_{\alpha-1}^-} (\rho^\alpha)^{-1}(\rho - N_\alpha) = +\infty \quad , \quad \lim_{\rho \rightarrow N_\alpha^+} (\rho^\alpha)^{-1}(\rho - N_\alpha) = 0. \quad (133)$$

The derivative of  $\rho^\alpha$  does not vanish. Hence its inverse lies in  $C^1([0, n_\alpha])$  and can be extended into an element of  $C^0([0, n_\alpha])$ . By (125) and (126), the restriction of  $\tilde{\rho}_i$  to  $[N_\alpha, N_{\alpha-1}]$  is given by

$$\tilde{\rho}_i(\rho) = \rho_i^{\alpha, c}, \quad \text{with } c = (\rho^\alpha)^{-1}(\rho - N_\alpha) \quad (134)$$

and thus lies in  $C^1([N_\alpha, N_{\alpha-1})) \cap C^0([N_\alpha, N_{\alpha-1}])$ . It follows that

$$\tilde{\rho}'_i(\rho) = \frac{\frac{d}{dc} \rho_i^{\alpha, c}}{\frac{d}{dc} \rho^\alpha(c)} \quad (135)$$

with  $c$  given by (134). From this, a direct computation shows that

$$\lim_{\rho \rightarrow N_{\alpha-1}^-} \tilde{\rho}'_i(\rho) = f_i^\alpha > 0, \quad \lim_{\rho \rightarrow N_{\alpha+}} \tilde{\rho}'_i(\rho) = e_i^{\alpha+1} > 0 \quad (136)$$

where  $e_i^\alpha$  and  $f_i^\alpha$  are defined by (99). Finally, by the last line of (126), the restriction of  $\tilde{\rho}_i$  to  $[N_\alpha, N_{\alpha-1}]$  depends only on  $\lambda^\alpha$ , which depends only on the restriction of  $q(\cdot, \cdot)$  to  $\mathcal{C}_\alpha \times \mathcal{C}_\alpha$ .  $\square$

*Proof of Proposition 4.1.* (i) By (59) and Lemma 5.3,  $G$  is continuously differentiable on  $[N_\alpha, N_{\alpha-1}]$ . Besides, for  $\rho$  in this interval, only indices  $i \in \mathcal{C}_\alpha$  may contribute to (59), because  $\tilde{\rho}_i(\rho) \in \{0, 1\}$  for other indices.

(ii) Since  $\tilde{\rho}_i$  increases from 0 to 1 on  $[N_\alpha, N_{\alpha-1}]$ , (98) follows from (59)–(60) and (136).

(iii) For  $\rho = \rho^\alpha(c)$  defined in (130), we have

$$G(\rho) = \sum_{i \in \mathcal{C}_\alpha} \gamma_i \frac{c\lambda_i^\alpha}{(1 + c\lambda_i^\alpha)^2} =: \tilde{G}^\alpha(c) \quad (137)$$

and  $G'$  vanishes at  $\rho = \rho^\alpha(c) \in [N_\alpha, N_{\alpha-1}]$  if and only if  $(\tilde{G}^\alpha)'$  vanishes at  $c$ . The latter derivative is a rational fraction, hence it has finitely many zeroes.  $\square$

## 6 Proof of Theorem 4.1

For the following computations we assume without loss of generality that  $d_0 + d_1 = 1$  (which amounts to a time change), and that  $p + q = 1$  (which allows all the possible values for the flux  $G$ , since  $G$  depends on  $p/q$ ), with  $0 < p < 1/2$ . We rely on the expression for  $G$  given in (101)–(103), where we set  $\gamma_0 = d$  and  $\gamma_1 = 1 - d$ . We assume that  $d \geq 1/2$ . Note that  $\psi(\rho) \leq 1$ . The following values and equalities are independent of  $d$ :

$$G(0) = G(2) = 0 \quad (138)$$

$$G(1) = \frac{1}{4} - \varphi(1)^2 \quad (139)$$

$$\psi(2 - \rho) = \psi(\rho), \quad \text{hence} \quad \varphi(2 - \rho) = \varphi(\rho). \quad (140)$$

We then compute

$$\begin{aligned}
\psi'(\rho) &= \left(\frac{r-1}{r+1}\right)^2 2(\rho-1) \\
\varphi'(\rho) &= -\frac{1}{2} \left(\frac{r-1}{r+1}\right) \frac{(\rho-1)}{\sqrt{\psi(\rho)}} \\
\varphi''(\rho) &= -2r \left(\frac{r-1}{r+1}\right)^3 \psi(\rho)^{-3/2} \\
\varphi^{(3)}(\rho) &= 6r(\rho-1) \left(\frac{r-1}{r+1}\right)^5 \psi(\rho)^{-5/2} \\
G'(\rho) &= \frac{1}{2}(1-\rho) + (2d-1) [-\varphi(\rho) + (1-\rho)\varphi'(\rho)] - 2\varphi(\rho)\varphi'(\rho) \\
G''(\rho) &= -\frac{1}{2} + (2d-1) [-2\varphi'(\rho) + (1-\rho)\varphi''(\rho)] - 2\varphi'(\rho)^2 - 2\varphi(\rho)\varphi''(\rho) \\
G^{(3)}(\rho) &= (2d-1) [-3\varphi''(\rho) + (1-\rho)\varphi^{(3)}(\rho)] - 6\varphi'(\rho)\varphi''(\rho) - 2\varphi(\rho)\varphi^{(3)}(\rho) \\
&= 6r \frac{(r-1)^2}{(r+1)^4} \psi(\rho)^{-5/2} \left[ (2d-1) \frac{4r}{(r-1)(r+1)} + (1-\rho) \right].
\end{aligned}$$

We have that  $G^{(3)}(\rho)$  changes sign for the value

$$\tilde{\rho}_0 = \tilde{\rho}_0(r, d) = 1 + (2d-1) \frac{4r}{(r-1)(r+1)} \geq 1. \quad (141)$$

Therefore  $G''$  is increasing before  $\tilde{\rho}_0$  and decreasing after. We compute

$$G''(0) = -\frac{r^2+1}{(r+1)^2} - (2d-1) \frac{(r-1)}{(r+1)} \left[ 1 + \frac{2r}{(r+1)^2} \right] < 0 \quad (142)$$

$$G''(1) = -1 + \frac{r+1}{4\sqrt{r}} \quad (\text{this value is independent of } d) \quad (143)$$

$$\psi(\tilde{\rho}_0) = \frac{4r}{(r+1)^2} \left[ (2d-1)^2 \frac{4r}{(r+1)^2} + 1 \right] \quad (144)$$

$$G''(\tilde{\rho}_0) = -1 + \frac{1}{\sqrt{\psi(\tilde{\rho}_0)}} \left[ -\frac{1}{2} + \frac{(r+1)^2}{4r} \psi(\tilde{\rho}_0) \right] \quad (145)$$

$$G''(2) = -\frac{r^2+1}{(r+1)^2} + (2d-1) \frac{(r-1)}{(r+1)} \left[ 1 + \frac{2r}{(r+1)^2} \right]. \quad (146)$$

Moreover we have that

$$G''(2) < 0 \Leftrightarrow g(r) := r^3(1-d) + r^2(2-3d) + r(3d-1) + d > 0 \quad (147)$$

$$\tilde{\rho}_0 < 2 \Leftrightarrow (2d-1)4r < r^2 - 1 \quad (148)$$

$$\Leftrightarrow r > \tilde{r}_1 = \tilde{r}_1(d) := 2(2d-1) + \sqrt{4(2d-1)^2 + 1}. \quad (149)$$

Note that  $d \mapsto \tilde{r}_1(d)$  is an increasing function. Since  $G''(0) > 0$ , to determine the number of inflexion points of  $G$  depending on  $r$  for given  $d$ , it is enough to



determine the sign of  $G''(2)$  (i.e. of  $-g(r)$ ), and that of  $G''(\tilde{\rho}_0(r))$ .

*Sign of  $g(r)$  and  $G''(2)$ .*

*First case:  $d = 1$ .* Then,

$$g(r) = -r^2 + 2r + 1, \quad g'(r) = -2r + 2 < 0 \quad (150)$$

thus  $g$  is decreasing, with  $g(r) = 0$  for  $\tilde{r}_0 = 1 + \sqrt{2} < \tilde{r}_1(1)$  (the other root,  $1 - \sqrt{2}$ , is less than 1). Thus for  $r < \tilde{r}_0$ ,  $G''(2) < 0$  and for  $r > \tilde{r}_0$ ,  $G''(2) > 0$ .

*Second case:  $1/2 \leq d < 1$ .* We have

$$g'(r) = 3r^2(1-d) + 2r(2-3d) + (3d-1) \quad (151)$$

$$g(1) = 2; \quad g'(1) = 6(1-d) > 0. \quad (152)$$

The sign of  $g'(r)$  depends on that of

$$\delta = \delta(d) := 18d^2 - 24d + 7 \quad (153)$$

which has a unique root in  $[1/2; +\infty)$ , namely

$$d_1 := \frac{4 + \sqrt{2}}{6} > \frac{5}{6}. \quad (154)$$

Note that  $d_1 > 5/6$ . For  $d < d_1$ ,  $\delta(d) \leq 0$ , thus  $g'(r)$  has constant sign; by (152),  $g'(r) < 0$ ,  $g$  is decreasing negative and  $G''(2) > 0$ . For  $d \geq d_1$ , there are two values  $r_1(d), r_2(d)$  such that  $g'(r_1) = g'(r_2) = 0$ :

$$r_1 = \frac{3d-2-\sqrt{\delta}}{3(1-d)} = r_0 - \frac{\sqrt{\delta}}{3(1-d)} \quad (155)$$

$$r_2 = \frac{3d-2+\sqrt{\delta}}{3(1-d)} = r_0 + \frac{\sqrt{\delta}}{3(1-d)}, \quad \text{where} \quad (156)$$

$$r_0 = \frac{3d-2}{3(1-d)}. \quad (157)$$

Then  $g$  is increasing between 1 and  $r_1$ , then decreasing between  $r_1$  and  $r_2$ , and finally increasing after  $r_2$ . Therefore if  $g(r_2) > 0$ , then  $G''(2) < 0$ . We have

$$g(r_2) > 0 \Leftrightarrow \phi(d) < 0, \quad \text{where} \quad (158)$$

$$\phi(d) = \delta\sqrt{\delta} - 3(1-d) + (3d-2)\delta. \quad (159)$$

Since  $d > \frac{5}{6}$ , the function  $\phi$  is increasing. We have  $\phi(d_1) < 0$ , and  $\phi(\frac{14}{15}) > 0$ , thus  $\phi$  changes sign for some value  $\tilde{d}_1$  between  $d_1$  and  $\frac{14}{15}$  such that  $\phi(\tilde{d}_1) = 0$ , given by:

$$\tilde{d}_1 := \frac{1}{2} + \frac{1}{6}\sqrt{3 + 2\sqrt{3}}. \quad (160)$$

Gathering the above lines, we have that

$$\text{if } d \leq \tilde{d}_1, \quad \text{then } G''(2) < 0. \quad (161)$$

For  $d \geq \tilde{d}_1$ , there are two values  $r_3(d) < r_2(d) < r_4(d)$  such that  $g(r_3) = g(r_4) = 0$ ,  $g$  is positive outside  $(r_3(d), r_4(d))$  hence  $G''(2) < 0$ , and negative in  $(r_3(d), r_4(d))$  hence  $G''(2) > 0$ .

Third case:  $d < 1/2$ . Then  $r_2(d) < 0$  and  $0 < r_1(d) < 1$ . Then  $g$  is non-decreasing on  $[1; +\infty)$  and  $\lim_{r \rightarrow +\infty} g(r) = -\infty$ . Thus  $g$  has a single zero on  $[1; +\infty)$ , that we still denote by  $r_3(d)$ . Then for  $r < r_3(d)$ ,  $G''(2) > 0$  and for  $r > r_3(d)$ ,  $G''(2) < 0$ .

*Sign of  $G''(\tilde{\rho}_0(r))$ .* Let

$$\bar{d}_1 = \frac{2 + \sqrt{2\sqrt{3}}}{4} \quad \text{and} \quad (162)$$

$$A_1 = A_1(d) = \frac{1}{2(2d-1)^2} \left( -1 + \frac{1}{2\sqrt{d(1-d)}} \right) \quad (163)$$

$$A_1(\bar{d}_1) = 1 \quad \text{and} \quad A_1 < 1 \Leftrightarrow d < \bar{d}_1 \quad (164)$$

$$\text{if } d \leq \bar{d}_1, \quad \text{let } \bar{r}_1 = \bar{r}_1(d) = \frac{(1 + \sqrt{1 - A_1})^2}{A_1} \quad (165)$$

$$\text{if } d = \bar{d}_1, \quad \text{then } G''(\tilde{\rho}_0) \geq 0 \quad (166)$$

$$\text{if } d > \bar{d}_1, \quad \text{then } G''(\tilde{\rho}_0) \geq 0 \quad (167)$$

$$\text{if } \frac{1}{2} < d \leq \bar{d}_1 \quad \text{and} \quad r \geq \bar{r}_1, \quad \text{then } G''(\tilde{\rho}_0) \geq 0 \quad (168)$$

$$\text{if } \frac{1}{2} < d \leq \bar{d}_1 \quad \text{and} \quad r \leq \bar{r}_1, \quad \text{then } G''(\tilde{\rho}_0) \leq 0. \quad (169)$$

Moreover we have that

$$\bar{r}_1 > \tilde{r}_1 \Leftrightarrow d < \tilde{d}_0 := \frac{1}{2} + \frac{\sqrt{3}}{4}. \quad (170)$$

Hence

$$\frac{5}{6} < \tilde{d}_1 < \tilde{d}_0 < \frac{14}{15} < \bar{d}_1 < \frac{39}{40}. \quad (171)$$

*Proof of 1, (i).* Properties of  $\tilde{r}_1(d)$  and  $\bar{r}_1(d)$  follow from their expressions (149), resp. (163)–(165). Monotonicity properties of  $r_3(d)$  and  $r_4(d)$  hold because for fixed  $r > 1$ ,  $g$  is a decreasing function of  $d$ . By implicit function Theorem,  $r_3(d)$  and  $r_4(d)$  are continuously differentiable on  $(\tilde{d}_1; 1)$ , and due to joint  $(r, d)$  continuity of  $g$ , they extend continuously to  $\tilde{d}_1$ . When  $d \rightarrow 1^-$ , (114) holds because for  $r > 1$ ,  $g(r)$  tends to  $-\infty$  as  $d \rightarrow 1^-$ , whereas  $r_3(d) \rightarrow \tilde{r}_0 = 1 + \sqrt{2}$ , the unique zero of  $g(r)$  found above when  $d = 1$ . The unique zero  $r_3(d)$  of

$g(r)$  for  $d > 1$  tends to  $\tilde{r}_0$  as  $d \rightarrow 1+$ , and on  $(1; +\infty)$ ,  $r_3(d)$  is continuously differentiable by the implicit function theorem.

*Proof of 1, (ii).* For the purpose of 2., we prove the following larger set of properties:

$$d = \tilde{d}_0 \Rightarrow \tilde{r}_1(d) = \bar{r}_1(d) = r_3(d) \quad (172)$$

$$d < \tilde{d}_0 \Rightarrow \tilde{r}_1(d) < \bar{r}_1(d) < r_3(d) \quad (173)$$

$$d > \tilde{d}_0 \Rightarrow r_3(d) < \tilde{r}_1(d) < r_4(d). \quad (174)$$

Proof of (172): For  $d = \tilde{d}_0$ , and  $r = \tilde{r}_1(d) = \bar{r}_1(d)$ , by definition of these quantities, we have  $\tilde{\rho}_0 = 2$  and  $G''(\tilde{\rho}_0) = 0$ , hence  $G''(2) = 0$ , implying  $r \in \{r_3(d); r_4(d)\}$ ; that  $r = r_3(d)$ , i.e. (173), follows from (174), proven right below.

Proof of (173): Assume  $\bar{r}_1(d) > r_3(d)$ ; then for  $r_3(d) < r < \inf(\bar{r}_1(d), r_4(d))$ , we have  $G''(2) < 0$  and  $G''(\tilde{\rho}_0) \geq 0$ , hence a contradiction. Monotonicities proved in 1.(i), and (172), imply  $\tilde{r}_1(d) < \bar{r}_1(d)$ .

Proof of (174): that  $r_3(d) < \tilde{r}_1(d)$  follows from (172) and monotonicities. Since  $r_3(d)$  and  $\tilde{r}_1(d)$  are continuous, assuming  $\tilde{r}_1(d) \geq r_4(d)$  for some  $d$  implies  $\tilde{r}_1(d') = r_4(d')$  for some  $d'$ . Arguing as for (172), this implies  $\bar{r}_1(d') = r_4(d')$ , which contradicts (172)–(173).

*Proof of 2.* The following holds regardless of the value of  $d$ . First, if  $r \leq \bar{r}_1(d)$ , since  $G''(\tilde{\rho}_0) \leq 0$  and  $\tilde{\rho}_0$  is a global strict maximizer of  $G''$  on  $[0; +\infty)$ ,  $G'' \leq 0$  and vanishes at most once, hence  $G$  is strictly concave. Next, if  $r_3(d) < r < r_4(d)$ ,  $G''(\tilde{\rho}_0) > G''(2) > 0$ . Thus  $G$  has a single inflexion point (note that this does not depend on the position of  $\tilde{\rho}_0$  with respect to 2). Finally, if  $r > r_4(d)$ ,  $G''(2) < 0 < G''(\tilde{\rho})$ . Since  $r_4(d) > \tilde{r}_1(d)$ ,  $\tilde{\rho}_0 < 2$ , hence  $G$  has two inflexion points.

For other positions of  $r$ , the conclusion depends on  $d$ :

Proof of (i): If  $r > \bar{r}_1(d)$ , by (173),  $\bar{r}_1(d) > \tilde{r}_1(d)$ , thus  $G''(\tilde{\rho}_0) > 0$  and  $\tilde{\rho}_0 < 2$ . Since  $d < \tilde{d}_1$ , we also have  $G''(2) < 0$ . Recalling  $G''(0) < 0$  and variations of  $G''$ , we conclude that  $G''$  vanishes once on either side of  $\tilde{\rho}_0$ .

Proof of (ii): If  $\bar{r}_1(d) < r < r_3(d)$ , then  $G''(\tilde{\rho}_0) > 0$ ,  $G''(2) < 0$ , and by (174),  $r > \tilde{r}_1(d)$ , thus  $\tilde{\rho} < 2$ , implying two points of inflexion.

Proof of (iii)–(v): If  $r < r_3(d)$ , then  $G''(2) < 0$ . Since  $r_3(d) < \tilde{r}_1(d)$ , we also have  $\tilde{\rho}_0 > 2$ , thus  $G'' < 0$  on  $[0; 2]$  and  $G$  is strictly concave.

The sign of  $G$  in (v) follows from item 3 of Proposition 4.2, the value  $\frac{d}{d-1}$

arising from (106) with  $\gamma_0 = d$ ,  $\gamma_1 = 1 - d$  and  $q > p$ . Note that  $r_3(d) < \frac{d}{d-1}$ , because if  $G$  is strictly concave with  $G(0) = G(2) = 0$ , it cannot vanish on  $(0; 2)$ .

As regards local extrema of  $G$  in  $(v)$ , denoting the location of the zero by  $\rho_0$ , since  $G(0) = G(\rho_0) = G(2) = 0$ , there is at least one extremum on either side of  $\rho_0$ , and there cannot be more because there is a single inflexion point; the extremum on the left of  $\rho_0$  is a maximum because  $G''(0) < 0$ .

*Complement.* We can locate inflexion points with respect to 1. Recall that  $\tilde{\rho}_0 > 1$ , cf. (141). From the above discussion, whenever there are two inflexion points, they lie on either side of  $\tilde{\rho}_0$ , hence the larger one lies to the right of 1; whereas a single inflexion point lies to the left of  $\min(\tilde{\rho}_0; 2)$ . Further, by (143),

$$G''(1) > 0 \Leftrightarrow r > (2 + \sqrt{3})^2 =: \bar{r}_1. \quad (175)$$

Note that

$$\tilde{r}_1 < \bar{r}_1. \quad (176)$$

Let

$$\tilde{d}_0 < \bar{d}_1 := \frac{1}{2} \left( 1 + \frac{14\sqrt{3}}{27} \right) < \bar{d}_1.$$

Using (146), we have that for  $r = \bar{r}_1$ ,  $G''(2) > 0$  if and only if  $d > \bar{d}_1$ , and  $G''(2) < 0$  if and only if  $d < \bar{d}_1$ . Since  $r_3(d)$  is decreasing and  $r_4(d)$  increasing, this implies

$$d < \bar{d}_1 \Rightarrow \bar{r}_1 > r_4(d) \quad (177)$$

$$d > \bar{d}_1 \Rightarrow r_3(d) < \bar{r}_1 < r_4(d). \quad (178)$$

We can deduce that:

1) If  $d < \bar{d}_1$ : a) whenever  $G$  has two inflexion points, if  $r > \bar{r}_1$ , the smaller inflexion point lies to the left of 1; if  $r < \bar{r}_1$ , it lies to the right of 1. b) Whenever  $G$  has a single inflexion point, it lies between 1 and  $\min(\tilde{\rho}_0; 2)$ .

2) If  $d > \bar{d}_1$ : a) whenever  $G$  has two inflexion points, the smaller one lies to the left of 1. b) Whenever  $G$  has a single inflexion point, if  $r > \bar{r}_1$ , it lies to the left of 1; if  $r < \bar{r}_1$ , it lies between 1 and  $\min(\tilde{\rho}_0; 2)$ .

## 7 Proof of hydrodynamics: Theorem 3.1

We follow the approach of [5]. The principle of this approach is to reduce the hydrodynamic limit for the Cauchy problem to that for the Riemann problem (41), (61)–(62) thanks to an approximation scheme.

To derive Cauchy hydrodynamics from Riemann hydrodynamics, we need two

properties. The first one is the *finite propagation* property ([5, Lemma 3.1, Lemma 3.2]), whose proof carries over here with minor modifications, and the similar property for the hydrodynamic equation. In our context, the statements read as follows:

**Proposition 7.1.** *There exist constants  $\sigma, C > 0$  such that the following holds:*

(i) *Assume  $(\eta_t^N)_{t \geq 0}$  and  $(\xi_t^N)_{t \geq 0}$  are two coupled multilane SEP's with common generator (54) such that  $\eta_0^N(z, i) = \xi_0^N(z, i)$  for all  $z \in [a, b]$  and  $i \in W$  (cf. (13)), where  $a, b \in \mathbb{Z}$  and  $a < b$ . Then outside probability  $e^{-CNt}$ ,  $\eta_0^N(z, i) = \xi_0^N(z, i)$  for all  $z \in [a + \sigma Nt, b - \sigma Nt]$  and  $i \in W$ .*

(ii) *Let  $u_0, v_0$  be initial data for (61). Then the associated entropy solutions  $u(\cdot, t)$  and  $v(\cdot, t)$  at time  $t < (b - a)/(2\sigma)$  satisfy*

$$\int_{a+\sigma t}^{b-\sigma t} [u(x, t) - v(x, t)]^\pm dx \leq \int_a^b [u_0(x) - v_0(x)]^\pm dx. \quad (179)$$

*In particular, if  $u_0(\cdot)$  and  $v_0(\cdot)$  coincide a.e. on  $[a, b]$ , then  $u(\cdot, t)$  and  $v(\cdot, t)$  coincide a.e. on  $[a + \sigma t, b - \sigma t]$ .*

The second property, proved in Subsection 7.3, is *macroscopic stability* ([5, Definition 3.1]). It states that if two coupled configurations are initially macroscopically close, they remain so at later times. This will be a consequence of the following property, which is a substantial multilane refinement of [9, Lemma 3.1]. The refinement involves condition (52) to control the possibly slower transverse dynamics.

**Proposition 7.2.** *Let  $\eta_0^N, \xi_0^N \in \mathcal{X}$  be such that*

$$\eta_0^N(z, i) = \xi_0^N(z, i) = 0, \quad \forall (z, i) \in [-aN; aN] \times W \quad (180)$$

*for some constant  $a > 0$ . Define, for  $t \geq 0$ , the function  $\phi_t^N : \mathbb{Z} \rightarrow \mathbb{Z} \cap [-n, n]$  by*

$$\phi_t^N(z) = \sum_{i \in W} [\eta_t^N(z, i) - \xi_t^N(z, i)] \quad (181)$$

*where  $(\eta_t^N)_{t \geq 0}$  and  $(\xi_t^N)_{t \geq 0}$  denote the processes with generator  $L^{\theta(N)/N}$  starting respectively from  $\eta_0^N$  and  $\xi_0^N$ . Then, under condition (52), for every  $\gamma > 0$  and  $t > 0$ ,*

$$\lim_{N \rightarrow +\infty} \mathbb{P} \left( \sup_{z \in \mathbb{Z}, t \geq 0} N^{-1} \left| \sum_{u \in \mathbb{Z}: u \geq z} \phi_t^N(u) \right| > \sup_{z \in \mathbb{Z}} N^{-1} \left| \sum_{u \in \mathbb{Z}: u \geq z} \phi_0^N(u) \right| + \gamma \right) = 0. \quad (182)$$

Following [5, Theorem 3.2], in order to prove statement (i) of Theorem 3.1, it is enough to prove it for Riemann initial profiles (41), (62), and to verify that our model satisfies Propositions 7.1–7.2.

**Remark 7.1.** *Since local equilibrium implies hydrodynamics (see [18, Proposition 0.4]), statement (i) of Theorem 3.1 for Riemann profiles is implied by statement (iii) of Theorem 3.1. Thus the latter will be the actual purpose of Subsection 7.1.*

For the Riemann problem, the hydrodynamic limit was first addressed in [2] in the case of a strictly concave flux function, for which the entropy solution is very explicit. For more general flux functions, the Riemann solution is not so explicit, but the hydrodynamic limit was addressed in [5] thanks to a variational representation.

In the present case, the microscopic derivation of this variational formula cannot be carried out as in [5] because the vertical part of the generator has a different scaling than the horizontal part. The microscopic derivation here is carried out in Subsection 7.1 and relies on a two-block estimate which we derive in Subsection 7.4 from an *approximate interface property*. This property is a suitable generalization of the known *exact interface property* ([20]) for the single-lane asymmetric exclusion process (which no longer holds exactly for multilane exclusion). The two-block estimate will also allow us in Subsection 7.2 to derive the weak local equilibrium property for the Cauchy problem. The Riemann strong local equilibrium in statement (iii) of Theorem 3.1 shall be derived in Subsection 7.1 together with Riemann hydrodynamics, see Remark 7.1.

To prove Propositions 7.1 and 7.2 and conclude the proof of Theorem 3.1, we need coupling tools recalled hereafter.

**Coupling and discrepancies.** The Harris construction allows to couple the evolutions from different initial configurations through *basic coupling*, that is, by using the same Poisson processes for them. There is a natural partial order on  $\mathcal{X}$ , namely, for  $\eta, \xi \in \mathcal{X}$ ,

$$\eta \leq \xi \quad \text{if and only if} \quad \forall x \in V, \eta(x) \leq \xi(x). \quad (183)$$

Such a coupling shows that the simple exclusion process is *attractive*, that is, the partial order (183) is conserved by the dynamics. In other words,

$$\forall \eta_0, \xi_0 \in \mathcal{X}, \eta_0 \leq \xi_0 \Rightarrow \forall t \geq 0, \eta_t \leq \xi_t \text{ a.s.} \quad (184)$$

The order (183) endows an order on the set  $\mathcal{M}_1$  of probability measures on  $\mathcal{X}$  in the following way. A function  $f$  on  $\mathcal{X}$  is said to be increasing if and only if  $\eta \leq \xi$  implies  $f(\eta) \leq f(\xi)$ . For two probability measures  $\mu_0, \mu_1$  on  $\mathcal{X}$ , we write  $\mu_0 \leq \mu_1$  if and only if for every increasing function  $f$  on  $\mathcal{X}$  we have  $\int f(\eta)\mu_0(d\eta) \leq \int f(\eta)\mu_1(d\eta)$ . We shall write  $\mu_1 < \mu_2$  if  $\mu_1 \leq \mu_2$  and  $\mu_1 \neq \mu_2$ .

Thus (184) implies, for two probability measures  $\mu, \nu$  on  $\mathcal{X}$ ,

$$\mu \leq \nu \Rightarrow \mu S_t \leq \nu S_t \quad (185)$$

where  $(S_t)_{t \geq 0}$  denotes the semigroup of the process  $(\eta_t)_{t \geq 0}$ . In such a coupling, we say that at  $x$  there is an  *$\eta$  discrepancy* (or blue particle) if  $\eta(x) > \xi(x)$ ,

a  $\xi$  discrepancy (or red particle) if  $\eta(x) < \xi(x)$ , a coupled particle (or black particle) if  $\eta(x) = \xi(x) = 1$ , a hole (or white particle) if  $\eta(x) = \xi(x) = 0$ . Blue and red are called opposite (type) particles (or opposite (type) discrepancies). The evolution of the coupled process can be formulated as follows. At a time  $t \in \mathcal{N}_{(x,y)}$ , a blue, red or black particle at  $x$  exchanges with a hole at  $y$ ; a black particle at  $x$  exchanges with a blue or red particle at  $y$ ; if there is a pair of opposite (type) particles at  $x$  and  $y$ , they are replaced by a hole at  $x$  and a black particle at  $y$ . We call this a *coalescence*. This shows that no new discrepancy can ever be created.

### 7.1 The Riemann problem: Proof of Theorem 3.1, (iii)

For the Riemann problem (61)–(62), the following result can be found in [5].

**Proposition 7.3.** *The entropy weak solution of (61)–(62) is the self-similar function given by*

$$u(t, x \pm 0) = u\left(1, \frac{x}{t} \pm 0\right) = (G_{\alpha, \beta}^*)' \left(\frac{x}{t} \pm 0\right) \quad (186)$$

where

$$G_{\alpha, \beta}^*(v) := \begin{cases} \inf_{\rho} [v\rho - G(\rho)] & \text{if } \alpha \geq \beta \\ \sup_{\rho} [v\rho - G(\rho)] & \text{if } \alpha \leq \beta. \end{cases} \quad (187)$$

If  $\alpha \leq \beta$  (resp.  $\alpha \geq \beta$ ),  $u(1, v-)$  is the smallest (resp. largest) and  $u(1, v+)$  the largest (resp. smallest) optimizer in (187). In particular,  $u(1, \cdot)$  is continuous at  $v$  if and only if the optimizer is unique, and is then equal to this optimizer.

The proof of Theorem 3.1, (iii) partly follows the scheme of [2, 5], but significant differences are involved due to the fact that the horizontal and vertical parts of the generator have different scalings. In particular, we need a block estimate (Proposition 7.4 below) for which we first introduce relevant notation. The one block part, see (189) below, will be used here, while the two-block part (190) will be necessary for Cauchy hydrodynamics. Let  $f$  be a local function on  $\mathcal{X}$ . For  $z \in \mathbb{Z}$ ,  $\xi \in \mathcal{X}$ ,  $l \in \mathbb{N}$ , we define the local block average of  $f$  by

$$M_{z, l} f(\eta) := \frac{1}{2l+1} \sum_{u \in \mathbb{Z}: |u-z| \leq l} \tau_u f(\eta). \quad (188)$$

By a slight abuse of notation, we write  $M_{x, l} \eta^i$ , resp.  $M_{x, l} \bar{\eta}$ , to denote  $M_{x, l} f(\eta)$  for  $f$  defined by  $f(\eta) = \eta^i(0)$ , resp.  $f(\eta) = \bar{\eta}(0)$ .

**Proposition 7.4.** *Let  $u(\cdot) \in C_K^0(\mathbb{R})$  such that for all  $c \in [0, n]$ ,  $u(\cdot) - c$  changes sign finitely many times. Assume  $\eta_0^N$  is a local Gibbs state with global profile  $u(\cdot)$ , defined by (39) with  $u_x^N := u(x/N)$ . Then, for  $i \in W$ ,  $A < B$  and  $s > 0$ ,*

$$\lim_{l \rightarrow +\infty} \limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{x \in \mathbb{Z} \cap [NA, NB]} \Delta_{x, l}(\eta_{Ns}^N) \right\} = 0. \quad (189)$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{x \in \mathbb{Z} \cap [NA, NB]} \Delta_{x, N\varepsilon}(\eta_{Ns}^N) \right\} = 0 \quad (190)$$

where

$$\Delta_{x,l}(\eta) := |M_{x,l}f(\eta) - \bar{f}(M_{x,l}\bar{\eta})| \quad (191)$$

and, as in Theorem 3.1,  $(\eta_s^N)_{s \geq 0}$  denotes a process with initial state  $\eta_0^N$  and generator  $L^{\theta(N)/N}$ .

Proposition 7.4 is proved in Subsection 7.4.

*Proof of Theorem 3.1, (iii).*

We assume  $\beta \leq \alpha$ , the case  $\alpha \leq \beta$  being similar. In the sequel, we denote by  $(S_t^N)_{t \geq 0}$  the semigroup generated by  $L^{\theta(N)/N}$ , and  $(S_{Nt}^N)_{t \geq 0}$  the semigroup generated by  $\mathcal{L}^N$  defined in (54).

*Step 1.* We show that there exists a dense subset  $D$  of  $\mathbb{R}$  with the following property: for every sequence  $N_\ell \rightarrow +\infty$  as  $\ell \rightarrow +\infty$ , there exists a subsequence  $N_{\ell_k}$  such that, for  $N = N_{\ell_k} \rightarrow +\infty$ :

$$\lim_{N \rightarrow +\infty} \int_0^t \mu_{\alpha, \beta} \tau_{[Nvs]} S_{Ns}^N ds = \mu_v, \quad (192)$$

for a family of measures  $\mu_v$ , nondecreasing with respect to  $v$ , of the form

$$\mu_v = \int_{[0;n]} \nu_\rho \lambda_v(d\rho) = \int_{[\alpha, \beta]} \nu_\rho \lambda_v(d\rho) \quad (193)$$

where  $\lambda_v$  is a probability measure on  $[0, n]$  supported on  $[\alpha, \beta]$ , and  $\nu_\rho$  is the measure defined by (32) and (27).

To this end, for  $\varepsilon > 0$ , we define

$$\mu_v(N) := \frac{1}{t} \int_0^t \mu_{\alpha, \beta} \tau_{[Nvs]} S_{Ns}^N ds, \quad (194)$$

$$\mu_{v, \varepsilon}^+(N) := \frac{1}{[\varepsilon N] + 1} \sum_{z \in \mathbb{Z} \cap [0, \varepsilon N]} \tau_z \mu_v(N), \quad (195)$$

$$\mu_{v, \varepsilon}^-(N) := \frac{1}{[\varepsilon N] + 1} \sum_{z \in \mathbb{Z} \cap [-\varepsilon N, 0]} \tau_z \mu_v(N). \quad (196)$$

Since  $\beta \leq \alpha$ ,  $\tau \mu_{\alpha, \beta} \leq \mu$ . Thus by attractiveness, we have, for  $0 < \varepsilon < \varepsilon'$ ,

$$\mu_{v, \varepsilon'}^-(N) \geq \mu_{v, \varepsilon}^-(N) \geq \mu_v(N) \geq \mu_{v, \varepsilon}^+(N) \geq \mu_{v, \varepsilon'}^+(N). \quad (197)$$



Since the above measures are supported on the compact set  $\mathcal{X}$ , by diagonal extraction, we can find a dense subset  $D_0$  of  $\mathbb{R}$  and a subsequence  $(N_{\ell_k})_k$  of  $(N_\ell)_\ell$  such that, for each  $v \in D_0$ , the following weak limits exist for  $N = N_{\ell_k} \rightarrow +\infty$ :

$$\mu_v := \lim_{N \rightarrow +\infty} \mu_v(N), \quad (198)$$

$$\mu_{v,\varepsilon}^\pm := \lim_{N \rightarrow +\infty} \mu_{v,\varepsilon}^\pm(N), \quad (199)$$

$$\mu_v^\pm := \lim_{\varepsilon \rightarrow 0} \mu_{v,\varepsilon}^\pm. \quad (200)$$

By (197), for  $u < v < w$  in  $D$ , we have

$$\mu_u^+ \geq \mu_v^- \geq \mu_v \geq \mu_v^+ \geq \mu_w^-. \quad (201)$$

From this we can conclude that there exists a dense subset  $D \subset D_0$  of  $\mathbb{R}$  such that  $\mu_v = \mu_v^+ = \mu_v^-$  for all  $v \in D$  (to this end integrate (201) against a dense countable subset of non-decreasing local functions, and recall that a non-decreasing function from  $\mathbb{R}$  to  $\mathbb{R}$  has countably many discontinuities).

It remains to show that for  $v \in D$ , the measure  $\mu_v$  is of the form (193). To prove this, we consider a countable dense subset  $\mathcal{D}$  of local functions of  $\mathcal{X}$ . Let  $f \in \mathcal{D}$ . By (194) and (195),

$$\int_{\mathcal{X}} f(\eta) d\mu_{v,\varepsilon}^+(N)(\eta) = \mathbb{E} \left\{ \frac{1}{t} \int_0^t \frac{1}{[N\varepsilon] + 1} \sum_{z \in \mathbb{Z}: 0 \leq z \leq \varepsilon N} f(\tau_{z+[Nvs]} \eta_{Ns}^N) ds \right\}. \quad (202)$$

By (189) in Proposition 7.4 (for  $A = 0, B = \varepsilon$ ), we can rewrite the right-hand side of (202) as

$$\mathbb{E} \left\{ \frac{1}{t} \int_0^t \frac{1}{[N\varepsilon] + 1} \sum_{z \in \mathbb{Z}: 0 \leq z \leq \varepsilon N} \bar{f}(M_{z+[Nvs],l} \bar{\eta}_{Ns}^N) ds \right\} + \delta_{N,l,\varepsilon} \quad (203)$$

where  $\lim_{l \rightarrow +\infty} \limsup_{N \rightarrow +\infty} \delta_{N,l,\varepsilon} = 0$ . The expectation in (203) is of the form

$$\int_{[0,n]} \bar{f}(\rho) d\lambda_{N,l,\varepsilon}(\rho) = \int_{\mathcal{X}} f(\eta) d\mu_{N,l,\varepsilon}(\eta) \quad (204)$$

for some probability measure  $\lambda_{N,l,\varepsilon}$  on  $[0, n]$  (not depending on  $f$ ), where (recall definition (44) of  $\bar{f}$ )

$$\mu_{N,l,\varepsilon} := \int_{[0,n]} \nu_\rho d\lambda_{N,l,\varepsilon}(\eta). \quad (205)$$

It follows from (200) that

$$\int_{\mathcal{X}} f(\eta) d\mu_v(\eta) - \delta_{N,l,\varepsilon} = \int_{\mathcal{X}} f(\eta) \int_{[0,n]} \nu_\rho d\lambda_{N,l,\varepsilon}(\rho). \quad (206)$$

By diagonal extraction we can find a probability measure  $\lambda_v$  on  $[0, n]$  and a subsequence of  $(N_{\ell_k})$  such that, simultaneously for each  $f \in \mathcal{D}$ ,  $\lambda_{N, l, \varepsilon}$  converges weakly to  $\lambda_v$  as  $N \rightarrow +\infty$  along this subsequence,  $l \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ . This yields the first equality in (193). The second one holds because the process is attractive and  $\nu_\alpha, \nu_\beta$  are invariant for  $\mathcal{L}^N$ , implying  $\nu_\alpha \leq \mu_v \leq \nu_\beta$ .

*Step 2.* We prove that for  $N = N_{\ell_k} \rightarrow +\infty$ ,

$$\lim_{N \rightarrow \infty} \mu_{\alpha, \beta} S_{Nt}^N \left( \frac{1}{Nt} \sum_{x \in V: \lfloor uNt \rfloor \leq x(0) \leq \lfloor vNt \rfloor} \bar{\eta}(x) \right) = F(v) - F(u), \quad (207)$$

where

$$F(w) = \int [w\rho - G(\rho)] \lambda_w(d\rho). \quad (208)$$

Define the function

$$\begin{aligned} g_N(s) &= \sum_{\lfloor uNs \rfloor + 1 \leq x(0) \leq \lfloor vNs \rfloor} \mu_{\alpha, \beta} S_t^N(\bar{\eta}(x)) + (vNs - \lfloor vNs \rfloor) \mu_{\alpha, \beta} S_s^N(\bar{\eta}(\lfloor vNs \rfloor + 1)) \\ &\quad + (\lfloor uNs \rfloor + 1 - uNs) \mu_{\alpha, \beta} S_s^N(\bar{\eta}(\lfloor uNs \rfloor)). \end{aligned}$$

This function  $g_N$  is absolutely continuous and therefore  $g_N(t) = \int_0^t g'_N(s) ds$ . For  $u, v \in D$  and  $s$  such that  $sNv, sNu \notin \mathbb{Z}$  we have

$$\begin{aligned} \frac{d}{ds} [g_N(Ns)] &= Nv \mu_{\alpha, \beta} S_{Ns}^N(\bar{\eta}(\lfloor vNs \rfloor + 1)) - Nu \mu_{\alpha, \beta} S_{Ns}^N(\bar{\eta}(\lfloor uNs \rfloor)) \\ &\quad + \sum_{\lfloor uNs \rfloor + 1 \leq x(0) \leq \lfloor vNs \rfloor} \mu_{\alpha, \beta} S_{Ns}^N(\mathcal{L}^N \bar{\eta}(x)) \\ &\quad + (vNs - \lfloor vNs \rfloor) \mu_{\alpha, \beta} S_{Ns}^N(\mathcal{L}^N \bar{\eta}(\lfloor vNs \rfloor + 1)) \\ &\quad + (\lfloor uNs \rfloor + 1 - uNs) \mu_{\alpha, \beta} S_{Ns}^N(\mathcal{L}^N \bar{\eta}(\lfloor uNs \rfloor)) \\ &= Nv \mu_{\alpha, \beta} S_{Ns}^N(\bar{\eta}(\lfloor vNs \rfloor + 1)) - Nu \mu_{\alpha, \beta} S_{Ns}^N(\bar{\eta}(\lfloor uNs \rfloor)) \\ &\quad + \mu_{\alpha, \beta} S_{Ns}^N(\tau_{\lfloor uNs \rfloor} Nj(\eta)) - \mu_{\alpha, \beta} S_{Ns}^N(\tau_{\lfloor vNs \rfloor} Nj(\eta)) \\ &\quad + (vNs - \lfloor vNs \rfloor) \mu_{\alpha, \beta} S_{Ns}^N(\mathcal{L}^N \bar{\eta}(\lfloor vNs \rfloor + 1)) \\ &\quad + (\lfloor uNs \rfloor + 1 - uNs) \mu_{\alpha, \beta} S_{Ns}^N(\mathcal{L}^N \bar{\eta}(\lfloor uNs \rfloor)) \end{aligned}$$

where we have used (57) in the second equality. Note that by the translation invariance of  $\mu_v$  (the last two terms vanish in the limit) we have

$$\frac{g_{N_{\ell_k}}(N_{\ell_k} t)}{N_{\ell_k}} = \frac{1}{N_{\ell_k}} \int_0^t \frac{d}{ds} [g_N(N_{\ell_k} s)] ds \rightarrow F(v) - F(u) \quad k \rightarrow \infty,$$

and that the difference between  $\frac{g_{N_{\ell_k}}(N_{\ell_k} t)}{N_{\ell_k}}$  and the l.h.s. of (207) is at most  $O(N_{\ell_k}^{-1})$ , and the proof of (207)–(208) is complete.

*Step 3.* Consider the measure  $\lambda_v(d\rho)$  in (208). As in [5, Theorem 2.1], using attractiveness and processes starting from  $\mu_{\theta,\beta}$  and  $\mu_\theta$  for  $\theta \in [\alpha, \beta]$ , we can show that

$$\int_{[\alpha,\beta]} [v\rho - G(\rho)]\lambda_v(d\rho) = \inf_{\beta \leq \theta \leq \alpha} \{v\theta - G(\theta)\}.$$

By Proposition 7.3 it follows that

$$\lambda_v = \delta_{u(v,1)}.$$

Next, as in [2], one can show that the convergence of the Cesaro mean in fact implies

$$\lim_{N \rightarrow \infty} \mu_{\alpha,\beta} \tau_{[Nvt]} S_{Nt}^N = \nu_{u(v,1)}. \quad (209)$$

This completes the proof of statement (iii) of Theorem 3.1.  $\square$

**Remark 7.2.** When  $\theta(N) = N$ , that is  $\mathcal{L}^N = N(L_h + L_v)$ , the dynamics follows a time rescaling of a fixed generator. In this case, the above proof can be simplified by using the original argument of [2]. The limit (192)–(193) can be obtained by observing that Cesaro limits of the process distribution are contained in the subset  $\mathcal{I} \cap \mathcal{S}$  of invariant measures of this fixed generator, and using Theorem 2.1.

## 7.2 Local equilibrium: proof of Theorem 3.1, part (ii)

In the sequel, we use the following realization of a local Gibbs state ( $\xi^N$ ) with distribution (40). Take an i.i.d. sequence  $(U_x)_{x \in V}$  of  $\mathcal{U}(0, 1)$  random variables, and define, for  $x = (z, i) \in V$ ,

$$\xi^N(z, i) = \mathbf{1}_{\{U_x \leq u_z^{N,i}\}}. \quad (210)$$

The construction (210) has the property that if we construct two Gibbs states ( $\xi^N$ ) and ( $\zeta^N$ ) with respective profiles  $u(\cdot)$  and  $v(\cdot)$  with the same  $\mathcal{U}(0, 1)$  family, then

$$\mathbb{E} \left[ (\xi^N(z, i) - \zeta^N(z, i))^{\pm} \right] = (u_x^{N,i} - v_x^{N,i})^{\pm}. \quad (211)$$

In particular,  $\xi^N(z, i)$  and  $\zeta^N(z, i)$  are ordered a.s. like  $u_x^N$  and  $v_x^N$ .

*Proof of Theorem 3.1, part (ii).*

*Step one.* First assume that  $u_0(\cdot)$  has compact support and satisfies the assumption of Proposition 7.4. The triangle inequality and Proposition 7.4 imply, for  $\varphi \in C_K^0(\mathbb{R})$ ,

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left\{ \left| N^{-1} \sum_{x \in \mathbb{Z}} \varphi \left( \frac{x}{N} \right) \tau_x f(\eta_{Nt}^N) - N^{-1} \sum_{x \in \mathbb{Z}} \varphi \left( \frac{x}{N} \right) \bar{f}(M_{x,i} \bar{\eta}_{Nt}^N) \right| dx \right\} = 0. \quad (212)$$

Indeed, in the sum in (212) we may replace  $\tau_x f(\eta_{Nt}^N)$  by the spatial average of  $\tau_y f(\eta_{Nt}^N)$  over  $y \in \mathbb{Z}$  such that  $|y - x| \leq N\varepsilon$ . The replacement error vanishes as  $N \rightarrow +\infty$  followed by  $\varepsilon \rightarrow 0$ , because by an exchange of summation this is equivalent to replacing  $\varphi(x/N)$  with its spatial average. We can then apply Proposition 7.4 to replace the spatial average of  $\tau_y f(\eta_{Nt}^N)$  by  $\bar{f}(M_{x,l}\bar{\eta}_{Nt})$ .

Next, by part (i) of Theorem 3.1, for every  $A > 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ \int_{-A}^A |M_{\lfloor Nx \rfloor, N\varepsilon} \bar{\eta}_{Nt} - u(x, t)| dx \right\} = 0. \quad (213)$$

Since  $\bar{f}$  is continuous,  $\bar{f}$  allows us to replace the second sum in (212) by the last integral in (45).

*Step two.* Next assume  $u_0(\cdot)$  is a Borel function supported in  $[a, b] \subset \mathbb{R}$ . It can be approximated in  $L^1(\mathbb{R})$  by a sequence  $(u_{k,0}(\cdot))_{k \in \mathbb{N}}$  of continuous functions supported in  $[a, b]$  and satisfying the assumption of Proposition 7.4. For each  $k \in \mathbb{N}$ , let  $(\eta_{k,0}^N)_{N \in \mathbb{N}^*}$  denote a local Gibbs state with profile  $u_{k,0}$ . Since the coupling cannot create discrepancies, using (211), we have

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{(x,i) \in V} |\eta_{k,Nt}^N(x, i) - \eta_{Nt}^N(x, i)| \right\} \leq \sum_{i \in W} \int |u_{k,0}^i(x) - u_0^i(x)| dx. \quad (214)$$

Denoting (for any  $\eta, \xi \in \mathcal{X}$ )

$$\begin{aligned} A_{\varphi, f}^N(\eta) &:= N^{-1} \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{N}\right) \tau_x f(\eta) \\ B^N(\eta; \xi) &:= N^{-1} \sum_{(x,i) \in V} |\eta(x, i) - \xi(x, i)|. \end{aligned}$$

we have that

$$|A_{\varphi, f}^N(\eta_{k, Nt}^N) - A_{\varphi, f}^N(\eta_{Nt}^N)| \leq \|\varphi\|_\infty \|f\|_\infty B^N(\eta_{k, Nt}^N; \eta_{Nt}^N). \quad (215)$$

For each  $k \in \mathbb{N}$ ,  $\eta_{k, Nt}^N$  satisfies w.l.e. with profiles  $u_k^i(\cdot, t) := \tilde{\rho}_i[u_k(\cdot, t)]$ , and by contraction property (179) for the hydrodynamic equation (61),  $u_k^i(\cdot, t) \rightarrow u_0^i(\cdot, t)$  in  $L^1_{\text{loc}}(\mathbb{R})$ . Then (214)–(215) implies weak local equilibrium for  $(\eta_{Nt}^N)$ .

*Step three.* Assume  $u_0(\cdot)$  is a Borel function on  $\mathbb{R}$ . For  $k \in \mathbb{N}$ , define  $u_{k,0}$  by truncating  $u_0$  to 0 outside  $[-k, k]$ . Denote by  $u_k(\cdot, \cdot)$  the corresponding entropy solution. The associated l.g.s.  $\eta_{k,0}^N$  can be coupled to  $\eta_0^N$  so that  $\eta_{k,0}^N(x, i) = \eta_0^N(x, i)$  for all  $x \in [-kN, kN]$  and  $i \in W$ . Choosing  $k$  large enough, using w.l.e. for  $(\eta_{k, Nt}^N)$  and finite propagation property (Proposition 7.1) yields the conclusion.  $\square$

### 7.3 Macroscopic stability: proof of Proposition 7.2

Recall the definition of discrepancies after equation (185). For the proof of Proposition 7.2, we introduce a labeling of discrepancies. We initially label blue particles ( $\eta$  discrepancies) using successive integers. We denote by  $X_t^k \in V$  the position of the blue particle with label  $k$  (in short, we shall simply say: the position of blue label  $k$ ) at time  $t$ . This position is defined until the particle possibly coalesces with a red one. The labeling is chosen initially so that for any pair of labels  $u, v$ ,  $u < v \Rightarrow X_0^u(0) \leq X_0^v(0)$ . We define the motion of labeled particles as follows from the Harris construction so that at time  $t > 0$ , we still have, for any pair of labels  $u, v$ ,

$$u < v \Rightarrow X_t^u(0) \leq X_t^v(0). \quad (216)$$

Let now  $k$  be a blue label. If  $X_{t-}^k = x = (z, i)$ ,  $t \in \mathcal{N}_{(x,y)}$ , and there is a hole at  $y$  at time  $t-$ , then label  $k$  first jumps to  $y$ , so that  $X_t^k = y$ . This may in some cases break the ordering relation (216). If so, an exchange of labels is performed after the jump to prevent this without modifying the particle configurations. Namely:

(i) if  $y = (z', i)$  with  $z' > z$ , label  $k$  is exchanged with the biggest label  $l$  such that  $X_{t-}^l(0) = z$ , if  $l \neq k$ ; so that in the end we have  $X_t^k = x$  and  $X_t^l = y$ .

(ii) If  $y = (z', i)$  with  $z' < z$ , label  $k$  is exchanged with the smallest label  $l$  such that  $X_{t-}^l(0) = z$ , if  $l \neq k$ ; so that in the end we have  $X_t^k = x$  and  $X_t^l = y$ .

Similarly If  $X_{t-}^k = y = (z', j)$ ,  $t \in \mathcal{N}_{(x,y)}$  and there is a black particle at time  $t-$  at  $x$ , the black particle exchanges with label  $k$ , but as above, the latter exchanges with another label  $l$  initially at  $(z', j)$  for some  $j \neq i$ , if necessary to maintain (216).

Finally, if a coalescence occurs following a jump from a site  $(z, i)$  to  $(z, j)$  with  $z \in \mathbb{Z}$  and  $i \neq j$ , we exchange the label  $k$  of the newly coalesced blue particle with a label  $l$  chosen randomly (independently of anything else) among labels of all blue particles currently on  $\{z\} \times W$ , so that after the jump,  $l$  will be coalesced everafter, and  $k$  still uncoalesced if  $l \neq k$ . It is important to notice that this redistribution of labels does not affect the spatial ordering of blue labels.

We also label red particles according to similar rules and denote by  $Y_t^k$  the position at time  $t$  of the red particle with label  $k$ .

As will appear below, the quantities in (182) are closely related to crossings of blue particles by red particles as we now define.

**Definition 7.1.** (Crossings.) *Let  $k$  and  $l$  be two labels (of the same or different colours) and  $Z_t^k, Z_t^l$  their positions at time  $t$  (e.g.  $Z_t^k = X_t^k$  if  $k$  is blue, or  $Z_t^k = Y_t^k$  if it is red). We shall say that  $l$  has crossed  $k$  to the right between times  $s$  and  $t$  (where  $s < t$  or  $s = t-$ ) if  $Z_s^l(0) < Z_s^k(0)$  and  $Z_t^l(0) \geq Z_t^k(0)$ ; and that  $l$  has crossed  $k$  to the left if  $Z_s^l(0) \geq Z_s^k(0)$  and  $Z_t^l(0) < Z_t^k(0)$ .*

Observe that due to the non-strict inequality in the above definition, it is indeed possible for two blue particles to “cross” in spite of (216).

We introduce the following definitions of relevant crossings:

**Definition 7.2.** Let  $\mathcal{V} = \mathcal{V}_0$  denote the set of blue labels initially in  $[-aN, aN]$ , and  $\mathcal{V}_t$  the set of blue labels  $v$  that are uncoalesced at time  $t$ .

$$L_t^v := \mathbf{1}_{\mathcal{V}_t}(v) \left\{ \sum_{w \in \mathcal{W}} \mathbf{1}_{\{Y_0^w(0) < X_0^v(0)\}} \mathbf{1}_{\{Y_t^w(0) \geq X_t^v(0)\}} \right\}, \quad (217)$$

$$R_t^v := \mathbf{1}_{\mathcal{V}_t}(v) \left\{ \sum_{w \in \mathcal{W}} \mathbf{1}_{\{Y_0^w(0) \geq X_0^v(0)\}} \mathbf{1}_{\{Y_t^w(0) < X_t^v(0)\}} \right\}. \quad (218)$$

That is,  $L_t^v$  is the number of red particles having crossed  $v$  to the right up to time  $t$ , and  $R_t^v$  the number of red particles having crossed  $v$  to the left, on the time interval  $(0; t]$ .

The proof of Proposition 7.2 follows the ideas of [9, Lemma 3.1]. However in our setting, to take into account the possibly slower vertical scaling, a significant refinement involving condition (52) is necessary, see (230) and Lemma 7.2 below.

Most of the proof is contained in the following two results. Lemma 7.1 says that it is unlikely for a *given* label to be crossed by many labels of opposite colour, while Corollary 7.1 says it is simultaneously unlikely for *all* relevant blue labels. We point out that *these results do not in themselves require (180)*, but their application to Proposition 7.2 does. However, they will be used to prove the quasi-interface property, cf. Lemma 7.5 below, under a set of assumptions that does not in general imply (180).

**Lemma 7.1.** Let  $F_v(t) = \{v \in \mathcal{V}_t\}$  be the event that label  $v$  has not coalesced by time  $t$ . Then for any  $\gamma > 0$ ,

$$\mathbb{P} \left( F_v(t); C_t^v > \gamma N \right) < \max \left[ e^{-CN}, \exp \left\{ -C \frac{\theta(N)^{m^*}}{N^{m^*-1}} \right\} \right] \quad (219)$$

for some constant  $C = C(\gamma) > 0$ , where  $C_t^v := L_t^v + R_t^v$ .

Observe that the right-hand side of (219) vanishes if and only if (52) holds.

**Corollary 7.1.** Under assumptions of Lemma 7.1, we have

$$\lim_{N \rightarrow +\infty} \mathbb{P} \left( \sup_{v \in \mathcal{V}, t > 0} \mathbf{1}_{\mathcal{V}_t}(v) C_t^v > \gamma N \right) = 0. \quad (220)$$

**Remark 7.3.** Lemma 7.1 and Corollary 7.1 equally hold for red particles, since exchanging the roles of  $\eta$  and  $\xi$  exchanges the colours of blue and red particles.

We first conclude the proof of Proposition 7.2, then prove Lemma 7.1 and Corollary 7.1.

*Proof of Proposition 7.2. Notational remark.* In the sequel,  $\phi_t^N$  will sometimes be denoted simply by  $\phi_t$ . Similarly, dependence on  $N$  will sometimes be implicit for other quantities such as  $\eta_0, \xi_0, X_t^k, Y_t^l, \mathcal{V}_t$  defined above.

It is enough to show that for  $\gamma > 0$ ,

$$\lim_{N \rightarrow +\infty} \mathbb{P} \left( \sup_{z \in \mathbb{Z}} \sum_{u \in \mathbb{Z}: u \geq z} \phi_t^N(u) - \sup_{z \in \mathbb{Z}} \sum_{u \in \mathbb{Z}: u \geq z} \phi_0^N(u) > \gamma N \right) = 0. \quad (221)$$

Then, (182) follows by exchanging the roles of  $(\eta_t^N)_{t \geq 0}$  and  $(\xi_t^N)_{t \geq 0}$ .

Notice that, if there is initially no blue particle in  $[-aN, aN]$ , then  $\eta_0 \leq \xi_0$ , the suprema in (221) are achieved for any sufficiently large  $z$ , and are equal to 0, thereby implying the conclusion of the proposition. Hence, we will now assume that there is initially at least one blue particle in  $[-aN; aN]$ .

Remark that there exists a label  $v \in \mathcal{V}_t$  such that

$$\sup_{z \in \mathbb{Z}} \sum_{u \in \mathbb{Z}: u \geq z} \phi_t(u) = \sum_{u \in \mathbb{Z}: u \geq X_t^v(0)} \phi_t(u). \quad (222)$$

It follows that

$$\sup_{z \in \mathbb{Z}} \sum_{u \in \mathbb{Z}: u \geq z} \phi_t(u) - \sup_{z \in \mathbb{Z}} \sum_{u \in \mathbb{Z}: u \geq z} \phi_0(u) \leq \sup_{v \in \mathcal{V}} \mathbf{1}_{\mathcal{V}_t}(v) \Delta_t^v$$

where, for  $v \in \mathcal{V}$ , we set

$$\Delta_t^v := \sum_{u \in \mathbb{Z}: u \geq X_t^v(0)} \phi_t(u) - \sum_{u \in \mathbb{Z}: u \geq X_0^v(0)} \phi_0(u).$$

Define

$$B_t^v := \mathbf{1}_{\{v \in \mathcal{V}_t\}} \left\{ \sum_{x \in V: x(0) \geq X_t^v(0)} (\eta_t^N(x) - \xi_t^N(x))^+ - \sum_{x \in V: x(0) \geq X_0^v(0)} (\eta_0^N(x) - \xi_0^N(x))^+ \right\}. \quad (223)$$

In words,  $B_t^v$  is the number of blue particles that crossed  $v$  to the right minus the number that crossed  $v$  to the left. It follows from definitions (217)–(218) and (223) that

$$\Delta_t^v = R_t^v - L_t^v + B_t^v. \quad (224)$$

Moreover,

$$\begin{aligned}
B_t^v &= (v_t^* - V_t + 1) - (v_0^* - V_0 + 1) \\
&= (v - V_t) - (v - V_0) + (v_t^* - v_0^*) \\
&\leq n - 1
\end{aligned} \tag{225}$$

where  $v_t^* := \max \mathcal{V}_t$  denotes the highest of all blue labels at time  $t$ , and  $V_t$  the lowest among blue labels lying at time  $t$  at the same location as  $v$ . In the first equality in (225), each of the quantities in parenthesis is equal to the corresponding sum in the definition (218) of  $B_t^v$ , measuring the number of blue particles to the right of  $v$  (including  $v$  itself), hence the equality. The first two terms on the second line of (225) are each positive by definition, and they are bounded from above by  $n - 1$  due to (216). The last term on the second line is nonpositive because blue labels cannot be created, hence the final inequality. Finally, (221) follows from Corollary 7.1, (224) and (225).  $\square$

*Proof of Lemma 7.1.* For each blue label  $v$ , define the stopping times

$$S_0^v = 0 \leq T_1^v \leq S_1^v \leq T_2^v \dots \tag{226}$$

(hereafter denoted simply by  $T_i, S_i$ ), by

$$T_i = \inf \{t \geq S_{i-1} : \text{there is a red particle at } (X_t^v(0), j) \text{ for some } j \neq X_t^v(1) \text{ or } v \text{ has coalesced by time } t\} \tag{227}$$

$$S_i = \inf \left\{ t > T_i : t \in \bigcup_{(x,y) \in V_i} \mathcal{N}_{(x,y)} \text{ or } v \text{ has coalesced by time } t \right\}, \text{ where}$$

$$V_i := \{(z, j), (z', j) \in V^2 : j \in W, z \neq z', \{z, z'\} \subset \{X_{T_i}^v(0), X_{T_i}^v(0) \pm 1\}\} \tag{228}$$

In words, the first part of the event defining  $T_i$  says that at time  $T_i$ , a red label finds itself at the same spatial position as  $X_t^v$  in  $\mathbb{Z}$  but on another lane (this only makes sense as long as  $v$  has not coalesced, hence the second part of the event). The first part of the event defining  $S_i$  says that a Poisson clock rings for one of the  $2n$  horizontal edges connecting the position of  $X^v$  to a neighbouring position on one of the lanes. Let  $(\mathcal{F}_t)_{t \geq 0}$  be the filtration generated by  $(\eta_s^N, \xi_s^N)_{0 \leq s \leq t}$ . Note that  $T_i = S_{i-1}$  is possible only if and only if the first event in  $S_{i-1}$  had brought a red particle or left an existing one at the same location as  $v$ , or  $v$  was already coalesced at time  $T_i$ , whereas  $S_i = T_i$  is possible if and only if  $v$  was already coalesced at time  $T_i$ . In particular, the sequence (226) becomes stationary as soon as  $v$  coalesces.

An important fact is that on the time interval  $(S_i, S_{i+1}]$ , at most  $(n-1)$  red/blue crossings may occur, i.e.

$$C_{S_{i+1}} - C_{S_i} = L_{S_{i+1}} - L_{S_i} + R_{S_{i+1}} - R_{S_i} \leq n - 1. \tag{229}$$

Indeed on  $(S_i, T_{i+1})$  there is no red particle at the same spatial location as  $v$ , hence no crossing may occur. One crossing possibly occurs at time  $T_{i+1}$  if the



red particle was on the left of  $v$  at  $T_{i+1}^-$ . On the time interval  $(T_{i+1}, S_{i+1})$ ,  $v$  cannot move and no red particle can move from or to  $z = X_{T_{i+1}^-}$ , so no crossing may occur. Finally at time  $S_{i+1}$  a red particle may move from or to  $z$ , which generates one crossing; or  $v$  may jump, which may generate up to  $(n-1)$  crossings. We now show that on  $F_v(t)$ ,

$$\mathbb{P}(\text{label } v \text{ coalesces in } (T_i, S_i] | \mathcal{F}_{T_i}) \geq \frac{p^*}{n-1} \left( \frac{q^*}{q^* + \frac{N}{\theta(N)}} \right)^{m^*} \quad (230)$$

where  $m^*$  is given by (51) and  $p^*, q^* > 0$  are independent of  $i, v$  and  $N$ . This is where condition (52) is involved through Lemma 7.2 below. First note that on the time interval  $[T_i, S_i)$ , in the coupled process  $(\eta_t, \xi_t)_{t \geq 0}$ ,  $v$  does not move and the vertical layer  $\{X_{T_i}^v\} \times W$  is isolated. By strong Markov property, conditioned on  $\mathcal{F}_{T_i}$ , this layer evolves following the simple exclusion process on  $\{X_{T_i}^v\} \times W$  with kernel  $q(\cdot, \cdot)$  accelerated by a factor  $\theta(N)$ , with initial condition given by the restriction of  $(\eta_{T_i}, \xi_{T_i})$ . Up to time  $S_i$ , it can be coupled to an exclusion process on the same layer (ignoring the rest of space). Let us denote by  $U_i = U_i^*/\theta(N)$  the first coalescence time of a blue and red particle in this process (where  $U_i^*$  is the first coalescence time for the non accelerated process). If  $U_i$  is smaller than the exponential first times of all Poisson processes involved in  $S_i$ , the coalescence occurs before  $S_i$ , unless  $v$  has coalesced before  $S_i$ . Further (conditioned on  $\mathcal{F}_{T_i}$ ), these exponential times are independent of  $U_i$ , because the latter depends only on vertical Poisson processes on our layer. The total rate of these Poisson processes involved in  $S_i$  is

$$\lambda^* := \sum_{i \in W} (d_i + l_i)$$

and thus the first of the corresponding exponential times is an exponential random variable  $W_i = W_i^*/\lambda^*$ , where  $W_i^* \sim \mathcal{E}(1)$ . For simplicity and without loss of generality we may assume  $\lambda^* = 1$ . Thus, the probability that our coalescence occurs before  $S_i$  satisfies

$$\mathbb{P}\left(U_i^* < \frac{\theta(N)}{N} W_i^*\right) = \mathbb{E}\left(e^{-\frac{N}{\theta(N)} U_i^*}\right) \geq p^* \left( \frac{q^*}{q^* + \frac{N}{\theta(N)}} \right)^{m^*} \quad (231)$$

where the equality follows from conditioning on  $U_i^*$ , and the inequality from Lemma 7.2, where  $q^*$  is defined (the random variable  $U_i^*$  in (231) has the same law as the random variable  $T^*$  in Lemma 7.2). The proof of (230) is concluded by observing that there is a probability at least  $1/(n-1)$  that the coalesced label selected after the vertical redistribution of labels defined above is eventually  $v$ .

Moreover, we have

$$\frac{q^*}{q^* + \frac{N}{\theta(N)}} \geq \min\left(a, b \frac{\theta(N)}{N}\right) \quad (232)$$

for constants  $a, b$  independent of  $i, v, N$ . Using strong Markov property and (229)–(232), we conclude that for every  $\gamma > 0$ , there are constants  $A, C$  independent of  $i, v$  and  $N$ , such that

$$\begin{aligned} \mathbb{P}\left(F_v(t); C_t^v > \gamma N\right) &< \left[1 - \frac{p^*}{n-1} \left(\frac{q^*}{q^* + \frac{N}{\theta(N)}}\right)^{m^*}\right]^{\frac{\gamma N}{n-1}} \\ &< \max\left[e^{-CN}, \exp\left\{-C\frac{\theta(N)^{m^*}}{Nm^{*-1}}\right\}\right] \end{aligned} \quad (233)$$

for some constant  $C > 0$ , where we used the inequality  $1 - x \leq e^{-x}$ .  $\square$

*Proof of Corollary 7.1.* First observe that a union bound from Lemma 7.1 over the  $O(N)$  relevant labels  $v$  (as done in [9, Lemma 3.1]) leads to a slightly worse condition than (52) to get a vanishing bound, namely

$$\lim_{N \rightarrow +\infty} \frac{\theta(N)}{N^{1 - \frac{1}{m^*}} (\log N)^{\frac{1}{m^*}}} = +\infty. \quad (234)$$

Note that when  $m^* = 1$ , (52) reduces to  $\theta(N) \rightarrow +\infty$ , whereas (234) introduces a superfluous growth condition.

To prove (220), we will show that with negligible error, we can restrict the supremum to a set of labels whose size is bounded uniformly with respect to  $N$ . To this end, we prove below that for every  $\varepsilon > 0$ , there exists a finite subset  $\mathcal{V}' = \mathcal{V}'(\eta_0, \xi_0, \mathcal{V})$  such that

$$|\mathcal{V}'| \leq \frac{2an}{\varepsilon}, \quad \sup_{v \in \mathcal{V}} \mathbf{1}_{\mathcal{V}'}(v) L_t^v \leq 2(N\varepsilon + n) + \sup_{v \in \mathcal{V}'} \mathbf{1}_{\mathcal{V}'}(v) L_t^v \quad (235)$$

$$\sup_{v \in \mathcal{V}} \mathbf{1}_{\mathcal{V}'}(v) R_t^v \leq 2(N\varepsilon + n) + \sup_{v \in \mathcal{V}'} \mathbf{1}_{\mathcal{V}'}(v) R_t^v. \quad (236)$$

The limit (220) then follows from Lemma 7.1, (235)–(236) and a union bound, lastly letting  $\varepsilon \rightarrow 0$ . In order to prove (235)–(236), we define a subdivision of  $[-aN; aN] \cap \mathbb{Z}$ , denoted

$$\mathcal{E} = \mathcal{E}(\eta_0, \xi_0) = \{z_k : k = 1, \dots, m\}, \quad m \leq \frac{2an}{\varepsilon},$$

as follows: For  $k \geq 1$ , we denote by  $v_k$  the label of a blue particle initially at  $z_k$ , i.e.

$$X_0^{v_k}(0) = z_k, \quad k = 1, \dots, m.$$

In words,  $z_1$  is the spatial location of the leftmost blue particle in  $[-aN, aN]$ , and  $v_{k+1}$  is the first blue particle to the right of  $v_k$  such that there are at least  $N\varepsilon$  red particles on  $(z_k, z_{k+1}]$ . For the first value of  $k$  such that the set in the second minimum is empty, we set  $m = k$  if  $v_k$  is the rightmost blue particle, otherwise we set  $m = k + 1$  and  $z_{k+1}$  to be the position of the next blue particle

on its right. Note that the minimum defining  $z_1$  always exists since there is initially at least one blue particle, thus  $m \geq 2$ . Let  $v \in \mathcal{V} \setminus \{v_k : k = 1, \dots, m\}$ . Then we have

$$z_k < X_0^v(0) < z_{k+1}$$

for a unique  $k = k(v) \in \{1, \dots, m\}$ . Now, let  $v \in \mathcal{V} \cap (z_k; z_{k+1})$  for  $k = 1, \dots, m-1$ . Observe that a red particle that crossed  $v$  from right to left in the time interval  $(0; t]$  and did not lie initially in  $[z_k; z_{k+1}]$  must have crossed  $v_{k+1}$  in this time interval. Similarly, a red particle that crossed  $v_{k+1}$  from left to right and did not start in  $[z_k; z_{k+1}]$  must have crossed  $v$ . Note also that by definition of  $z_k$ , there are initially at most  $N\varepsilon + n$  red particles in  $[z_k; z_{k+1}]$ . It follows from this and a similar comparison with  $v_k$  that

$$\begin{aligned} R_t^v &\leq R_t^{v_{k+1}} + N\varepsilon + n, & L_t^v &\geq L_t^{v_{k+1}} - N\varepsilon - n \\ R_t^v &\geq R_t^{v_k} - N\varepsilon - n, & L_t^v &\leq L_t^{v_k} + N\varepsilon + n. \end{aligned}$$

Thus (235)–(236) hold for  $\mathcal{V}' = \mathcal{E}$ .  $\square$

We conclude by the following lemma required to derive (230) above. Recall the definitions of discrepancies, coloured particles and coalescence after Proposition 7.2.

**Lemma 7.2.** *Consider the simple exclusion process on  $W = \{0, \dots, n-1\}$  with weakly irreducible jump kernel  $q(\cdot, \cdot)$ . Let  $(\zeta_t^1, \zeta_t^2)$  be a coupling of two such processes via a common Harris system. Assume  $\zeta_0^1$  and  $\zeta_0^2$  have at least two opposite discrepancies, and denote by  $T^*$  the first coalescence time of two opposite discrepancies. Then, under condition (52), there exist  $p^* > 0$ , and  $q^* = q^*(n, q(\cdot, \cdot)) > 0$  (not depending on the initial configurations), such that, for every  $\theta > 0$  and  $m^*$  given by (51),*

$$\mathbb{E} \left( e^{-\theta T^*} \right) \geq p^* \left( \frac{q^*}{q^* + \theta} \right)^{m^*}. \quad (237)$$

**Example 7.1.** *When  $n = 2$ ,  $m^*$  and  $q^*$  are explicit, as a coalescence occurs after the first jump. Indeed, looking into the proof below shows that the statement holds with  $m^* = 1$  and  $q^* = q(0; 1) + q(1; 0)$ . More generally under (53), we have  $m^* = 1$  and  $q^*$  given by (53).*

*Proof of Lemma 7.2.* By Lemma 7.3 below, there exists  $j \leq m^*(n, q(\cdot, \cdot))$  and a sequence  $(x_0, \dots, x_j) \in W^{j+1}$ , such that for  $k = 0, \dots, j-1$  and  $i \in \{1; 2\}$ ,

$$\zeta^{i,0} := \zeta_0^i \quad (238)$$

$$\zeta^{i,k+1} := (\zeta^{i,k})_{x_k, x_{k+1}} \quad (239)$$

$$q(x_k, x_{k+1}) \max \{ \zeta^{i,k}(x_k) [1 - \zeta^{i,k}(x_{k+1})], \quad i \in \{1; 2\} \} > 0 \quad (240)$$

and that the jump from  $x_{j-1}$  to  $x_j$  for  $(\zeta^{1,j-1}, \zeta^{2,j-1})$  generates a coalescence, i.e.

$$\begin{aligned} & \left| \zeta^{1,j}(x_{j-1}) - \zeta^{2,j}(x_{j-1}) \right| + \left| \zeta^{1,j}(x_j) - \zeta^{2,j}(x_j) \right| \\ & < \left| \zeta^{1,j-1}(x_{j-1}) - \zeta^{2,j-1}(x_{j-1}) \right| + \left| \zeta^{1,j-1}(x_j) - \zeta^{2,j-1}(x_j) \right|. \end{aligned} \quad (241)$$

Let  $(T_k)_{k \geq 0}$  denote the successive jump times of the Markov process  $(\zeta_t^1, \zeta_t^2)_{t \geq 0}$ , and  $(\tilde{\zeta}_k^1, \tilde{\zeta}_k^2) := (\xi_{T_k}^1, \xi_{T_k}^2)$  the discrete-time skeleton of this process. Let

$$T^s := \inf\{k \geq 1 : \text{a coalescence occurs for } (\zeta_t^1, \zeta_t^2) \text{ at time } t = T_k\} \quad (242)$$

denote the discrete coalescence time of this process (we do not know a priori whether  $T^s < +\infty$  and do not need this information, but this follows from (243) below and strong Markov property). Conditioned on  $(\tilde{\zeta}_k^1, \tilde{\zeta}_k^2)_{k \geq 0}$ , the transition times  $T_k$  are separated by intervals  $\Delta_k := T_k - T_{k-1}$  that are independent exponentials with parameters  $\lambda_k = \lambda(\tilde{\zeta}_k^1, \tilde{\zeta}_k^2)$  bounded from below by a constant  $q^*$  independent of  $(\zeta_0^1, \zeta_0^2)$  (indeed if we start with at least two opposite discrepancies, before  $T^s$ , we can never reach a blocked configuration with all white or all black particles). By Lemma 7.3, uniformly over initial configurations,

$$p^* := \mathbb{P}(T^s \leq m^*) > 0 \quad (243)$$

Hence, using the moment generating function of  $\Gamma(m^*, q^*)$  distribution,

$$\begin{aligned} \mathbb{E}\left(e^{-\theta T^s}\right) &= \mathbb{E}\left(e^{-\theta \sum_{k=1}^{T^s} \Delta_k}\right) \\ &= \sum_{j=1}^{+\infty} \mathbb{E}\left(e^{-\theta \sum_{k=1}^j \Delta_k}\right) \mathbb{P}(T^s = j) \\ &\geq \sum_{j=1}^{+\infty} \left(\frac{q^*}{\theta + q^*}\right)^j \mathbb{P}(T^s = j) \\ &\geq \sum_{j=1}^{m^*} \left(\frac{q^*}{\theta + q^*}\right)^j \mathbb{P}(T^s = j) \\ &\geq p^* \left(\frac{q^*}{\theta + q^*}\right)^{m^*}. \end{aligned}$$

□

**Lemma 7.3.** *Under assumptions of Lemma 7.2, starting from any two coupled configurations  $(\zeta_0^1, \zeta_0^2)$  with at least two opposite discrepancies, there exists  $l \leq m^*(n, q(\cdot, \cdot))$  (with  $m^*$  defined in (51)) and a sequence  $(x_0, \dots, x_l) \in W^{l+1}$  such that for  $k = 0, \dots, l-1$  and  $i \in \{1, 2\}$ , (238)–(239) holds, and that the last jump from  $x_{l-1}$  to  $x_l$  for  $(\zeta^{1,l-1}, \zeta^{2,l-1})$  generates a coalescence.*

*Proof of Lemma 7.3.* Let  $i, j$  denote locations of two opposite discrepancies (say a blue one at  $i$ , i.e. a  $\zeta_0^1$  discrepancy, and a red one at  $j$ , i.e. a  $\zeta_0^2$  discrepancy) between  $\zeta_0^1$  and  $\zeta_0^2$ . In short, to describe the transition (239)–(240), we will say that has jumped from  $x_k$  to  $x_{k+1}$  a blue particle if  $\zeta^{1,k}(x_k) = 1$ ,  $\zeta^{2,k}(x_k) = 0$ ; a red particle if  $\zeta^{1,k}(x_k) = 0$ ,  $\zeta^{2,k}(x_k) = 1$ ; a black particle if  $\zeta^{1,k}(x_k) = \zeta^{2,k}(x_k) = 1$ .

Let  $(x_0 = i, \dots, x_D = j)$  be a path connecting  $i$  to  $j$ , with  $D \leq n^*$ , either for

$q(\cdot, \cdot)$  or  $\check{q}(\cdot, \cdot)$ . Without loss of generality, we assume the first case. First we define a sequence of admissible jumps leading from  $(\zeta_0^1, \zeta_0^2)$  to  $((\zeta_0^1)^{x_0, x_1}, (\zeta_0^2)^{x_0, x_1})$ . If  $x_1$  is occupied by a red particle, the jump is directly feasible and leads to a coalescence, hence  $l = 1$ . If  $x_1$  is occupied by a white particle, i.e.  $\zeta_0^1(x_1) = \zeta_0^2(x_1) = 0$ , the jump is also feasible, but no coalescence occurs. Finally, if there is a black particle at  $x_1$ , i.e.  $\zeta_0^1(x_1) = \zeta_0^2(x_1) = 1$ , the blue particle at  $x_0 = i$  may not jump directly to  $x_1$  and we need intermediate jumps. Let  $x_{k_1}, \dots, x_{k_p}$  denote positions of black particles along the path  $(x_0, \dots, x_D)$ , where  $p \leq D - 1$  is the number of such particles, and  $k_1 < \dots < k_p$ . The black particle at  $x_{k_p}$  can be moved to  $x_{D-1}$  in  $D - 1 - k_p \leq n^* - 1 - p$  steps along the path, since there are no more black particles there. At the end of this the black particle is at  $x_{D-1}$ . Next, the black particle at  $x_{k_{p-1}}$  can be similarly moved to  $x_{D-2}$  in at most  $n^* - p - 1$  steps. At the end of these procedures, all black particles have been moved to  $x_{D-1}, \dots, x_{D-p}$  in at most  $p(n^* - 1 - p)$  steps. After this, the blue particle at  $x_0$  can either be moved to  $x_{D-p-1}$  (if  $p = D - 1$ , none of the black particle moves has actually occurred because the whole path between  $x_1$  and  $x_{D-1}$  was occupied by black particles) or meet a red particle along this path before, in which case the coalescence is achieved. In the former case, in  $p$  steps, we can successively let each of the black particles jump one step further along the path, exchanging with the red particle, until the latter finds itself at  $x_{D-p}$ . Finally, the blue particle at  $x_0$  can be moved in at most  $D - p$  steps to the first red particle it encounters along the path (which is either at  $x_{D-p}$ , or possibly earlier on the path. The total number of steps performed to achieve coalescence was at most  $p(n^* - 1 - p) + n^*$ . This quantity achieves maximum value given by (51) for  $p = \lfloor \frac{n^*}{2} \rfloor$ .  $\square$

#### 7.4 Two-block estimate: proof of Proposition 7.4

The proof of Proposition 7.4 relies on the *quasi-interface* property stated in Lemma 7.5 below. The latter is an extension to multilane SEP of the *exact* interface property for single-lane SEP, see [20], which states that the number of sign changes of discrepancies between two coupled processes cannot increase in time. Such a property does not hold in the multilane case, but we will show that it holds approximately with high probability. The new difficulty here with respect to the single-lane setting is that discrepancies of opposite type may cross each other (in the sense of Definition 7.1) without coalescing by using different lanes.

**Definition 7.3.** We call admissible a coupled configuration  $(\eta, \xi)$  such that:

(i) For every  $z \in \mathbb{Z}$ , the vertical layer  $\{z\} \times W$  does not contain two opposite discrepancies.

(ii) There exists at least a pair of opposite discrepancies between  $\eta$  and  $\xi$  (by (i) these are necessarily located at different vertical layers).

(iii) There exists  $A \in \mathbb{N}$  such that  $\eta$  and  $\xi$  are ordered on  $[A, +\infty) \cap \mathbb{Z}$  and on  $(-\infty, A] \cap \mathbb{Z}$ .

The above admissibility property will be used as an assumption for Lemma 7.5. Before proceeding further, we give an important example of admissible configurations for the sequel.

**Lemma 7.4.** *The coupled Gibbs state  $(\eta^N, \xi^N)$  defined by (210) is admissible if the following conditions are (both) satisfied: (a) there exists  $x, y \in \mathbb{R}$  such that  $u_0(x) < v_0(x)$  and  $u_0(y) > v_0(y)$ ; (b) there exists  $a > 0$  such that  $u_0(\cdot)$  and  $v_0(\cdot)$  are ordered on  $(-\infty; a]$  and on  $[a; +\infty)$ .*

*Proof of Lemma 7.4.* Let  $(z, i) \in \mathbb{Z} \times W$ . Since  $\tilde{\rho}_i(\cdot)$  is increasing (cf. Proposition 2.1), (211) and (39) imply that the ordering between  $\eta^N(z, i)$  and  $\xi^N(z, i)$  is the same as between  $u_z^N$  and  $v_z^N$ , hence independent of  $i$ . This implies condition (i) of Definition 7.3. Moreover, assumption (a) of the lemma implies condition (ii) of Definition 7.3, and assumption (b) implies condition (iii).  $\square$

Given an admissible pair  $(\eta, \xi)$ , a finite family  $\mathcal{C} = \mathcal{C}(\eta, \xi)$  of at least 2 nonempty subintervals of  $\mathbb{Z}$  is called a *separating family* if: (i) it forms a partition of  $\mathbb{Z}$ ; (ii) no interval  $I \in \mathcal{C}$  contains two opposite discrepancies; (iii) each interval  $I \in \mathcal{C}$  contains at least a discrepancy, and the discrepancies in two successive intervals are of opposite type. The interval  $I$  is named *blue* or *red* according to the colour of the discrepancies it contains. In particular, a blue or red interval may contain black or white particles. Note that such a family exists thanks to (i)–(ii) above, but is not uniquely defined.

We say an interval  $I \in \mathcal{C}$  *has survived* by time  $t$  if at least one discrepancy initially in  $I$  has not coalesced. If so, consider the (possibly coinciding) leftmost and rightmost such discrepancies. We denote by  $I_t$  the subinterval lying between them (including them). Hence, in view of (216),  $I_t$  has leftmost label  $v_t$  and rightmost label  $w_t$ , where  $v_0 = v$ ,  $w_0 = w$ , and  $v_t$  (resp.  $w_t$ ) is a nondecreasing (resp. nonincreasing) function of  $t$ . Note that due to possible crossings of opposite discrepancies,  $I_t$  may contain discrepancies opposite to its initial colour, even though it initially did not; however we still call it blue or red according to its initial colour. If  $I$  has not survived by time  $t$ , we set  $I_t = \emptyset$ . We denote by  $\mathcal{C}_t$  the collection of intervals  $I_t$  having survived by time  $t$ .

At time  $t > 0$ , the spatial ordering is conserved within blue and within red intervals; however, condition (i) of Definition 7.3 is not necessarily conserved by the dynamics, hence two intervals of the same colour may overlap through their endpoints. In contrast, spatial ordering is not necessarily conserved between blue and red intervals. More precisely, at time  $t$ , it is possible for an interval of one colour to have an overlap (not reduced to an endpoint) with one of the other colour, be contained in it, or have entirely crossed it to the other side. Consider the set of points at time  $t$  that belong to no blue or red interval. Connected components of this set are called *black and white* intervals at time  $t$ .

We point out that the multilane case is quite different from the single-lane one in several respects. First, while the above intervals are *one-dimensional*, they involve a global (and not lane-by-lane) inspection of the *two-dimensional* microscopic model. Indeed, the property of being an admissible coupled configuration is in general not conserved by the evolution. Thus at later times, at a given spatial location, there may be different colours on different lanes, so that even at a given spatial location “the colour” is not clearly defined. Next, in the single-lane case, due to non-crossings of opposite discrepancies, a blue or red interval may never contain a discrepancy of the other colour, a black and white interval may never contain a blue or red discrepancy, and no overlap is possible.

These two differences make the definition and control of a one-dimensional interface more challenging. However, the following lemma shows that with several lanes, there remains but little invasion of blue or red intervals by the opposite colour.

**Lemma 7.5.** *Assume that  $(\eta^N, \xi^N)_{N \in \mathbb{N}}$  is a sequence of admissible coupled configurations in the sense of Definition 7.3. Assume further that, for some  $a > 0$  independent of  $N$ , we have  $A = A^N = aN$  in condition (iii). For  $m \in \mathbb{N}^*$ , let  $\mathcal{E}_m$  be the event that at all times  $t \geq 0$ , the following holds: for every  $I \in \mathcal{C}$  such that  $I_t \neq \emptyset$ ,  $I_t$  contains at most  $m$  discrepancies of type opposite to the ones initially in  $I$ ; and every black and white interval contains at most  $m$  discrepancies. Then:*

(o) *The number of black and white intervals at any time is at most  $M^N - 1$ , where  $M^N$  is the number of intervals in the initial separating family.*

(i) *for every  $h > 0$ ,*

$$\lim_{N \rightarrow +\infty} \mathbb{P}(\mathcal{E}_{Nh}) = 1 \quad (244)$$

*Proof of Lemma 7.5 .*

*Proof of (o).* The number of black and white intervals is highest when there is no interlap between blue and red intervals, in which case we have to count the number of intermediate intervals between  $M^N$  consecutive non-overlapping intervals.

*Proof of (i).* Consider a given  $I \in \mathcal{C}$ , for instance a blue interval. Let  $v_t, w_t$  denote labels of the leftmost and rightmost blue particles in  $I_t$ . The number of red particles having invaded  $I_t$  at time  $t$  is given by

$$L^{v_t} + R_t^{w_t} \leq C_t^{v_t} + C_t^{w_t} \leq 2 \sup_{v \in \mathcal{V}} \mathbf{1}_{\mathcal{V}_t}(v) C_t^v \quad (245)$$

The assumptions of the lemma imply that on each side of  $[-aN, aN]$ , there cannot be both blue and red particles. Thus there exist  $z_-, z_+$  in  $\mathbb{Z} \cap [-aN, aN]$  such that each of the intervals  $\mathbb{Z} \cap [z_+, +\infty)$  and  $\mathbb{Z} \cap (-\infty, z_-]$  is (initially) a

blue or a red interval. Hence invasions of blue intervals only involve crossings of blue particles initially in  $[-aN, aN]$  by red particles. The result then follows from Corollary 7.1. For a red interval, we replace  $L_t^{v_t}$  and  $R_t^{v_t}$  by  $L_t'^{v_t}$  and  $R_t'^{v_t}$ , and similarly for  $w_t$ , where  $L_t'^v$  and  $R_t'^v$  are defined analogously to (218) for crossings of a red particle labelled  $v$  by blue particles (exchanging the roles of  $X$  and  $Y$ ).

The same argument holds for black and white intervals. Indeed, such an interval  $J$  at time  $t$  must lie between two blue or red (not necessarily the same colour) intervals. The rightmost label  $w_t$  at time  $t$  of the interval on the left of  $J$  and leftmost label  $v_t$  of the interval on the right were initially in  $[-aN, aN]$ . At time  $t$ , depending on colours of the intervals around,  $J$  contains  $L_t^{w_t}$  or  $L_t'^{w_t}$  (resp.  $R_t^{v_t}$  or  $R_t'^{v_t}$ ) particles of the colour opposite to the interval on its left (resp. right). Hence the result again follows from Corollary 7.1.  $\square$

*Proof of Proposition 7.4.* Every local function on  $\mathcal{X}$  is the sum of a constant and a linear combination of nonconstant nondecreasing functions. Thus it is enough to consider nonconstant nondecreasing functions  $f$ . For such functions  $f$ , we will show that, for every  $A > 0$ ,  $s > 0$ ,  $r \in [0, 2]$  and  $\delta > 0$ ,

$$\lim_{l \rightarrow +\infty} \limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \Delta_{x,l}^{\delta \pm}(\eta_{Ns}^N; r) \right\} = 0 \quad (246)$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \Delta_{x, N\varepsilon}^{\delta \pm}(\eta_{Ns}^N; r) \right\} = 0 \quad (247)$$

where

$$\Delta_{x,l}^{\delta, \pm}(\eta; r) := \psi^{\delta, \pm} [M_{x,l} f(\eta), \bar{f}(M_{x,l} \bar{\eta}); r]$$

and the functions  $\psi^{\delta, \pm}$  are defined by

$$\psi^{\delta, +}(u, v; r) = \mathbf{1}_{\{u > \bar{f}(r) + \delta, v < \bar{f}(r) - \delta\}}, \quad \psi^{\delta, -}(u, v; r) = \mathbf{1}_{\{u < \bar{f}(r) - \delta, v > \bar{f}(r) + \delta\}}$$

We will then show at the end of this proof that there exists a constant  $C > 0$  (depending only on  $f$ ) such that

$$\int_0^2 [\psi^{\delta, +}(u, v; r) + \psi^{\delta, -}(u, v; r)] dr \geq C(|u - v| - 2\delta) \quad (248)$$

for every  $u, v \in [\min f, \max f]$ , so that (190) follows from (247) by integration with respect to  $r$  and dominated convergence. The inequality (248) says that if  $u$  and  $v$  lie almost on the same side of  $\bar{f}(r)$  for all  $r$ , then they must be close to each other.

To establish (247), we consider the coupled process  $(\eta_{Nt}^N, \xi_{Nt}^N)_{t \geq 0}$ , where  $\xi_0^N =$



$\xi_0 \sim \nu_r$ . Since  $\nu_r$  is invariant, we have  $\xi_t \sim \nu_r$  for all  $t > 0$ . We couple  $\eta_0^N$  and  $\xi_0^N$  via (210)–(211), the latter being a local Gibbs state with uniform profile  $r$ . Lemma 7.4 ensures that  $(\eta_0^N, \xi_0^N)$  is admissible in the sense of Definition 7.3.

By assumption on  $u_0(\cdot)$ , for  $N$  large enough, there is a (deterministic) separating family  $\mathcal{C}^N$  of intervals for  $(\eta_0^N, \xi_0)$  with a fixed number  $M$  of intervals depending only on  $\rho_0(\cdot)$  and  $r$  (namely the number of sign changes between  $\rho_0(\cdot)$  and  $r$ ). We apply Lemma 7.5 and denote simply by  $\mathcal{E}^N = \mathcal{E}_{N_h}$  the event in the lemma, and by  $\mathcal{C}_t = \mathcal{C}_t^N$  the evolution of  $\mathcal{C}^N = \mathcal{C}$  at time  $t$ . By (o) of Lemma 7.5, the number of intervals in  $\mathcal{C}_t$  plus the number of black and white intervals as time  $t$  is at most  $2M - 1$ . We divide the interval  $\mathbb{Z} \cap [-NA, NA]$  into the following (random) partition. We denote by  $B_{N_s}^{N,l}$ , resp.  $R_{N_s}^{N,l}$ ,  $W_{N_s}^{N,l}$ , the set of  $x \in [-NA, NA] \cap \mathbb{Z}$  such that  $[x-l, x+l] \cap \mathbb{Z}$  lies inside a blue, resp. red, resp. black and white interval. We denote by  $E_{N_s}^{N,l}$  the set of  $x \in [-NA, NA] \cap \mathbb{Z}$  that belong to none of the sets  $B_{N_s}^{N,l}$ , resp.  $R_{N_s}^{N,l}$ ,  $W_{N_s}^{N,l}$ .

The idea to obtain (247) is that most  $x \in [-NA, NA]$  lie in one of these sets, so  $\eta_{N_s}^N$  and  $\xi_{N_s}$  are almost ordered on  $[x-l, x+l] \cap \mathbb{Z}$ . This implies  $M_{x,l}f(\eta_{N_s}^N)$  and  $M_{x,l}f(\xi_{N_s}) \simeq \bar{f}(r)$  are ordered like  $\bar{f}(M_{x,l}\bar{\eta}_{N_s}^N)$  and  $\bar{f}(M_{x,l}\bar{\xi}_{N_s}) \simeq \bar{f}(r)$ , where the approximations follow from the law of large numbers for the stationary process; we have also used that  $\bar{f}$  is continuous and nondecreasing, see remark after (44). This makes  $\Delta^{\delta,\pm}(\eta_{N_s}^N; r)$  small.

Let us now make this precise. With the above definitions, we have

$$|E_{N_s}^{N,l}| \leq 2(2M - 1)l$$

because for any subinterval  $J$  of  $\mathbb{Z}$ , there are at most  $2l$  points  $x \in \mathbb{Z}$  for which  $[x-l, x+l]$  crosses the boundary of  $J$ . It follows that

$$\mathbb{E} \left\{ \mathbf{1}_{\mathcal{E}^N} N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \Delta_{x,l}^{\delta,\pm}(\eta_{N_s}^N; r) \mathbf{1}_{E_{N_s}^{N,l}}(x) \right\} \leq 4A(2M - 1) \frac{l}{N} \quad (249)$$

We will next show that

$$\lim_{l \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \mathbf{1}_{\mathcal{E}^N} \Delta_{x,l}^{\delta,\pm}(\eta_{N_s}^N; r) \mathbf{1}_{B_{N_s}^{N,l}}(x) \right\} = 0, \quad (250)$$

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left\{ N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \mathbf{1}_{\mathcal{E}^N} \Delta_{x,N\varepsilon}^{\delta,\pm}(\eta_{N_s}^N; r) \mathbf{1}_{B_{N_s}^{N,N\varepsilon}}(x) \right\} = 0 \quad (251)$$

and the same limits when replacing  $B_{N_s}^{N,\cdot}$  with  $R_{N_s}^{N,\cdot}$  or  $W_{N_s}^{N,\cdot}$ . Since  $\Delta^{\delta,\pm}$  is bounded, summing these three limits with (249), letting  $N \rightarrow +\infty$  followed by either  $l \rightarrow +\infty$ , or  $\varepsilon \rightarrow 0$  with  $l = N\varepsilon$ , and using Lemma 7.5, we obtain (247).

Let  $n_0$  denote the smallest integer such that  $f$  only depends on sites in  $[-n_0, n_0] \times W$ . Then for all  $\eta, \xi \in \mathcal{X}$ ,

$$|f(\eta) - f(\xi)| \leq \|f\|_\infty \sum_{x \in V: |x(0)| \leq n_0} |\eta(x) - \xi(x)| \quad (252)$$

for  $x \in \mathbb{Z} \cap [-NA, NA]$ , we denote by  $n_t^l(x)$  the number of wrong discrepancies in  $[x-l-n_0, x+l+n_0]$ . By ‘‘wrong’’ discrepancies we mean red particles in a blue interval, blue particles in a red interval, or blue and red particles in a black and white interval. Hence

$$n_t^l(x) = \begin{cases} \sum_{z \in \mathbb{Z} \cap [x-l, x+l]} \sum_{i \in W} (\eta_t(z, i) - \xi_t(z, i))^- & \text{if } x \in B_t^{N, l}, \\ \sum_{z \in \mathbb{Z} \cap [x-l, x+l]} \sum_{i \in W} (\eta_t(z, i) - \xi_t(z, i))^+ & \text{if } x \in R_t^{N, l}, \\ \sum_{z \in \mathbb{Z} \cap [x-l, x+l]} \sum_{i \in W} |\eta_t(z, i) - \xi_t(z, i)| & \text{if } x \in W_t^{N, l}. \end{cases} \quad (253)$$

By triangle inequality,

$$(M_{x, l} \bar{\eta}_{N_s}^N - M_{x, l} \bar{\xi}_{N_s})^\pm \leq (2l+1)^{-1} \sum_{z \in \mathbb{Z} \cap [x-l, x+l]} \sum_{i \in W} (\eta(z, i) - \xi(z, i))^\pm. \quad (254)$$

Thus by (252)–(254), for all  $x \in B_{N_s}^N$ , we have

$$M_{x, l} \bar{\eta}_{N_s}^N \geq M_{x, l} \bar{\xi}_{N_s} - \frac{n_{N_s}^l(x)}{2l+1} \geq r - \frac{n_{N_s}^l(x)}{2l+1} - \varepsilon_{N_s}^{N, l, 1}(x) \quad (255)$$

$$\bar{f}(M_{x, l} \bar{\eta}_{N_s}^N) \geq \bar{f}(r) - \|\bar{f}'\|_\infty \left( \frac{n_{N_s}^l(x)}{2l+1} + \varepsilon_{N_s}^{N, l, 1}(x) \right) \quad (256)$$

$$\begin{aligned} M_{x, l} f(\eta_{N_s}^N) &\geq M_{x, l} f(\xi_{N_s}) - \|f\|_\infty \frac{n_{N_s}^l(x)}{2l+1} \\ &\geq \bar{f}(r) - \|f\|_\infty \frac{n_{N_s}^l(x)}{2l+1} - \varepsilon_{N_s}^{N, l, 2}(x) \end{aligned} \quad (257)$$

where

$$\varepsilon_{N_s}^{N, l, 1} := M_{x, l} \bar{\xi}_{N_s} - r, \quad \varepsilon_{N_s}^{N, l, 2} := M_{x, l} f(\xi_{N_s}) - f(r). \quad (258)$$

Similarly, for  $x \in R_{N_s}^N$ ,

$$M_{x, l} \bar{\eta}_{N_s}^N \leq M_{x, l} \bar{\xi}_{N_s} + \frac{n_{N_s}^l(x)}{2l+1} \leq r + \frac{n_{N_s}^l(x)}{2l+1} + \varepsilon_{N_s}^{N, l, 1}(x) \quad (259)$$

$$\bar{f}(M_{x, l} \bar{\eta}_{N_s}^N) \leq \bar{f}(r) + \|\bar{f}'\|_\infty \left( \frac{n_{N_s}^l(x)}{2l+1} + \varepsilon_{N_s}^{N, l, 1}(x) \right) \quad (260)$$

$$\begin{aligned} M_{x, l} f(\eta_{N_s}^N) &\leq M_{x, l} f(\xi_{N_s}) + \|f\|_\infty \frac{n_{N_s}^l(x)}{2l+1} \\ &\leq \bar{f}(r) + \|f\|_\infty \frac{n_{N_s}^l(x)}{2l+1} + \varepsilon_{N_s}^{N, l, 2}(x). \end{aligned} \quad (261)$$

For  $[x-l, x+l] \subset W_{N_s}^N$ , both (255)–(257) and (259)–(261) hold. It follows from (255)–(257) that

$$\begin{aligned}
& N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \mathbf{1}_{\mathcal{E}^N} \Delta_{x,l}^{\delta, \pm}(\eta_{N_s}^N; r) \mathbf{1}_{B_{N_s}^N}(x) \\
& \leq N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \mathbf{1}_{\mathcal{E}^N} \mathbf{1}_{B_{N_s}^N}(x) \mathbf{1}_{\left\{ \frac{n_{N_s}^l(x)}{2l+1} + |\varepsilon_{N_s}^{N,l,1}(x)| > \delta \right\}} \\
& \leq \delta^{-1} N^{-1} \sum_{x \in \mathbb{Z}: |x| \leq NA} \mathbf{1}_{\mathcal{E}^N} \mathbf{1}_{B_{N_s}^N}(x) \left( \frac{n_{N_s}^l(x)}{2l+1} + |\varepsilon_{N_s}^{N,l,1}(x)| \right) \quad (262)
\end{aligned}$$

and similar relations hold with  $R_{N_s}^N$  and  $W_{N_s}^N$ . By Lemma 7.5,

$$\mathbf{1}_{\mathcal{E}^N} \sum_{x \in \mathbb{Z} \cap [-NA, NA]} \mathbf{1}_{B_{N_s}^N \cup R_{N_s}^N \cup W_{N_s}^N}(x) n_{N_s}^l(x) \leq (2l+1)(2M-1)Nh, \quad (263)$$

while by the law of large numbers in  $L^1$ ,

$$\begin{aligned}
& \mathbb{E} \left\{ \lim_{N \rightarrow +\infty} N^{-1} \sum_{x \in \mathbb{Z} \cap [-NA, NA]} (|\varepsilon^{N,l,1}(x)| + |\varepsilon^{N,l,2}(x)|) \right\} \\
& = 2A \mathbb{E} (|M_{x,l} \xi_{N_s} - r| + |M_{x,l} f(\xi_{N_s}) - f(r)|) \xrightarrow{l \rightarrow +\infty} 0 \quad (264)
\end{aligned}$$

Gathering (262)–(264), letting  $N \rightarrow +\infty$ , then  $l \rightarrow +\infty$ , or  $l = N\varepsilon$  with  $\varepsilon \rightarrow 0$ , and finally  $h \rightarrow 0$ , we obtain (250)–(251). Similar limits are obtained replacing  $B_{N_s}$  by  $R_{N_s}$  or  $W_{N_s}$ . Combining these with (249) yields (246)–(247).

We finally prove the claim (248). Set

$$\Psi^{\delta,+}(u, v; \rho) = \mathbf{1}_{\{u > \rho + \delta, v < \rho - \delta\}}, \quad \Psi^{\delta,-}(u, v; \rho) = \mathbf{1}_{\{u < \rho - \delta, v > \rho + \delta\}}$$

so that  $\psi^{\delta, \pm}(u, v, r) = \Psi^{\delta, \pm}(u, v, \bar{f}(r))$ . Since  $f$  is nondecreasing,  $\rho \mapsto \nu_\rho$  is stochastically nondecreasing, and  $\nu_0$  and  $\nu_n$  are supported respectively on the empty and full configuration, the minimum  $m$  and maximum  $M$  of  $\bar{f}$  coincide with those of  $f$ , and are given respectively by the value of  $f$  at the empty and full configuration. Setting

$$C := \frac{1}{\sup_{r \in [0, 2]} \bar{f}'(r)}, \quad (265)$$

we have

$$\begin{aligned}
\int_0^2 \psi^{\delta,+}(u, v; r) dr & \geq C \int_0^2 \Psi^{\delta,+}(u, v; \bar{f}(r)) \bar{f}'(r) dr \\
& = C \int_m^M \Psi^{\delta,+}(u, v; \gamma) d\gamma = c(u - v - 2\delta)^+.
\end{aligned}$$

Similarly, we obtain the lower bound  $c(v - u - 2\delta)^-$  when replacing  $\psi^{\delta,+}$  by  $\psi^{\delta,-}$ . The bound (248) follows by summing these two bounds.  $\square$

## 8 Proof of Theorem 3.2

We begin with an outline of the proof. Some useful properties of weakly coupled systems (69) are next gathered in Subsection 8.1, and the complete proof follows in Subsection 8.2.

As mentioned in Subsection 3.3, we must close the conservation law obtained after adding the equations in (76), namely

$$\partial_t R_\varepsilon(t, x) + \partial_x \left( \sum_{i=0}^{n-1} f_i(\rho_{i,\varepsilon}(t, x)) \right) = 0. \quad (266)$$

To this end, we must use the balance term to show that the relaxation to local equilibrium (82) holds approximately for  $\rho_\varepsilon$  as  $\varepsilon \rightarrow 0$ , thereby allowing to approximate the flux in (266) as follows (with  $f$  given by (79)):

$$\rho_{i,\varepsilon}(t, x) \simeq \tilde{\rho}_i[R_{i,\varepsilon}(t, x)], \quad \sum_{i=0}^{n-1} f_i(\rho_{i,\varepsilon}(t, x)) \simeq f(R_\varepsilon(t, x)). \quad (267)$$

We must also establish entropy inequalities for the limiting conservation law. Following a usual scheme in relaxation theory (see e.g. [10, 22]), the proof of (82) entropy inequalities involves constructing a suitable *dissipative* entropy, here of the form

$$H(\rho) = \sum_{i=0}^{n-1} h_i(\rho_i) \quad (268)$$

for the balance system (76), where  $\rho = (\rho_0, \dots, \rho_{n-1})$ . By dissipative, we mean the following:

$$\langle \nabla H(\rho), c(\rho) \rangle \leq 0 \quad (269)$$

$$\langle \nabla H(\rho), c(\rho) \rangle = 0 \Rightarrow c(\rho) = 0 \quad (270)$$

where  $c(\rho) = (c_0(\rho), \dots, c_{n-1}(\rho))$ . The left-hand side of (269) is the entropy dissipation due to the balance term  $c(\rho)$ , and (270) expresses that zero dissipation implies local equilibrium. Regarding the latter, we will show that (with  $\mathcal{F}$  defined in (29))

$$c(\rho) = 0 \Leftrightarrow \rho \in \mathcal{F} \quad (271)$$

which implies (82). Note that the right to left implication in (271) follows directly from (77)–(78) and (28). The corresponding entropy inequality writes (cf. (70))

$$\partial_t \left[ \sum_{i=0}^{n-1} h_i(\rho_{i,\varepsilon}) \right] + \partial_x \left[ \sum_{i=0}^{n-1} g_i(\rho_{i,\varepsilon}) \right] = \varepsilon^{-1} \langle \nabla H(\rho_\varepsilon), c(\rho_\varepsilon) \rangle \leq 0 \quad (272)$$

where  $g_i$  is the entropy flux of  $h_i$  for (76). By space integration of (272) and using (269)–(270) and (271), we will arrive at (267). This approximation can

also be applied to entropies, hence in the limit, we will obtain the approximate entropy inequality

$$\partial_t h[R_\varepsilon(t, x)] + \partial_x g[R_\varepsilon(t, x)] \lesssim 0 \quad (273)$$

where the equilibrium entropy-flux pair is given by

$$h(R) := \sum_{i=0}^{n-1} h_i[\tilde{\rho}_i(R)], \quad g(R) := \sum_{i=0}^{n-1} g_i[\tilde{\rho}_i(R)]. \quad (274)$$

Then we have to find a large enough family of entropies of the form (268) so that (274) selects the unique entropy solution to (75).

It turns out that the special structure (77)–(78) of the balance term  $c(\rho)$  allows us to use for the above purposes a combination of Kruřkov entropies whereby the solution is coupled to stationary solutions supported on the equilibrium manifold. Namely, we will consider, cf. (73),

$$h_i(\rho) = (\rho - r_i)^+, \quad g_i(\rho) = \mathbf{1}_{\{\rho > r_i\}}[f_i(\rho) - f_i(r_i)], \quad r = (r_0, \dots, r_{n-1}) \in \mathcal{F}. \quad (275)$$

## 8.1 Preliminary material

We recall here some properties of entropy solutions of (69) established in [15]. In the following,  $\|\cdot\|_1$  denotes the  $L^1(\mathbb{R})$  norm, and for notational convenience, when  $u^+ := \max(u, 0) \in L^1(\mathbb{R})$ , we set (though this does not define a norm)

$$\|u\|_1^+ := \int_{\mathbb{R}} u(x)^+ dx.$$

**Theorem 8.1.** *Let  $\rho = (\rho_0, \dots, \rho_{n-1})$  and  $r = (r_0, \dots, r_{n-1})$  be entropy solutions to (69) with respective initial data  $\rho^0 = (\rho_0^0, \dots, \rho_{n-1}^0)$  and  $r^0 = (r_0^0, \dots, r_{n-1}^0)$ . Then*

1. *There exists a constant  $C > 0$  such that, for every  $t \geq 0$ , (77)–(78).*

$$\|\rho(t, \cdot) - r(t, \cdot)\|_1 \leq e^{Ct} \|\rho^0(\cdot) - r^0(\cdot)\|_1. \quad (276)$$

2. *(Finite propagation and uniqueness). Let*

$$V := \max_{i=0, \dots, n-1} \|f'_i\|_\infty \quad (277)$$

*If  $\rho^0(\cdot)$  and  $r^0(\cdot)$  coincide on the space interval  $[a, b]$  where  $-\infty \leq a < b \leq +\infty$ , then for every  $0 \leq t \leq (b - a)/(2V)$ ,  $\rho(t, \cdot)$  and  $r(t, \cdot)$  coincide on the space interval  $[a + Vt, b - Vt]$ .*

*In particular, there exists a unique entropy solution to (69) with given initial datum  $\rho^0(\cdot) := (\rho_0^0(\cdot), \dots, \rho_{n-1}^0(\cdot))$ .*

3. Assume that for each  $i = 0, \dots, n-1$ ,

$$\forall j \neq i, (\rho_0, \dots, \rho_{n-1}) \mapsto c_i(\rho_0, \dots, \rho_{n-1}) \text{ is nondecreasing w.r.t. } \rho_j \quad (278)$$

Then there exists a constant  $C > 0$  such that, for every  $t \geq 0$ ,

$$\|\rho(t, \cdot) - r(t, \cdot)\|_1^+ \leq e^{Ct} \|\rho^0(\cdot) - r^0(\cdot)\|_1^+ \quad (279)$$

In particular the solution semigroup for (69) is monotone: if  $\rho_i^0(\cdot) \leq r_i^0(\cdot)$  for every  $i = 0, \dots, n-1$ , then  $\rho_i(t, \cdot) \leq r_i(t, \cdot)$  for every  $i = 0, \dots, n-1$ .

In particular, looking at (77)–(78), we have

**Lemma 8.1.** Assumption (278) is satisfied by the relaxation terms (77)–(78).

The following corollary will be used afterwards.

**Corollary 8.1.** Let  $\rho = (\rho_0, \dots, \rho_{n-1})$  and  $r = (r_0, \dots, r_{n-1})$  be the entropy solutions to (69) with initial datum  $\rho^0 = (\rho_0^0, \dots, \rho_{n-1}^0)$ . Assume  $\rho^0(\cdot)$  has locally bounded space-time variation. Then  $\rho(\cdot, \cdot)$  has locally bounded space-time variation, and for every  $t > 0$ ,  $\rho(t, \cdot)$  has locally bounded space variation. Besides, for every  $a, b \in \mathbb{R}$  with  $a < b$ , and every  $t > 0$ ,

$$\text{TV}_{[a;b]}[\rho(t, \cdot)] \leq e^{Ct} \text{TV}_{[a-Vt; b+Vt]}[\rho^0(\cdot)] \quad (280)$$

and

$$\text{TV}_{[0;t] \times [a;b]}[\rho(\cdot, \cdot)] \leq (1+V)C^{-1}(e^{Ct}-1)\text{TV}_{[a-Vt; b+Vt]}[\rho^0(\cdot)] + \max_{i=0, \dots, n-1} \|c_i\|_\infty t \quad (281)$$

*Proof of Corollary 8.1.*

Applying (276) to  $\rho^0(\cdot)$  and  $\rho^0(\cdot + \varepsilon)$  yields

$$\begin{aligned} \int_a^b |\rho(t, x + \varepsilon) - \rho(t, x)| dx &\leq e^{Ct} \int_{a-Vt}^{b+Vt} |\rho^0(x + \varepsilon) - \rho^0(x)| dx \\ &\leq e^{Ct} \varepsilon \text{TV}_{[a-Vt; b+Vt]}[\rho^0(\cdot)]. \end{aligned} \quad (282)$$

Thus for a test function  $\varphi \in C_R^\infty((0; t) \times (a; b))$ ,

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \left| \varepsilon^{-1} \iint_{(0; +\infty) \times \mathbb{R}} \varphi(s, x) [\rho(s, x + \varepsilon) - \rho(s, x)] dx ds \right| \\ &= |\langle \partial_x \rho, \varphi \rangle| \leq C^{-1}(e^{Ct} - 1) \|\varphi\|_\infty \text{TV}_{[a-Vt; b+Vt]}[\rho^0(\cdot)] \end{aligned} \quad (283)$$

where the bracket denotes integrating the distributional derivative  $\partial_x \rho$  against test function  $\varphi$ . Hence,  $\partial_x \rho$  is a locally bounded measure, and

$$\iint_{[a;b] \times [0;t]} |\partial_x \rho(s, x)| (ds, dx) \leq C^{-1}(e^{Ct} - 1) \text{TV}_{[a-Vt; b+Vt]}[\rho^0(\cdot)]. \quad (284)$$

By (69) and (277),  $\partial_s \rho(s, x)$  is also a locally finite measure, and

$$|\partial_t \rho(s, x)| \leq |\partial_x \rho(s, x)| + c_i[\rho(s, x)].$$

This yields (281). On the other hand, (280) follows from (282), since

$$\text{TV}_{[a;b]}[\rho(t, \cdot)] = \sup_{\varepsilon > 0} \int_a^b |\rho(t, x + \varepsilon) - \rho(t, x)| dx.$$

□

## 8.2 Proof of Theorem 3.2

The proof is decomposed into four steps.

*Step one: entropy inequality and relaxation.* Let  $r = (r_i)_{\{i=0, \dots, n-1\}} \in \mathcal{F}$ . We write entropy inequality (70) for the relaxation system (76) with the Kružkov entropy-flux pair  $(h_{r_i+}, g_{r_i+})$ , cf. (275), against a nonnegative test function  $\varphi \in C_K^0((0; +\infty) \times \mathbb{R})$ , and sum over  $i = 0, \dots, n-1$  to obtain (272):

$$\begin{aligned} & \sum_{i=0}^{n-1} \int_0^{+\infty} \int_{\mathbb{R}} \{ \mathbf{1}_{\{\rho_{i,\varepsilon}(t,x) > r_i\}} [\rho_{i,\varepsilon}(t,x) - r_i] \varphi'_t(t,x) \\ & + \mathbf{1}_{\{\rho_{i,\varepsilon}(t,x) > r_i\}} [f_i(\rho_{i,\varepsilon}(t,x)) - f_i(r_i)] \varphi'_x(t,x) \} dx dt \\ & \geq -\varepsilon^{-1} \int_0^{+\infty} \varphi(t,x) \int_{\mathbb{R}} \sum_{i=0}^{n-1} \mathbf{1}_{\{\rho_{i,\varepsilon}(t,x) > r_i\}} \sum_{j=0}^{n-1} [c_{ji}(\rho_\varepsilon(t,x)) - c_{ij}(\rho_\varepsilon(t,x))] dx dt \end{aligned} \quad (285)$$

where  $c_{ij}$  is given by (78). The right-hand side can be rewritten

$$\begin{aligned} & -\frac{\varepsilon^{-1}}{2} \int_0^{+\infty} \int_{\mathbb{R}} \left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} [\mathbf{1}_{\{\rho_{i,\varepsilon}(t,x) > r_i\}} - \mathbf{1}_{\{\rho_{j,\varepsilon}(t,x) > r_j\}}] \right. \\ & \quad \left. \times [c_{ji}(\rho_\varepsilon(t,x)) - c_{ij}(\rho_\varepsilon(t,x))] \right\} \varphi(t,x) dx dt \\ & = -\frac{\varepsilon^{-1}}{2} \int_0^{+\infty} \int_{\mathbb{R}} F(\rho_\varepsilon(t,x); r) \varphi(t,x) dx dt \end{aligned} \quad (286)$$

where we set, for  $\rho, r \in [0; 1]^n$ ,

$$F(\rho; r) := \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} F_{i,j}(\rho; r) \quad (287)$$

with

$$F_{i,j}(\rho; r) := [\mathbf{1}_{\{\rho_i > r_i\}} - \mathbf{1}_{\{\rho_j > r_j\}}] [c_{ji}(\rho) - c_{ij}(\rho)] \quad (288)$$

The following lemma establishes the dissipation-equilibrium properties (269)–(270) and (271).

**Lemma 8.2.** For  $\rho \in [0; 1]^n$ , let

$$\tilde{F}(\rho) := \int_0^n F(\rho; \psi^{-1}(k)) dk \quad (289)$$

with (cf. Proposition 2.1)

$$\psi^{-1}(k) = (\tilde{\rho}_i(k))_{i=0, \dots, n-1} \in \mathcal{F}. \quad (290)$$

Then  $\tilde{F}(\rho) \leq 0$  for every  $\rho \in [0; 1]^n$ , and  $\tilde{F}(\rho) = 0$  if and only if  $\rho \in \mathcal{F}$ .

Next proposition establishes relaxation (267) to the equilibrium manifold.

**Proposition 8.1.** For every  $i = 0, \dots, n-1$  and every  $T, R > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-A}^A |\rho_{i,\varepsilon}(t, x) - \tilde{\rho}_i[R_\varepsilon(t, x)]| dx dt = 0. \quad (291)$$

*Proof of Lemma 8.2.*

*Step one.* We show that, for all  $(\rho, r) \in [0; 1]^n \times \mathcal{F}$  and  $(i, j) \in \{0, \dots, n-1\}^2$ ,

$$F_{i,j}(\rho; r) \leq 0. \quad (292)$$

If  $\rho_i > r_i$  and  $\rho_j > r_j$ , the result follows from (288). Otherwise, one of (293)–(294) below holds:

$$\rho_i > r_i, \quad \rho_j \leq r_j, \quad (293)$$

$$\rho_i \leq r_i, \quad \rho_j > r_j. \quad (294)$$

We consider (293), (294) being similar. In view of (78) and (293),

$$\begin{aligned} & \sum_{j=0}^{n-1} \{q(j, i)\rho_j[1 - \rho_i] - q(i, j)\rho_i[1 - \rho_j]\} \\ & \leq q(j, i)r_j(1 - r_i) - q(i, j)r_i(1 - r_j) = 0 \end{aligned} \quad (295)$$

where the last equality follows from the fact that, by (303) and Lemma 2.1,  $r = (r_i)_{i \in W} \in \mathcal{F}$ , thus satisfies (28).

*Step two.* Assume  $\tilde{F}(\rho) = 0$ . In view of (292), this implies

$$F_{i,j}(\rho; \psi^{-1}(k)) = 0 \quad (296)$$

for a.e.  $k \in [0; n]$  and every  $i, j \in \{0, \dots, n-1\}$ . We argue by contradiction that  $\rho \in \mathcal{F}$ . Assume the contrary; then there exist  $i, j \in \{0, \dots, n-1\}$  such that (28) fails. Then by (296) and (288), for a.e.  $k \in [0; n]$ , one of (297)–(298) below holds:

$$\rho_i > \tilde{\rho}_i(k), \quad \rho_j > \tilde{\rho}_j(k) \quad (297)$$

$$\rho_i \leq \tilde{\rho}_i(k), \quad \rho_j \leq \tilde{\rho}_j(k) \quad (298)$$



Since (cf. Proposition 2.1)  $\tilde{\rho}_i(\cdot)$  is nondecreasing and continuous, there exists a unique  $k_0$  such that (297) holds for  $k < k_0$ , (298) holds for  $k > k_0$ , and

$$\rho_i = \tilde{\rho}_i(k_0), \quad \rho_j = \tilde{\rho}_j(k_0). \quad (299)$$

But (290) implies that  $\rho_i$  and  $\rho_j$  given by (299) satisfy (28), whence the contradiction.  $\square$

*Proof of Proposition 8.1.* Since the test function in (286) is arbitrary, using (286), we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-A}^A F[\rho_\varepsilon(t, x); r] dx dt = 0 \quad (300)$$

for every  $r \in \mathcal{F}$ . It follows by dominated convergence that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-A}^A \tilde{F}[\rho_\varepsilon(t, x)] dx dt = 0. \quad (301)$$

Define the non-negative measure (called *Young measure*) on  $[0; +\infty) \times \mathbb{R} \times \mathbb{R}^n$ ,

$$M_\varepsilon(dt, dx, d\rho) := \mathbf{1}_{[0, T]}(t) \mathbf{1}_{[-A, A]}(x) \delta_{(t, x, \rho_\varepsilon(t, x))}(d\rho) dx dt. \quad (302)$$

Since the family of measures  $M_\varepsilon$  has compact support contained in  $[0; T] \times [-A; A] \times [0; 1]^n$  and constant mass  $2AT$ , it is tight with respect to weak convergence. Up to taking subsequences we may assume without loss of generality that it converges to a limiting measure  $M(dt, dx, d\rho)$ . Since the integral in (301) can be written

$$\int_0^T \int_{-A}^A \tilde{F}[\rho_\varepsilon(t, x)] dx dt = \int \tilde{F}(\rho) M_\varepsilon(dt, dx, d\rho) \xrightarrow{\varepsilon \rightarrow 0} \int \tilde{F}(\rho) M(dt, dx, d\rho)$$

and  $\tilde{F}$  is continuous, it follows from (301) and Lemma 8.2 that  $M$  is supported on  $\mathcal{F}$ . On the other hand, the integral on the left-hand side of (291) can be written as the integral with respect to  $M_\varepsilon(dt, dx, d\rho)$  of the function

$$H_i(t, x, \rho) := \left| \rho_i - \tilde{\rho}_i \left( \sum_{j=0}^{n-1} \rho_j \right) \right|.$$

Thus (291) converges to the integral of  $H_i$  with respect to  $M(dt, dx, d\rho)$ . This integral is 0 because, by definition of  $\tilde{\rho}_i(\cdot)$ ,  $H_i$  vanishes on  $\mathcal{F}$ .  $\square$

The next step of the proof establishes the approximate entropy condition (273), its limit, and shows that the family of Kruřkov entropies we considered generates all Kruřkov entropies for the limiting equations.

*Step two: limiting entropy inequality.* Let  $c \in [0; +\infty)$ , and set

$$r_i := \tilde{\rho}_i(c). \quad (303)$$

In view of Corollary 8.1, since  $h_{r_i+}$  and  $g_{r_i+}$  are uniformly Lipschitz, on the left-hand side of (286), we can replace  $\rho_\varepsilon(t, x)$  with  $\tilde{\rho}_i[R_\varepsilon(t, x)]$  with vanishing error as  $\varepsilon \rightarrow 0$ . Since  $\tilde{\rho}_i$  is an increasing function, the indicator function on the left-hand side can be replaced with  $\mathbf{1}_{\{R_\varepsilon(t, x) > c\}}$  with vanishing error as  $\varepsilon \rightarrow 0$ . Thus, collecting the left-hand side and the vanishing replacement error, we obtain

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_0^{+\infty} \int_{\mathbb{R}} \mathbf{1}_{\{R_\varepsilon(t, x) > c\}} \sum_{i=0}^{n-1} \{[\tilde{\rho}_i[R_\varepsilon(t, x)] - r_i] \varphi'_t(t, x) \\ & + [f_i(\tilde{\rho}_i[R_\varepsilon(t, x)]) - f_i(r_i)] \varphi'_x(t, x)\} dx dt \geq 0. \end{aligned} \quad (304)$$

By (79) and definition (cf. Lemma 2.1) of  $\tilde{\rho}_i$  we have, for every  $\rho \in [0; 1]$ ,

$$\begin{aligned} \sum_{i=0}^{n-1} \tilde{\rho}_i(\rho) &= \rho \\ \sum_{i=0}^{n-1} f_i(\tilde{\rho}_i(\rho)) &= f(\rho) \end{aligned}$$

Thus, recalling (303) we arrive at

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_0^{+\infty} \int_{\mathbb{R}} \{[R_\varepsilon(t, x) - c]^+ \varphi'_t(t, x) \\ & + \mathbf{1}_{\{R_\varepsilon(t, x) > c\}} [f(R_\varepsilon(t, x)) - f(c)] \varphi'_x(t, x)\} dx dt \geq 0. \end{aligned} \quad (305)$$

*Step three: passing to the limit.* Here and in step four below we make the temporary assumption that the initial datum  $\rho^0(\cdot)$  has locally bounded space variation. By (281), the family  $(\rho_\varepsilon(\cdot, \cdot))_{\varepsilon > 0}$  has uniformly bounded space-time variation on any space-time rectangle. Thus is relatively compact in  $L^1_{\text{loc}}((0; +\infty) \times \mathbb{R})^n$ . This implies that the family  $(R_\varepsilon)_{\varepsilon > 0}$  is relatively compact in  $L^1_{\text{loc}}((0; +\infty) \times \mathbb{R})$ . Let  $\rho(\cdot, \cdot)$  be any subsequential limit of this family as  $\varepsilon \rightarrow 0$ . Then (305) implies

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} \{[\rho(t, x) - c]^+ \varphi'_t(t, x) \\ & + \mathbf{1}_{\{\rho(t, x) > c\}} [f(\rho(t, x)) - f(c)] \varphi'_x(t, x)\} dx dt \geq 0. \end{aligned} \quad (306)$$

Therefore,  $\rho(\cdot, \cdot)$  is an entropy solution to (75) with flux function (79). To ensure uniqueness of this subsequential limit and hence convergence of the whole sequence, we are left to prove the initial condition (71).

*Step four: initial condition.* By step two, for every  $a < b$  in  $\mathbb{R}$ ,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{t} \int_0^t \int_a^b R_\varepsilon(s, x) dx ds = \frac{1}{t} \int_0^t \int_a^b \rho(s, x) dx ds \quad (307)$$

where the limit is meant along the sequence of  $\varepsilon$ 's producing the limit point  $\rho(\cdot, \cdot)$ . By (76),

$$\partial_t R_\varepsilon(t, x) + \sum_{i=0}^{n-1} \partial_x [f_i(\rho_{i, \varepsilon}(t, x))] = \sum_{i=0}^{n-1} \varepsilon^{-1} c_i(\rho_\varepsilon(t, x)). \quad (308)$$

Note that

$$\sum_{i=0}^{n-1} c_i(\rho) = 0$$

for every  $\rho \in [0; 1]^n$ . Thus for every  $a < b$  in  $\mathbb{R}$ ,

$$\frac{d}{dt} \int_a^b R_\varepsilon(t, x) dx = \sum_{i=0}^{n-1} \{f[\rho_{i,\varepsilon}(t, a^\pm)] - f[\rho_{i,\varepsilon}(t, b^\pm)]\}.$$

Note that the limits  $\rho_\varepsilon(t, x^\pm)$  are always defined since  $\rho_\varepsilon$  has bounded space variation by Corollary 8.1. Besides, (76) implies that for every  $x \in \mathbb{R}$ ,  $\rho_{i,\varepsilon}(t, x^+) = \rho_{i,\varepsilon}(t, x^-)$  for a.e.  $t > 0$ . Integrating (309) in time and using the initial condition (71) for  $\rho_\varepsilon(\cdot, \cdot)$  yields

$$\int_a^b R_\varepsilon(t, x) dx = \sum_{i=0}^{n-1} \int_a^b \rho_i^0(x) dx + O_1[(b-a)t]$$

where  $|O_1[(b-a)t]| \leq C(b-a)t$  for a constant  $C > 0$  independent of  $\varepsilon$ ,  $a$ ,  $b$  and  $t$ . This and (307) imply

$$\frac{1}{t} \int_0^t \int_a^b R_\varepsilon(s, x) dx ds = \int_a^b \sum_{i=0}^{n-1} \rho_i^0(x) dx + O_2[(b-a)t].$$

Letting  $\varepsilon \rightarrow 0$ , using (307), and then letting  $t \rightarrow 0$ , we obtain

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \int_a^b \rho(s, x) dx ds = \int_a^b \sum_{i=0}^{n-1} \rho_i^0(x). \quad (309)$$

By (281),  $R_\varepsilon(\cdot, \cdot)$  satisfies a local variation bound independent of  $\varepsilon$ , thus  $\rho(\cdot, \cdot)$  has locally bounded variation with the same bound. Thus the limit

$$\lim_{t \rightarrow 0} \rho(t, x) =: \rho(0^+, x)$$

exists in  $L^1_{\text{loc}}(\mathbb{R})$ . It follows from (309) that

$$\lim_{t \rightarrow 0^+} \rho(t, x) = \sum_{i=0}^{n-1} \rho_i^0(x) = R^0(x) \quad (310)$$

in  $L^1_{\text{loc}}(\mathbb{R})$ , where  $R^0(\cdot)$  is defined by (81). The limit in (310) is the initial condition for entropy solutions of (75). Since the entropy solution to (75)–(81) is unique, this ensures uniqueness of a subsequential limit for  $R_\varepsilon$  as  $\varepsilon \rightarrow 0$ , hence convergence to this unique entropy solution.

*Step five: general initial condition.* Assume now  $\rho^0 \in L^\infty((0; +\infty); [0; 1]^n)$ . Then we can find a sequence  $(\rho^{0,k})_{k \in \mathbb{N}}$  such that  $\rho^{0,k}$  has locally bounded space

variation, and  $\rho^{0,k} \rightarrow \rho^0$  in  $L^1_{\text{loc}}(\mathbb{R})$ . Let  $\rho_\varepsilon^k$  denote the entropy solution to (76) with initial datum  $\rho^{0,k}$ , and

$$R_\varepsilon^k(t, x) := \sum_{i=0}^{n-1} \rho_{i,\varepsilon}^k(t, x).$$

Then, for  $a < b$  in  $\mathbb{R}$  and  $T > 0$ , we have

$$\begin{aligned} \int_0^T \int_a^b |R_\varepsilon(t, x) - \rho(t, x)| dx dt &\leq \int_0^T \int_a^b |R_\varepsilon(t, x) - R_\varepsilon^k(t, x)| dx dt \\ &+ \int_0^T \int_a^b |R_\varepsilon^k(t, x) - \rho^k(t, x)| dx dt \\ &+ \int_0^T \int_a^b |\rho^k(t, x) - \rho(t, x)| dx dt. \end{aligned} \quad (311)$$

Note that

$$\int_a^b |R^\varepsilon(t, x) - R^{\varepsilon,k}(t, x)| dx \leq \sum_{i=0}^{n-1} \int_a^b |\rho_{i,\varepsilon}(t, x) - \rho_{i,\varepsilon}^k(t, x)| dx. \quad (312)$$

Thus, by Theorem 8.1, the first term on the right-hand side of (311) is bounded above by

$$\sum_{i=0}^{n-1} \int_{a-Vt}^{b+Vt} |\rho_i^0(x) - \rho_i^{0,k}(x)| dx. \quad (313)$$

By specializing Theorem 8.1 to scalar conservation law (75), a similar bound holds for the third term on the right-hand side of (311). Finally, for fixed  $k \in \mathbb{N}$ , by the previous steps, the second term vanishes as  $\varepsilon \rightarrow 0$ . Thus, letting first  $\varepsilon \rightarrow 0$  and then  $k \rightarrow +\infty$  in (311), we obtain that the left-hand side vanishes as  $\varepsilon \rightarrow 0$ .

## 9 Proof of many-lane limit: Theorem 4.2

*Proof of Theorem 4.2.* It is equivalent to show that for every  $v \in \mathbb{R}$  and every sequence  $v_n \rightarrow v$  as  $n \rightarrow +\infty$ ,

$$\widehat{u}(v-, 1) \wedge \widehat{u}(v+, 1) \leq \liminf_{n \rightarrow +\infty} \widehat{u}^n(v_n, 1) \leq \limsup_{n \rightarrow +\infty} \widehat{u}^n(v_n, 1) \leq \widehat{u}(v-, 1) \vee \widehat{u}(v+, 1). \quad (314)$$

We argue by contradiction. Let  $\varepsilon > 0$  and assume there exists a subsequence, still denoted by  $(v_n)$ , such that

$$d[\widehat{u}^n(v_n, 1), \widehat{U}(v, 1)] > \varepsilon. \quad (315)$$

Assume first  $\alpha \geq \beta$ . Let

$$\rho_{n,i} := i/n, \quad \rho^{n,i} = (i + 1/2)/n$$

so that

$$\widehat{G}^n(\rho_{n,i}) = 0, \quad \widehat{G}^n(\rho^{n,i}) = F(\rho_{n,i}).$$

Recall that  $\rho^{n,i}$  and  $\rho_{n,i}$  respectively achieve the maximum and minimum of  $\widehat{G}^n$  over  $[\rho_{n,i}, \rho_{n,i} + n^{-1}]$ . Let  $u_n := \lfloor n\widehat{u}(v_n, 1) \rfloor$  and  $i_n := \lfloor nu \rfloor$ , so that

$$\rho_{n,i} \leq u_n \leq \rho_{n,i} + \frac{1}{n}. \quad (316)$$

Then, using (316),

$$\begin{aligned} v_n u_n - \widehat{G}^n(u_n) &\geq v_n u_n - \widehat{G}^n(\rho^{n,i_n}) \\ &\geq v_n \rho_{n,i_n} - F(\rho_{n,i_n}) - \frac{1}{n}. \end{aligned} \quad (317)$$

Since any minimizer of (187) for  $G = F$  lies in  $\widehat{U}(v, 1)$ , under (315), the last line of (317) remains bounded away above  $F_{\alpha,\beta}^*(v)$ . On the other hand, taking  $i'_n = \lfloor n\widehat{u}(v+, 1) \rfloor$  and  $u'_n = \rho^{n,i'_n}$ , we have

$$\begin{aligned} v_n u'_n - \widehat{G}^n(u'_n) &= v_n u'_n - F(\rho_{n,i'_n}) \\ &\leq v_n \rho_{i,n} - F(\rho_{n,i'_n}) + \frac{1}{n} \end{aligned} \quad (318)$$

and the last line of (318) converges to  $vu - F(u) = F_{\alpha,\beta}^*(v)$  for  $u = \widehat{u}(v\pm, 1)$ . This contradicts the fact that the l.s.h. of (317) achieves the minimum in (187) for  $G = \widehat{G}^n$ .

We now consider  $\alpha \leq \beta$ . By Proposition 7.3,  $\widehat{u}(\cdot, 1)$  is the entropy solution of the Riemann problem with initial datum (62) for (121) with  $F$  replaced by the null function  $F_0$ . Setting  $u'_n = \rho_{n,i'_n}$ , we have

$$v_n u_n - \widehat{G}^n(u_n) \leq v_n u_n - F_0(u_n) \quad (319)$$

$$v_n u'_n - \widehat{G}^n(u'_n) = v_n u'_n - F_0(u'_n). \quad (320)$$

The r.h.s. of (320) converges to  $vu - F_0(u) = (F_0)_{\alpha,\beta}^*(v)$  for  $u = \widehat{u}(v\pm, 1)$ . This with (315) contradicts the fact that the l.h.s. of (319) achieves the minimum in (187) for  $G = \widehat{G}^n$ .  $\square$

## A Phase transitions in Theorem 4.1

We present below figures illustrating the different phases of the flux function stated in Theorem 4.1. Figures 1 to 3 show the phase diagram of inflexion points with regions 2, (i) to 2, (v) of the theorem in the  $(d; r)$  plane. Figures 3 to 9 illustrate the flux function parametrized by  $r$  for a sample value of  $d$  in each of these regions. Figures 4-5, 6-8, 9, 10, 11 correspond respectively to (i), (ii), (iii), (iv), (v); while Figure 12 corresponds to the rescaled limit  $d = +\infty$ , cf.

(115).

While the theorem is stated for  $(d, r) \in [1; +\infty)^2$  and  $d \geq 1$ , by the symmetry property (104), all graphics can be extended to  $(d, r) \in [0; +\infty)^2$ . The transformation  $d \rightarrow 1 - d$  induces a symmetry with respect to  $d = 0.5$  in phase diagrams and around  $\rho = 1$  in fluxes. The transformation  $r \rightarrow r^{-1}$  induces a nonlinear "symmetry" with respect to  $r = 1$  on phase diagrams, and along parametrized curves in Figures 4-12, it induces a symmetry around  $\rho = 1$ ; in addition, the evolution of curves with respect to  $r$  must be seen "upside down".

We interpret  $d$  as an asymmetry parameter and  $r$  as a coexistence/segregation parameter. The latter measures the separation/interplay between lanes (the larger  $r \geq 1$  the stronger the separation and the weaker the interplay). When  $r \rightarrow +\infty$ , particles can only move to lane 0, and only particles initially on lane 1 use this lane until they eventually move to 0 when possible. Thus when the local density is less than 1, only the ASEP flux on lane 0 contributes, while a density above 1 only activates the ASEP flux on lane 1. When  $r \rightarrow 0$ , lanes 1 and 0 are inverted. As mentioned above, we interpret the number of inflexion points of  $G(\rho)$  between 0 and 2, as marking phase transitions in the  $(d, r)$  plane. Indeed, as explained below, this number reflects the degree of separation of lanes (the more inflexion points the more separation). We also view them as marking phase transitions in  $\rho$  for given  $(d, r)$ . When two inflexion points are present, the interval between them is a coexistence zone where both lanes contribute to the global flux, as particles begin to move significantly from lane 0 to lane 1 because of jamming. For  $r \geq 1$ , on the left of this interval, the low density zone is "dominated" by lane 0 in a sense explained below, while the high density zone to the right is dominated by lane 1, but to a lesser extent, since the drift on lane 1 is smaller.

Assume for instance the flux has two inflexion points, as in Figure 9 below the pink layer (that is  $r$  large enough). Between the two inflexion points, as the density on lane 0 gets higher, many particles depart from lane 0 to lane 1, where the density is still low; the flux on lane 1 is thus more favourable to these particles, which begins to compensate for the decay of the flux due to strong exclusion on lane 0, hence the transition from concave to convex. On the right of the second inflexion point, lane 0 is fairly jammed and no longer significantly contributes to the global flux. The density on lane 1 gets high enough for particle on lane 1 to feel the exclusion effect there, hence the transition from convex to concave. Between the pink and yellow layers (that is intermediate values of  $r$ ), the rate of particles moving to lane 1 is sufficient to trigger coexistence but not to trigger the high density/lane 1 phase. Above the pink layer (that is  $r$  close enough to 1), there is never sufficient jamming on lane 0 to significantly reduce its contribution, and too few particles on lane 1 to start compensating the decay on lane 1 at high density. When  $r = 1$ , there is complete coexistence as no lane is favoured. This results on Figures 3-11 in a rescaled TASEP flux where the maximum fluxes of the two lanes add up, cf.

(4.6). When  $r \rightarrow +\infty$ , the separation on Figures 3-12 is complete and there is a sharp transition from ASEP flux on lane 0 to ASEP flux on lane 1, cf. (4.5).

The above arguments suggest that the number of inflexion points should be a monotone function of  $r$ . Surprisingly, Theorem 4.1 proves this to be not always true (though "most" of the time). Figures 2 and 3 indeed exhibit a narrow region of the  $(d, r)$  plane, that we call the "anomalous zone", where the number of inflexion points evolves from 0 to 2, 1 and eventually 2 as  $r$  increases. This region, which corresponds to case 2, (ii) in Theorem 4.1, is almost invisible on the global phase diagram of Figure 1 and only revealed by zooming as in Figures 2,3. It is also hard to detect on the graph of  $G$  (cf. Figure 6), because the transitions occur on a very narrow density interval where the flux is moreover almost linear. A plot of the second derivative of  $G$  helps identify this region clearly (Figures 7,8).

When  $d > 1$ , lane 0 has a positive drift and lane 1 a negative one (Figure 11). The fully segregated phase  $r = +\infty$  has a positive TASEP flux followed by a negative one. The progressive separation as  $r$  grows does not exhibit more than one inflexion point, cf. Figures 1 and 11, which we can view as the transition point between low density/lane 0 and high density/lane 1 phases. The special case  $d \rightarrow +\infty$  (Figure 12) of two lanes with opposite drifts equal in absolute value, cf. (115), yields fluxes symmetric with respect to  $(1; 0)$ . When  $r = 1$ , we obtain a null flux. There is an immediate transition at  $r = 1+ = \lim_{d \rightarrow +\infty} r_3(d)$  (cf. Figure 1) towards a segregated behaviour where all fluxes have an inflexion point at  $\rho = 1$ . In the borderline case  $d = 1$  (Figure 10), one of the lanes has 0 drift, and the corresponding TASEP flux in the fully segregated phase is flat.

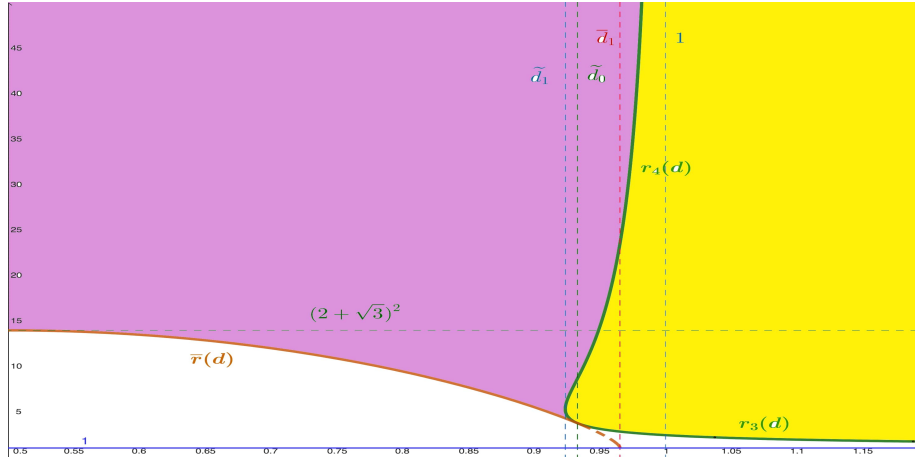


Figure 1: **Phase diagram 1:**  $\tilde{d}_1$ ,  $\tilde{d}_0$ ,  $\bar{d}_1$  are defined in (112). The rightmost dashed line is  $d = 1$ . In the white region, the flux function is strictly concave; in the yellow (resp. pink) region, it has one (resp. two) inflexion points. Intervals  $(-\infty; \tilde{d}_1)$ ,  $(\tilde{d}_1; \tilde{d}_0)$ ,  $(\tilde{d}_0; 1)$  and  $(1; +\infty)$  respectively correspond to cases (i), (ii), (iii) and (iv)–(v) in Theorem 4.1. Curves  $r_3(d)$  and  $r_4(d)$  are the lower and upper branches of the solution  $r$  of  $g(r) = 0$  (green curve), cf. (147), with  $r_4(d) \rightarrow +\infty$  as  $d \rightarrow 1^-$ . Curve  $\bar{r}_1(d)$  is defined by (162)–(168). The blue line at the bottom is  $r = 1$ .

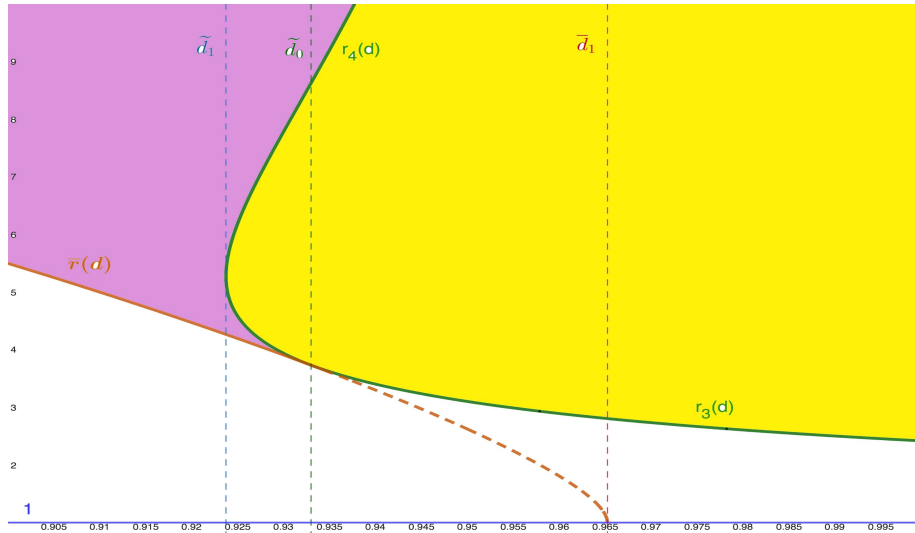


Figure 2: **Phase diagram 2:** zooming around the intersection of  $r_4(d)$  with  $\bar{r}(d)$  reveals the anomalous region between  $d = \tilde{d}_1$  and  $d = \tilde{d}_0$ .



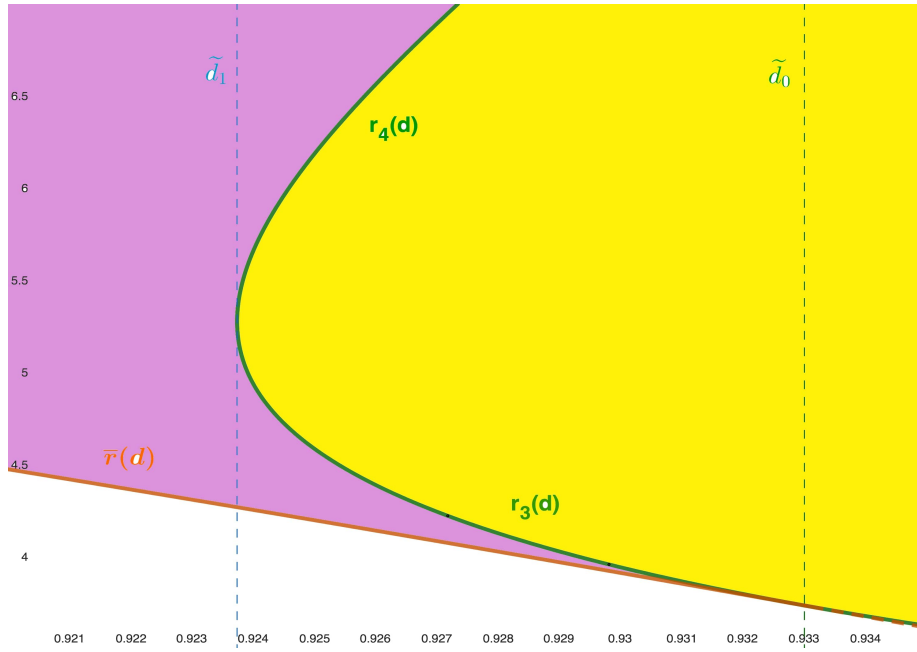


Figure 3: **Phase diagram 3:** zooming around the anomalous region.

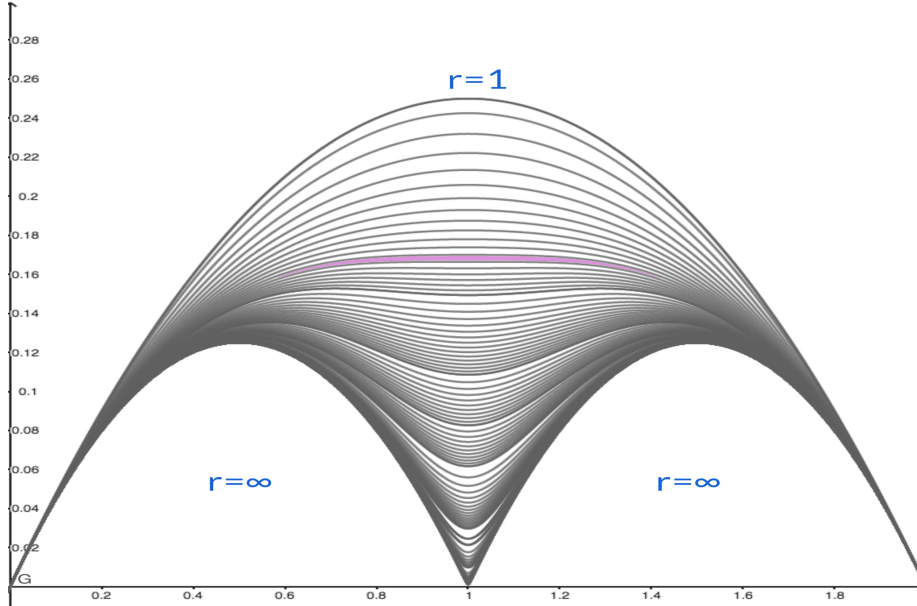


Figure 4: **Flux function 1:**  $d = 0.5$ ,  $\gamma_0 = \gamma_1 = 0.5$ , symmetric graph. Top concave curve:  $r = 1$ ; bottom double-bump curve  $r = +\infty$ . Unique phase transition from 0 to 2 inflexion points occurs at  $r = \bar{r}_1(0.5) = (2 + \sqrt{3})^2 \simeq 14$  (cf. fig 1) inside the pink layer.

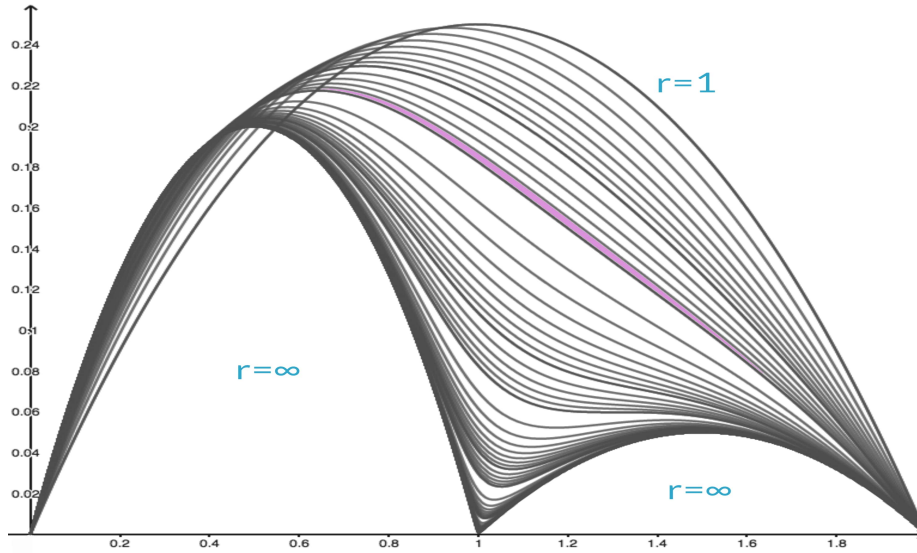


Figure 5: **Flux function 2:**  $d = 0.8$ ,  $\gamma_0 = 0.8$ ,  $\gamma_1 = 0.2$ . Top concave curve:  $r = 1$ ; bottom double-bump curve  $r = +\infty$ . Unique phase transition from 0 to 2 inflexion points occurs at  $r = \bar{r}_1(0.8) \simeq 9.4$  (pink layer).

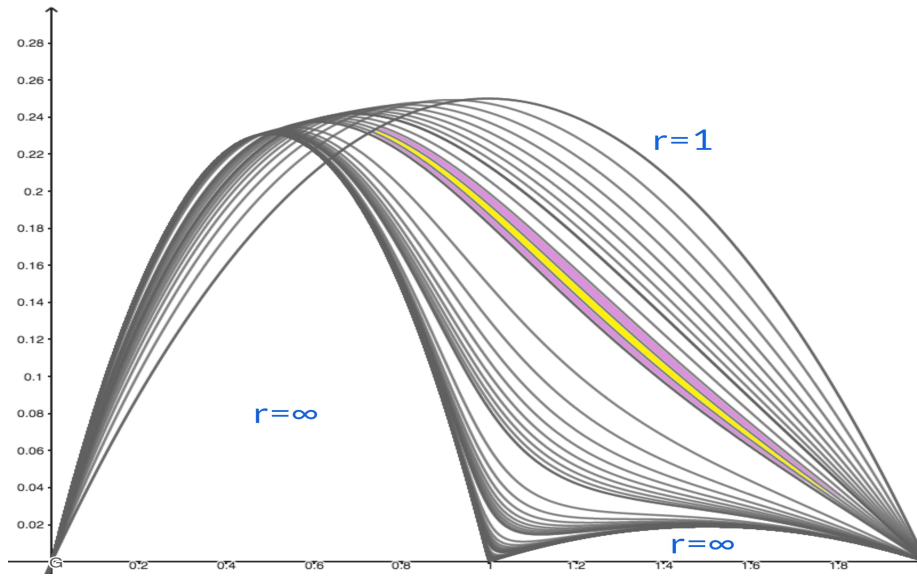


Figure 6: **Flux function 3a:** "Anomalous" zone in phase diagrams 2 and 3. Here  $\tilde{d}_1 < d = 0.924 < \tilde{d}_0$ ,  $\gamma_0 = 0.924$ ,  $\gamma_1 = 0.076$ . Top curve:  $r = 1$ ; bottom curve  $r = +\infty$ . Three transitions occur:  $\bar{r}_1(d) \simeq 4, 25$  (0 to 2 inflexion points, upper pink layer),  $r_3(d) \simeq 4, 9$  (2 to 1, yellow layer),  $r_4(d) \simeq 5, 7$  (1 to 2, lower pink layer). The anomalous interval is  $(\bar{r}_1(d); r_3(d))$ .

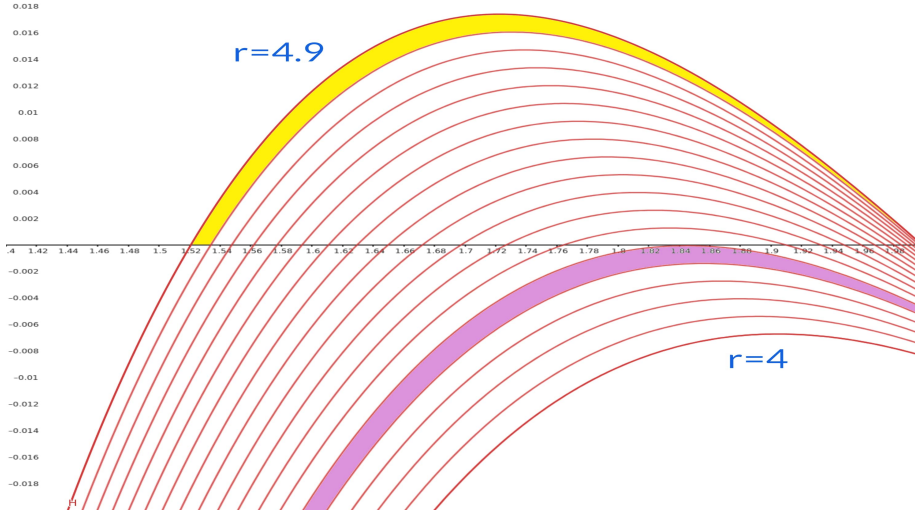


Figure 7: **Flux function 3b:** "Anomalous" zone continued with plot of  $G''(\rho)$  for  $r = 4$  (bottom curve) to  $r = 4.9 \simeq r_3(d)$  (top curve) spaced by  $\Delta r = 0.05$ . Again,  $\tilde{d}_1 < d = 0.924 < \tilde{d}_0$ ,  $\gamma_0 = 0.924$ ,  $\gamma_1 = 0.076$ . The transition at  $\bar{r}_1(d) \simeq 4.25$  (0 to 2 inflexion points) occurs on the top of the pink layer. The transition  $r_3(d) \simeq 4.9$  (2 to 1) occurs on the curve on top of the yellow layer. Intersections with  $x$ -axis indicate positions of inflexion points.

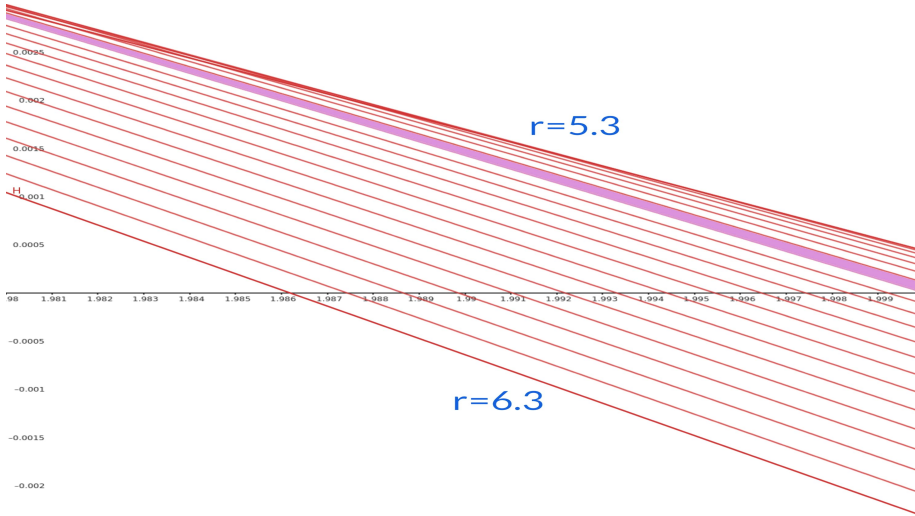


Figure 8: **Flux function 3c:** Anomalous zone continued. Zoom of  $G''(\rho)$  around its second inflexion point for  $r = 5.3$  (top curve) to  $r = 6.3$  (bottom curve) spaced by  $\Delta r = 0.05$ . The transition at  $r_4(d) \simeq 5.65$  (1 to 2 inflexion points) occurs on the bottom of the pink layer.

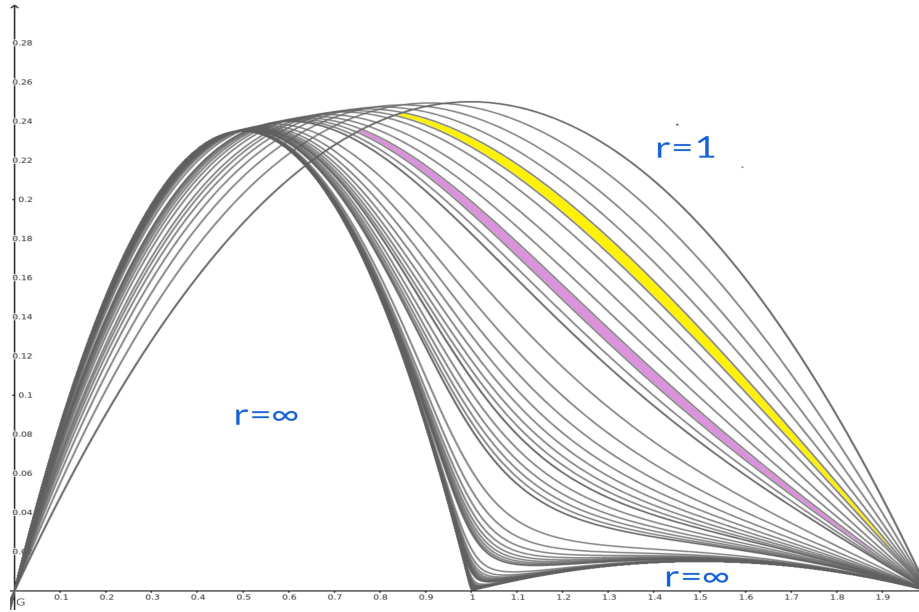


Figure 9: **Flux function 4:**  $\tilde{d}_0 < d = 0.94 = \gamma_0$ ,  $\gamma_1 = 0.06$ . Top concave curve:  $r = 1$ ; bottom double-bump curve  $r = +\infty$ . The transitions occur at  $r_3(d) \simeq 3.3$  (0 to 1 inflexion point, yellow) and  $r_4(d) \simeq 7.6$  (1 to 2 inflexion points, pink). and  $r = 10$ .

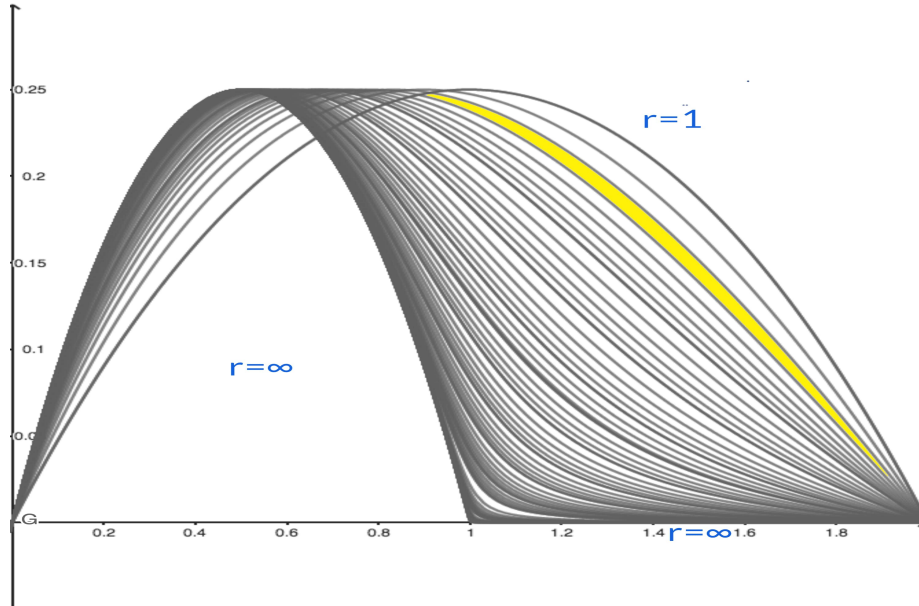


Figure 10: **Flux function 5:**  $d = 1 = \gamma_0$ ,  $\gamma_1 = 1 - d = 0$ . For  $r = 1$ : concave symmetric curve;  $r = +\infty$ : bump-flat curve. The transition (0 to 1 inflexion point) occurs at  $r_3(d) \simeq 2.4$  (yellow).

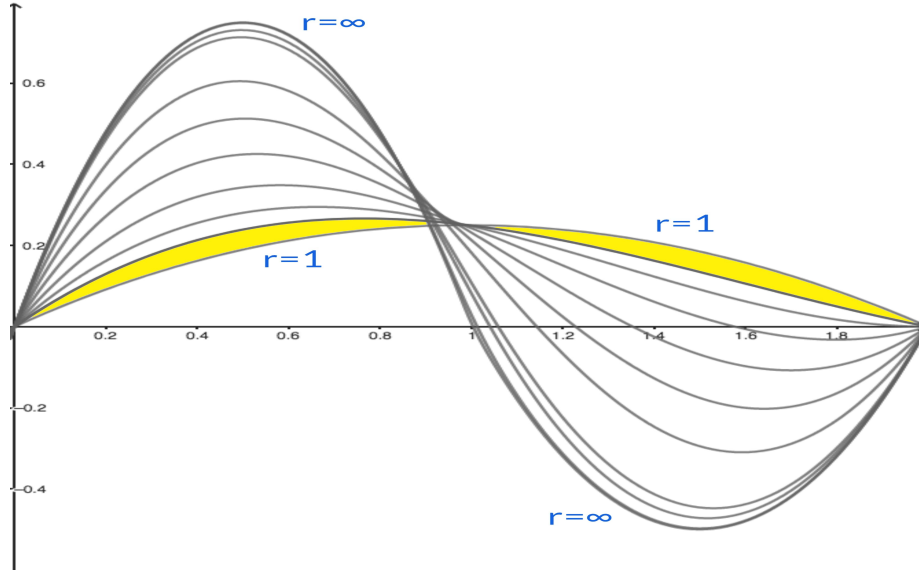


Figure 11: **Flux function 6:**  $d = 3 = \gamma_0$ ,  $\gamma_1 = 1 - d = -2$ . Outer curves are  $r = 1$  (concave curve) and  $r = +\infty$  (bump-well curve). The transition between 0 and 1 inflexion point occurs at  $r_3(d) \simeq 1.14$ , inside the yellow layer between curves  $r = 1$  (concave curve) and  $r = 1.25$  (concave-convex curve).

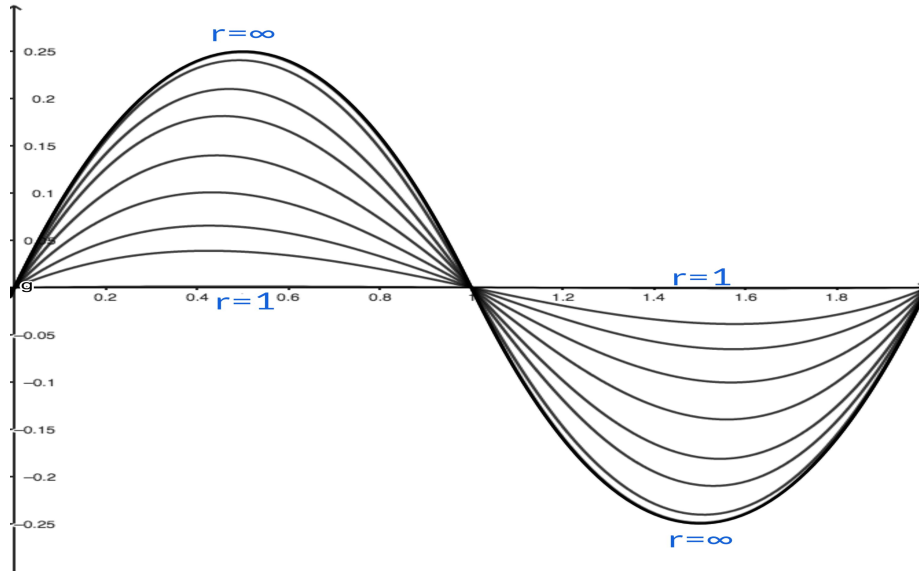


Figure 12: **Flux function 7:**  $d_0 = \infty$ ,  $\gamma_1 = 1 - d = -\infty$ , plot of  $d^{-1}G(\rho) = G_{1,-1}(\rho)$  (cf. (115));  $r = 1$  (identically 0 flux) to  $r = +\infty$  (in bold). Here  $r_3(\infty) = 1$ , there is an instant transition from 0 curve to 1 inflexion point. All curves are symmetric.

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