



HAL
open science

F4 and desmic quartic surfaces

Laurent Manivel

► **To cite this version:**

| Laurent Manivel. F4 and desmic quartic surfaces. 2025. <hal-05217453>

HAL Id: hal-05217453

<https://cnrs.hal.science/hal-05217453v1>

Preprint submitted on 21 Aug 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

F_4 AND DESMIC QUARTIC SURFACES

LAURENT MANIVEL

ABSTRACT. The desmic pencil of quartic surfaces is part of a beautiful, but mostly forgotten chapter of the classical theory of algebraic surfaces: it is the only non-degenerate pencil of surfaces in \mathbb{P}^3 containing at least three completely reducible members. We observe in this note that it is closely related to the Weyl group of the root system F_4 , and can be recovered from a series of symmetric spaces deduced from the exceptional Lie algebras. We discuss the main properties of the pencil from this Lie theoretic point of view.

1. INTRODUCTION

The desmic pencil was named this way by Stephanos in 1879 [Ste79]: it is the unique pencil of quartic surfaces, up to projectivity, containing three tetrahedra with no common face. It was studied in depth by Humbert [Hum91] and is devoted a chapter of Jessop's classical treatise on quartic surfaces [Jes16]. The general desmic surface has 12 singular points (the desmic points) and contains 16 lines and 16 conics. Its symmetry group is the Weyl group of the root system D_4 , but we observe that the symmetry group of the full desmic pencil is bigger: it is the Weyl group of F_4 .

In fact we will proceed in the reverse direction. In a recent series of works [BBFM23, BBFM24, Liu25, BBFM25], we studied gradings of simple complex Lie algebras as a powerful tool to construct interesting geometric objects, following the impulse given by [BH16] and [GSW13]. The main tools for dealing with such gradings were devised by Vinberg [Vin76], whose tables show that the Weyl group of F_4 is the so-called little Weyl group of a nice series of \mathbb{Z}_2 -gradings of the exceptional Lie algebras. Typically, there is \mathbb{Z}_2 -grading of \mathfrak{e}_6 with one piece isomorphic to $\mathbb{C}^2 \otimes \wedge^3 \mathbb{C}^6$, hence connected to lines in $\mathbb{P}(\wedge^3 \mathbb{C}^6)$. This projective space contains an invariant quartic hypersurface [Don77, Chapter 3], and such a line therefore defines an elliptic curve: this process, considered in [BH16, 6.4.2], was first suggested by Segre [Seg18]. Applying Vinberg theory, we construct in $\mathbb{C}^2 \otimes \wedge^3 \mathbb{C}^6$ a four-dimensional Cartan subspace (a generalization of the Cartan subalgebra). Restricting the invariant quartic, we get a $W(F_4)$ -invariant pencil of quartic surfaces in \mathbb{P}^3 , and this turns out to be the desmic pencil. This is another confirmation of the author's fantasy that part of the most beautiful objects of classical algebraic geometry are, in some sense, derived from the exceptional Lie algebras: including the Segre cubic, Kummer surfaces,

Coble hypersurfaces, and even cubic fourfolds and Debarre-Voisin fourfolds [IM19].

Interestingly, the three tetrahedra in the desmic pencil are in correspondence with the twelve long positive roots of F_4 , that one can uniquely split into three groups of mutually orthogonal roots. One can do the same with the short roots, which yields another copy of the desmic pencil, classically known as conjugate to the first one (its vertices are the 12 desmic points). From this point of view, the two desmic pencils are instances of (projectivized) Macdonald representations [Mac72], constructed from subroot systems of F_4 of type A_1^4 . It is worth noticing that little is known about linear systems of hypersurfaces containing many completely reducible members. For pencils, some progress have been made [PY08, Yuz09]; there is a combinatorial version called the desmic conjecture (see e.g. [Bor83]); but the general case seems wide open. Macdonald representations provide nice examples of such linear systems. In [LM25], we found a \mathbb{P}^4 (resp. a \mathbb{P}^8 and a \mathbb{P}^4) of hypersurfaces in \mathbb{P}^3 containing 15 tetrahedra (resp. 45 octahedra and 15 dodecahedra); one can wonder whether these examples are extremal and isolated.

Closely related to the desmic pencil is the Hesse pencil of plane cubics, which can also be derived from Vinberg's theory. As well as the Hesse pencil parametrizes elliptic curves, it was shown in [Hum91] that the quartic surfaces of the desmic pencil are birational to Kummer surfaces of squares of elliptic curves. We connect this wonderful observation to our approach through Vinberg's theory by checking that the elliptic curve in question is, as one could expect, the same as the one considered by Segre and Bhargava-Ho; this is the only original result of this modest note.

Acknowledgements. We warmly thank Willem de Graaf for his very useful hints, and Igor Dolgachev for his kind comments. Support is acknowledged from the ANR project FanoHK ANR-20-CE40-0023.

2. F_4 AND ITS ROOT SYSTEM

A convenient way to describe the root system of F_4 is to use the fact that \mathfrak{so}_9 embeds in \mathfrak{f}_4 as a simple subalgebra of maximal rank. More precisely, there is a \mathbb{Z}_2 -grading of \mathfrak{f}_4 given by

$$\mathfrak{f}_4 = \mathfrak{so}_9 \oplus \Delta,$$

where Δ is the sixteen dimensional spin representation of \mathfrak{so}_9 . As a consequence, the standard Cartan subalgebra of \mathfrak{so}_9 is also a Cartan subalgebra of \mathfrak{f}_4 , with respect to which the roots of the latter are

$$\pm\epsilon_i \pm \epsilon_j, \quad \pm\epsilon_k, \quad \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4),$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ is the usual notation for an orthogonal basis of the dual of the Cartan algebra. The first group gives the 24 long roots, the second and third group give the 8 + 16 short roots [Bou02]. In order to select a set

positive roots, we can fix a generic real linear form, say $\ell = \ell_1\epsilon_1^\vee + \ell_2\epsilon_2^\vee + \ell_3\epsilon_3^\vee + \ell_4\epsilon_4^\vee$; the positive roots are those on which ℓ takes positive values, and we deduce the simple roots as the indecomposable positive roots; fixing for example $\ell_1 \gg \ell_2 \gg \ell_3 \gg \ell_4 \gg 1$, we obtain the simple roots

$$(1) \quad \alpha_1 = \epsilon_2 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_4, \quad \alpha_3 = \epsilon_4, \quad \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

Lemma 1. *Any long (resp. short) root is orthogonal to three long (resp. short) positive roots and six short (resp. long) positive roots.*

Proof. Since the Weyl group acts transitively on long (resp. short) roots, it suffices to check the claim for one long root and one short root.

The long root $\epsilon_1 - \epsilon_2$ is orthogonal to the three long positive roots $\epsilon_1 + \epsilon_2$ and $\epsilon_3 \pm \epsilon_4$, and the six short positive roots ϵ_3, ϵ_4 and $\frac{1}{2}(\epsilon_1 + \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$.

The short root ϵ_1 is orthogonal to the three short positive roots $\epsilon_2, \epsilon_3, \epsilon_4$ and the six long positive roots $\epsilon_2 \pm \epsilon_3, \epsilon_2 \pm \epsilon_4, \epsilon_3 \pm \epsilon_4$. \square

An immediate consequence is the following

Proposition 2. *There is a unique way to split the twelve long (resp. short) positive roots of F_4 into three mutually orthogonal groups of four roots.*

Explicitly, the three groups of long roots are

$$\begin{aligned} &\epsilon_1 - \epsilon_2, \quad \epsilon_1 + \epsilon_2, \quad \epsilon_3 - \epsilon_4, \quad \epsilon_3 + \epsilon_4, \\ &\epsilon_1 - \epsilon_3, \quad \epsilon_1 + \epsilon_3, \quad \epsilon_2 - \epsilon_4, \quad \epsilon_2 + \epsilon_4, \\ &\epsilon_1 - \epsilon_4, \quad \epsilon_1 + \epsilon_4, \quad \epsilon_2 - \epsilon_3, \quad \epsilon_2 + \epsilon_3. \end{aligned}$$

The three groups of short roots are $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and

$$\begin{aligned} &\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4), \quad \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4), \\ &\frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4), \quad \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4), \\ &\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4), \quad \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4), \\ &\frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4), \quad \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4). \end{aligned}$$

Subsets of mutually orthogonal roots as in Proposition 2 are in correspondence with root subsystems of type A_1^4 . Such root subsystems have been extensively studied and classified [Osh06].

Remark 3. *The orthogonal matrix*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

sends the short roots to multiples of the long roots and conversely. This explains the symmetry between long and short roots in the previous statement.

3. $W(F_4)$ AS A LITTLE WEYL GROUP

Vinberg showed how to associate to a simple graded Lie algebra a *little Weyl group*, generalizing the construction of the usual Weyl group [Vin76]. It is a remarkable fact that, as can be observed in [Vin76, Table pp. 491–492], the Weyl group of F_4 is the little Weyl group of a series of \mathbb{Z}_2 -gradings, namely

$$\begin{aligned}\mathfrak{f}_4 &= \mathfrak{sl}_2 \times \mathfrak{sp}_6 \oplus \mathbb{C}^2 \otimes \wedge^{(3)}\mathbb{C}^6, \\ \mathfrak{e}_6 &= \mathfrak{sl}_2 \times \mathfrak{sl}_6 \oplus \mathbb{C}^2 \otimes \wedge^3\mathbb{C}^6, \\ \mathfrak{e}_7 &= \mathfrak{sl}_2 \times \mathfrak{so}_{12} \oplus \mathbb{C}^2 \otimes \Delta_+, \\ \mathfrak{e}_8 &= \mathfrak{sl}_2 \times \mathfrak{e}_7 \oplus \mathbb{C}^2 \otimes V_{56}.\end{aligned}$$

Here the fundamental representations $\wedge^{(3)}\mathbb{C}^6, \wedge^3\mathbb{C}^6, \Delta_+, V_{56}$ of $\mathfrak{sp}_6, \mathfrak{sl}_6, \mathfrak{so}_{12}, \mathfrak{e}_7$ have dimensions $6a + 8$, for $a = 1, 2, 4, 8$. This corresponds to the so called sub-exceptional, or sub-adjoint series, that has been studied by several authors, either from the representation-theoretic [DG02, LM02] or the geometric point of view [Fre63, LM01, Buc09, Liu25].

As in the classical theory, the little Weyl group comes with an action on the Cartan subspace (generalizing the usual action of the Weyl group on the Cartan algebra); for each grading of the sub-adjoint series, the Cartan subspace has dimension four and the action of the little Weyl group is equivalent to the action of $W(F_4)$ on a Cartan subalgebra of \mathfrak{f}_4 . In particular the associated symmetric spaces all have rank four, and the corresponding algebras of invariants are all freely generated by homogeneous polynomials of degrees 2, 6, 8, 12.

Cartan spaces can be computed explicitly. One can check that the sub-adjoint series of Lie algebras

$$\mathfrak{sp}_6 \subset \mathfrak{sl}_6 \subset \mathfrak{so}_{12} \subset \mathfrak{e}_7$$

comes with natural embeddings of representations

$$\wedge^{(3)}\mathbb{C}^6 \subset \wedge^3\mathbb{C}^6 \subset \Delta_+ \subset V_{56},$$

and that one can choose a Cartan subspace in $\mathbb{C}^2 \otimes \wedge^{(3)}\mathbb{C}^6$, that can serve as a Cartan subspace for the whole series.

Explicitly, choose a basis f_1, f_2 of \mathbb{C}^2 and a basis e_1, \dots, e_6 of \mathbb{C}^6 ; denote $e_{ijk} = e_i \wedge e_j \wedge e_k$. We claim that a Cartan subspace \mathfrak{c} can be chosen to be generated by

$$\begin{aligned}c_1 &= f_1 \otimes e_{135} + f_2 \otimes e_{246}, \\ c_2 &= f_1 \otimes e_{146} + f_2 \otimes e_{235}, \\ c_3 &= f_1 \otimes e_{245} + f_2 \otimes e_{136}, \\ c_4 &= f_1 \otimes e_{236} + f_2 \otimes e_{145}.\end{aligned}$$

Here e_{135} and the other trivectors can be seen as vectors in $\wedge^3\mathbb{C}^6$, or also in $\wedge^{(3)}\mathbb{C}^6$, which simply mean that they correspond to three-spaces that are isotropic with respect to the same symplectic form

$$\omega = e_1^\vee \wedge e_2^\vee + e_3^\vee \wedge e_4^\vee + e_5^\vee \wedge e_6^\vee.$$

In the sequel we represent elements of $W(F_4)$ as matrices encoding their action on \mathfrak{c} in the basis above.

Proposition 4. *The little Weyl group $W(F_4)$ is generated by the following transformations:*

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad s_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Proof. First observe that s_1, s_2, s_3, s_4 are hyperplane reflections with respect to $\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4$ and $\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$, which is our system (1) of simple roots for F_4 . As a consequence, they must generate a copy of $W(F_4)$. Since we already know that the little Weyl group is isomorphic to $W(F_4)$, we just need to check that it contains s_1, s_2, s_3, s_4 .

By definition, the little Weyl group is the quotient of the normalizer of \mathfrak{c} in $SL_2 \times SL_6$, by its centralizer. Therefore, we only need to check that we can lift s_1, s_2, s_3, s_4 to $SL_2 \times SL_6$. We provide explicit lifts $\sigma_1, \sigma_2, \sigma_3, \sigma_4$:

$$\sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\zeta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta \end{pmatrix}, \quad \zeta^4 = -1$$

$$\sigma_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The verification that these are lifts as claimed is left to the reader. \square

Considering the image in SL_2 we deduce:

Corollary 5. *The image in SL_2 of the normalizer of the Cartan subspace is a copy of the binary octahedral group.*

Remark 6. *The root system of type F_4 is sometimes described in terms of quaternions. Indeed, let us formally replace $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ by the quaternions $1, i, j, k$. Consider the cube whose vertices are $\pm 1, \pm i, \pm j, \pm k$. The group of unit quaternions that preserve this cube by conjugation is a copy of the binary octahedral group; it consists in the 24 short roots, and the 24 long roots, suitably normalized.*

4. SOME QUARTIC HYPERSURFACES

The sub-adjoint series has the form $\mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{h} \oplus \mathbb{C}^2 \otimes W$ where the representations W share the following properties [LM01, Buc09].

- (1) They are symplectic representations, being endowed with a non-degenerate invariant skew-symmetric form Ω .
- (2) They contain a unique invariant hypersurface, of degree four.
- (3) The orbit closures of $H = \text{Aut}(\mathfrak{h})$ in $\mathbb{P}(W)$ are the invariant quartic Q , its singular locus Y , and the singular locus X of Y , which is H -homogeneous.
- (4) The quartic Q is the tangent variety of X (the union of its tangent spaces), and also the projective dual variety $X^* \subset \mathbb{P}(W^\vee) \simeq \mathbb{P}(W)$ (the union of its tangent hyperplanes).
- (5) Any point of $\mathbb{P}(W)$, not belonging to the quartic Q , belongs to a unique bisecant to X .

The quartic can be constructed from the symplectic form. J.L. Clerc obtained a nice uniform expression by downgrading the \mathbb{Z}_2 -grading of \mathfrak{g} into a five-step \mathbb{Z} -grading with $\mathfrak{g}_2 \simeq \mathfrak{g}_{-2} \simeq \mathbb{C}$ and $\mathfrak{g}_1 \simeq W$ [Cle03]. A more direct statement is the following. Recall that \mathfrak{h} is self-dual through the Cartan-Killing quadratic form K .

Proposition 7.

- (1) *There is an invariant quadratic map $P : W \rightarrow \mathfrak{h}$ defined by*

$$K(P(w), h) = \Omega(hw, w), \quad \forall h \in \mathfrak{h}, w \in W.$$

(2) The invariant hypersurface Q is defined by the quartic equation

$$K(P(w), P(w)) = \Omega(P(w)w, w) = 0, \quad w \in W.$$

Proof. We check that there is a unique invariant linear map $S^2W \rightarrow \mathfrak{h}$, and a unique invariant $S^4W \rightarrow \mathbb{C}$, up to scalar. So any invariant expression, if nonzero, must be the right one. The claim easily follows. \square

Remark 8. A generic line $\delta \subset \mathbb{P}(W)$ cuts the quartic hypersurface Q at four points q_1, q_2, q_3, q_4 , and the double cover of δ branched at these four points is a genus one curve E_δ .

This is one of the constructions of elliptic curves from special representations considered in [BH16] and [Tho13], which ultimately led to major results on the statistics of elliptic curves over \mathbb{Q} . Modular interpretations are studied further in [Liu25].

Now consider a generic element w of the Cartan subspace \mathfrak{c} . For simplicity we consider \mathfrak{c} as a subspace of $\mathbb{C}^2 \otimes \wedge^3 \mathbb{C}^6$, but the other cases would lead to the same results. So we see w as a pencil of trivectors of the form

$$s(x_1e_{135} + x_2e_{146} + x_3e_{245} + x_4e_{236}) + t(x_1e_{246} + x_2e_{235} + x_3e_{136} + x_4e_{145}).$$

Our aim is to evaluate the quartic form Q on such a trivector. We first compute $P(w) \in \mathfrak{sl}_6$ using Proposition 7; for $\wedge^3 \mathbb{C}^6$, the invariant symplectic form is $\Omega(w, w') = w \wedge w' \in \wedge^6 \mathbb{C}^6$, that we trivialize by fixing a generator, say e_{123456} . If $X = e_i^\vee \otimes e_j$, $i \neq j$, the combinatorics of the Cartan subspace is such that $Xw \wedge w = 0$ when $(ij) \neq (12), (34), (45)$. The six nonzero terms are

$$\begin{aligned} (e_1^\vee \otimes e_2)w \wedge w &= 2(s^2x_1x_2 - t^2x_3x_4), \\ (e_2^\vee \otimes e_1)w \wedge w &= 2(s^2x_3x_4 - t^2x_1x_2), \\ (e_3^\vee \otimes e_4)w \wedge w &= 2(s^2x_1x_4 - t^2x_2x_3), \\ (e_4^\vee \otimes e_3)w \wedge w &= 2(s^2x_2x_3 - t^2x_1x_4), \\ (e_5^\vee \otimes e_6)w \wedge w &= 2(s^2x_1x_3 - t^2x_2x_4), \\ (e_6^\vee \otimes e_5)w \wedge w &= 2(s^2x_2x_4 - t^2x_1x_3). \end{aligned}$$

On the other hand, a traceless diagonal matrix H gives

$$\begin{aligned} Hw \wedge w &= -2st \left((h_1 + h_3 + h_5)x_1^2 + (h_1 + h_4 + h_6)x_2^2 + \right. \\ &\quad \left. + (h_2 + h_4 + h_5)x_3^2 + (h_2 + h_3 + h_6)x_4^2 \right). \end{aligned}$$

We deduce that $P(w)$ preserves each of the three planes $\langle e_1, e_2 \rangle$, $\langle e_3, e_4 \rangle$, $\langle e_5, e_6 \rangle$; so it is the sum of its restrictions to these three planes, which are given by the three following matrices in \mathfrak{sl}_2 :

$$\begin{pmatrix} st(x_3^2 + x_4^2 - x_1^2 - x_2^2) & 2(s^2x_1x_2 - t^2x_3x_4) \\ 2(s^2x_3x_4 - t^2x_1x_2) & st(x_1^2 + x_2^2 - x_3^2 - x_4^2) \end{pmatrix}$$

$$\begin{pmatrix} st(x_2^2 + x_3^2 - x_1^2 - x_4^2) & 2(s^2x_1x_4 - t^2x_2x_3) \\ 2(s^2x_2x_3 - t^2x_1x_4) & st(x_1^2 + x_4^2 - x_2^2 - x_3^2) \end{pmatrix}$$

$$\begin{pmatrix} st(x_2^2 + x_4^2 - x_1^2 - x_3^2) & 2(s^2x_1x_3 - t^2x_2x_4) \\ 2(s^2x_2x_4 - t^2x_1x_3) & st(x_1^2 + x_3^2 - x_2^2 - x_4^2) \end{pmatrix}$$

The contribution of the diagonal part is given by the formula above, with H being the diagonal part of $P(w)$; we get

$$\begin{aligned} & -2s^2t^2 \left((x_2^2 + x_3^2 + x_4^2 - 3x_1^2)x_1^2 + (x_1^2 + x_3^2 + x_4^2 - 3x_2^2)x_2^2 + \right. \\ & \left. + (x_1^2 + x_2^2 + x_4^2 - 3x_3^2)x_3^2 + (x_1^2 + x_2^2 + x_3^2 - 3x_4^2)x_4^2 \right). \end{aligned}$$

Using the formulas above, we obtain the contribution of the off-diagonal part of $P(w)$ as

$$\begin{aligned} & 8 \left((s^2x_1x_2 - t^2x_3x_4)(s^2x_3x_4 - t^2x_1x_2) + (s^2x_1x_4 - t^2x_2x_3)(s^2x_2x_3 - t^2x_1x_4) \right. \\ & \left. + (s^2x_1x_3 - t^2x_2x_4)(s^2x_2x_4 - t^2x_1x_3) \right). \end{aligned}$$

The sum of these two contributions finally yields:

Proposition 9.

$$Q(w) = 24(s^4 + t^4)x_1x_2x_3x_4 - 6s^2t^2 \left(2 \sum_{i < j} x_i^2x_j^2 - \sum_k x_k^4 \right).$$

We thus finally obtain the pencil of quartics of the form

$$(2) \quad 8Ax_1x_2x_3x_4 - B \left(2 \sum_{i < j} x_i^2x_j^2 - \sum_k x_k^4 \right).$$

This is classically known as the *desmic pencil*.

5. THE DESMIC PENCIL

The desmic pencil of quartic surfaces is characterized by the fact that it contains three tetrahedra [Ste79], as follows from the expression given by Proposition 9:

- **T₁**: $st = 0$,
faces $x_i = 0$,
vertices $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$,
- **T₂**: $s^2 - t^2 = 0$,
faces $\epsilon_1x_1 + \epsilon_2x_2 + \epsilon_3x_3 + \epsilon_4x_4 = 0$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$
vertices $[1, 1, 1, 1], [1, 1, -1, -1], [1, -1, 1, -1], [1, -1, -1, 1]$,
- **T₃**: $s^2 + t^2 = 0$,
faces $\epsilon_1x_1 + \epsilon_2x_2 + \epsilon_3x_3 + \epsilon_4x_4 = 0$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$
vertices $[1, 1, 1, -1], [1, 1, -1, 1], [1, -1, 1, 1], [1, -1, -1, -1]$.

Remark 10. *These three tetrahedra were called "desmiques" by Stephanos (from the Greek δέσμη, another word for "bundle" or "sheaf") precisely because they belong to a same pencil of quartic surfaces.*

We recognize in the three tetrahedra the splitting from Proposition 2 of the short roots of \mathfrak{f}_4 . The long roots can be used as well and yield another classical form of the desmic pencil, as the pencil of quartic surfaces with equations of the form

$$(3) \quad a(y_1^2 - y_2^2)(y_3^2 - y_4^2) - b(y_1^2 - y_3^2)(y_2^2 - y_4^2) + c(y_1^2 - y_4^2)(y_2^2 - y_3^2) = 0,$$

where $a + b + c = 0$. This can also be written as

$$(4) \quad (b - c)(y_1^2 y_2^2 + y_3^2 y_4^2) + (c - a)(y_1^2 y_3^2 + y_2^2 y_4^2) + (a - b)(y_1^2 y_4^2 + y_2^2 y_3^2) = 0.$$

In fact it was observed already in [Ste79, section 15] that the twelve desmic points are the vertices of three other tetrahedra forming another desmic system, called conjugate to the first one. The explicit computations of [Mat26, section 5] show that the desmic pencils (2) and (3) are conjugate. We conclude:

Proposition 11. *The two splitting of the long and short roots of \mathfrak{f}_4 into three tetrahedra yield two conjugate desmic pencils of quartic surfaces in \mathbb{P}^3 . As a consequence, the Weyl group $W(F_4)$ acts as a group of symmetries of these desmic pencils.*

Remark 12. *This was clearly known to Shephard and Todd, being briefly mentioned page 278 of [ST54]; this comes with a reference to [Cox91], although desmic surfaces do not appear there. However, Coxeter describes $W(F_4)$, rather denoted $[3, 4, 3]$, as the symmetry group of the 24-cell, whose vertices can readily be put in correspondence either with the long or the short roots of F_4 (see "Cesáro's construction" in [Cox91, 8.2]).*

A given desmic surface is of course not invariant under $W(F_4)$, but (in general) under changes of signs of the coordinates, and also double transpositions: these transformations generate a copy of $W(D_4)$. In particular, as Kummer surfaces, desmic surfaces belong to the family of Heisenberg invariant quartics studied in [Ekl18] (but they are never Kummer surfaces). A more classical study of these surfaces is given in [Edg49], where it is shown among other things that they contain 15 tetrahedra yielding the famous Klein configuration.

Remark 13. *Taking into account the relation*

$$(y_1^2 - y_2^2)(y_3^2 - y_4^2) - (y_1^2 - y_3^2)(y_2^2 - y_4^2) + (y_1^2 - y_4^2)(y_2^2 - y_3^2) = 0,$$

we see that fixing a relation like (3), for some given values of (a, b, c) , amounts to fixing the cross-ratio of $y_1^2, y_2^2, y_3^2, y_4^2$.

Remark 14. *The relationship with root subsystems of F_4 of type A_1^4 show that the two desmic pencils are incarnations of the so-called Macdonald representations. The general recipe is the following: given a subsystem of a root system, the product of the corresponding linear forms, and the translates of this product by the Weyl group, span an irreducible representation of the latter [Mac72]. In the case at hand, each subsystem has only three different*

translates (up to scalar), and they generate a two-dimensional Macdonald representation: after projectivization, a desmic pencil. In fact the generators s_3 and s_4 act trivially and the representation factors through the symmetric group S_3 . Taking the two desmic pencils into account yields the classical extension [CCN⁺85]

$$1 \rightarrow \mu_2^5 \rightarrow W(F_4) \rightarrow S_3 \times S_3 \rightarrow 1.$$

The stabilizer of $[T_1]$ in $W(F_4)$ clearly acts by signed permutations of the x_i 's, and is therefore a copy of $W(D_4)$. Restricting the previous extension we get

$$1 \rightarrow \mu_2^5 \rightarrow W(D_4) \rightarrow S_3 \rightarrow 1,$$

which is a manifestation of triality.

Let us now come back to the form (2) and recall a few classical properties of the desmic pencil.

- (1) The twelve vertices of the three desmic tetrahedra correspond to completely decomposable elements in the Cartan subspace (for example, $s = t$ and $x_1 = x_2 = x_3 = x_4$ yields $(e_1 + e_2) \wedge (e_3 + e_4) \wedge (e_5 + e_6)$). The desmic property is that any edge of one octahedron meets a pair of opposite edges of the second; the intersection points are the 12 special points $[0, 0, 1, \pm 1]$ and correspond to lines in Y (the orbit of partially decomposable forms), typically $[0, 0, 1, 1]$ yields $(te_1 + se_2) \wedge (e_{36} + e_{45}) \in Y$. These points are called *desmic points* in [Hum91]. Each desmic quartic of the pencil is singular at the 12 desmic points.
- (2) The base locus of the desmic pencil is the union of 16 lines with equations of the form

$$x_1 = \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 = 0, \quad \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1.$$

These 16 lines meet at the 12 desmic points: each line contains three of them and each desmic point belongs to four lines: hence a $(16_3, 12_4)$ configuration with a $W(F_4)$ -symmetry, which is nothing else than the classical *Reye configuration* [Dol04, Man06]. Moreover the tangent plane to each desmic surface in the pencil is constant along any of the 16 lines; so the residual intersection is a conic, one of the 16 conics contained in each desmic surface

- (3) The product $st(s^4 - t^4)$ of the quadratic forms associated to the three tetrahedra is Klein's invariant of the binary octahedral group, in agreement with Corollary 5.

6. DESMIC AND KUMMER SURFACES

Let E be the elliptic curve obtained as the quotient of \mathbb{C} by the lattice Λ generated by ω_1 and ω_2 . Recall that the Weierstrass \wp -function associated

to Λ is a doubly periodic meromorphic function with order two poles on Λ , and regular outside Λ . The half-periods

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), \quad e_2 = \wp\left(\frac{\omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right)$$

are such that \wp satisfies the differential equation

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

On the other hand, the Weierstrass σ -function is an odd entire function, quasi-periodic with respect to Λ , and such that σ'/σ is a primitive of $-\wp$. Traditionally (see e.g. [Law89, section 6.2]), one defines three other σ -functions, $\sigma_1, \sigma_2, \sigma_3$, such that the following relations are verified [Law89, (6.7.10-11-12)]:

$$\wp = \left(\frac{\sigma_1}{\sigma}\right)^2 + e_1 = \left(\frac{\sigma_2}{\sigma}\right)^2 + e_2 = \left(\frac{\sigma_3}{\sigma}\right)^2 + e_3.$$

From these relations, a straightforward computation (see [Hum91, section 6] or [Jes16, Chap. 2]) shows that, letting

$$(5) \quad y_1 = \frac{\sigma_1(u)}{\sigma_1(v)}, \quad y_2 = \frac{\sigma_2(u)}{\sigma_2(v)}, \quad y_3 = \frac{\sigma_3(u)}{\sigma_3(v)}, \quad y_4 = \frac{\sigma(u)}{\sigma(v)},$$

for some $u, v \in \mathbb{C}$, the following identity holds:

$$(6) \quad (e_1 - e_2)(y_1^2 y_2^2 + y_3^2 y_4^2) + (e_3 - e_1)(y_1^2 y_3^2 + y_2^2 y_4^2) + (e_2 - e_3)(y_1^2 y_4^2 + y_2^2 y_3^2) = 0.$$

As a consequence, (5) provides a rational parametrization of the desmic surface with parameters $(a, b, c) = (e_2, e_3, e_1)$. Since the σ function is odd, this parametrization is unchanged when $(u, v) \mapsto (-u, -v)$, and one deduces that the desmic surface is birational to the Kummer surface of $E \times E$ (see also [DK25] for an alternative approach).

But what is this elliptic curve? That's an easy guess in our context. Indeed a point in $x = [w] \in \mathbb{P}(\mathfrak{c})$, outside the three tetrahedra, belongs to a unique desmic surface Δ_x in the desmic pencil. On the other hand, it defines a line $\delta(x)$ in $\mathbb{P}(\wedge^3 \mathbb{C}^6)$, that cuts the invariant quartic at four points; hence defines an elliptic curve $E_{\delta(x)}$, as already noticed in Remark 8.

Proposition 15. *The desmic surface Δ_x is birational to $\text{Kum}(E_{\delta(x)} \times E_{\delta(x)})$.*

Proof. The desmic surface to which x belongs has parameters A, B as in (2). The four points of $\delta(x)$ correspond to the four points $[s, t]$ in \mathbb{P}^1 such that $B(s^4 + t^4) = 2As^2 t^2$. If $[s, t]$ is one of them, the three others are $[s, -t], [t, s], [t, -s]$; computing their cross-ratio, we find e.g. $-(s^2 - t^2)^2 / 4s^2 t^2$.

On the other hand, applying the transformation of Remark 3 we find a projectively equivalent desmic surface in the conjugate pencil, with equation $4s^2 t^2 (y_1^2 y_2^2 + y_3^2 y_4^2) + (s^2 - t^2)^2 (y_1^2 y_3^2 + y_2^2 y_4^2) - (s^2 + t^2)^2 (y_1^2 y_4^2 + y_2^2 y_3^2) = 0$.

Comparing with (6), we see that the half-periods (e_1, e_2, e_3) are such that $(e_1 - e_2, e_3 - e_1, e_2 - e_3)$ is proportional to $(4s^2 t^2, (s^2 - t^2)^2, -(s^2 + t^2)^2)$.

The elliptic curve E with these half-periods is the plane cubic of equations $y^2 = 4(x - e_1)(x - e_2)(x - e_3)$, hence a double cover of \mathbb{P}^1 branched at (e_1, e_2, e_3, ∞) . Computing the cross-ratio of these four points, we get e.g.

$$\frac{e_3 - e_1}{e_2 - e_1} = \frac{(s^2 - t^2)^2}{-4s^2t^2}.$$

This proves that E and the curve $E_{\delta(x)}$ are isomorphic. \square

Remark 16. *There is a similar story with the complex reflection group G_{26} of the Shephard-Todd classification, also known as the Hessian group. Indeed, this is the little Weyl group of a series of \mathbb{Z}_3 -gradings of the exceptional Lie algebras. This series is related to the second row of the Tits-Freudenthal magic square, while the sub-adjoint series is related to its third row [LM01]. For this series the rank (the dimension of the Cartan subspace) is equal to three, and the related classical geometric object is the Hesse pencil of cubic curves, which contains four triangles and has many similarities with the desmic pencil. Notwithstanding the fact the Hesse pencil parametrizes elliptic curves, while the desmic pencil parametrizes (essentially) squares of such elliptic curves, it was observed long ago that the two pencils are intimately related, see e.g. [Mat26]. A modern exposition of the geometry of the Hesse pencil can be found in [AD09].*

7. COMPLEMENTS

Starting from the desmic family $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$ one may consider the double cover X of the latter branched along \mathcal{X} . This fourfold has for canonical bundle the pull-back of $\mathcal{O}_{\mathbb{P}^3}(-2)$, and the projection to \mathbb{P}^3 is an elliptic fibration. Of course X is acted on by $W(F_4)$, but it is singular over the pre-images of the 16 desmic points. By base change over \mathbb{P}^1 , one gets a covering $X \rightarrow X_0$ with X_0 a $W(F_4)$ -invariant, also singular Fano fourfold. This is similar to the construction of [AD09, section 7]; in their Theorem 8.6 they obtain a K3 surface which is birational to the Kummer surface of the square of an elliptic curve, just as a desmic quartic. This is another way to see the geometry of the Hesse pencil as a degeneration of that of the desmic pencil.

REFERENCES

- [AD09] M. Artebani and I. Dolgachev, *The Hesse pencil of plane cubic curves*, Enseign. Math. (2) **55** (2009), no. 3-4, 235–273.
- [BBFM23] V. Benedetti, M. Bolognesi, D. Faenzi, and L. Manivel, *Hecke cycles on moduli of vector bundles and orbital degeneracy loci*, arXiv e-print math/2310.06482, to appear in J.A.G., 2023.
- [BBFM24] ———, *The Coble quadric*, Forum Math. Sigma **12** (2024), Paper No. e63.
- [BBFM25] ———, *Göpel varieties*, arXiv e-print math/2506.22030, 2025.
- [BH16] M. Bhargava and W. Ho, *Coregular spaces and genus one curves*, Camb. J. Math. **4** (2016), no. 1, 1–119.
- [Bor83] P. B. Borwein, *The Desmic conjecture*, J. Combin. Theory Ser. A **35** (1983), no. 1, 1–9.

- [Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics, Springer-Verlag, 2002.
- [Buc09] J. Buczyński, *Algebraic Legendrian varieties*, Dissertationes Math. **467** (2009).
- [CCN⁺85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of finite groups*, Oxford University Press, 1985, Maximal subgroups and ordinary characters for simple groups.
- [Cle03] J.-L. Clerc, *Special prehomogeneous vector spaces associated to F_4 , E_6 , E_7 , E_8 and simple Jordan algebras of rank 3*, J. Algebra **264** (2003), no. 1, 98–128.
- [Cox91] H. S. M. Coxeter, *Regular complex polytopes*, second ed., Cambridge University Press, 1991.
- [DG02] P. Deligne and B. H. Gross, *On the exceptional series, and its descendants*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 11, 877–881.
- [DK25] I. Dolgachev and S. Kondo, *Desmic quartic surfaces in arbitrary characteristic*, arXiv e-print math/2505.01033, 2025.
- [Dol04] I. Dolgachev, *Abstract configurations in algebraic geometry*, The Fano Conference, Univ. Torino, Turin, 2004, pp. 423–462.
- [Don77] R. Y. Donagi, *On the geometry of Grassmannians*, Duke Math. J. **44** (1977), no. 4, 795–837.
- [Edg49] W. L. Edge, *The Kummer quartic and the tetrahedroids based on the Maschke forms*, Proc. Cambridge Philos. Soc. **45** (1949), 519–535.
- [Ekl18] D. Eklund, *Curves on Heisenberg invariant quartic surfaces in projective 3-space*, Eur. J. Math. **4** (2018), no. 3, 931–952.
- [Fre63] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene. X, XI*, Indag. Math. **25** (1963), 457–471; 472–487, Nederl. Akad. Wetensch. Proc. Ser. A **66**.
- [GSW13] L. Gruson, S. V. Sam, and J. Weyman, *Moduli of abelian varieties, Vinberg θ -groups, and free resolutions*, Commutative algebra, Springer, New York, 2013, pp. 419–469.
- [Hum91] G. Humbert, *Sur la surface desmique du quatrième ordre*, J. Math. Pures App. **7** (1891), 353–398.
- [IM19] A. Iliev and L. Manivel, *Hyperkähler manifolds from the Tits-Freudenthal magic square*, Eur. J. Math. **5** (2019), no. 4, 1139–1155.
- [Jes16] C. M. Jessop, *Quartic surfaces with singular points*, Cambridge University Press, 1916.
- [Law89] D. F. Lawden, *Elliptic functions and applications*, Applied Mathematical Sciences, vol. 80, Springer-Verlag, 1989.
- [Liu25] Y. Liu, *Moduli of codimension two linear sections of subadjoint varieties*, Internat. J. Math. **36** (2025), no. 10, Paper No. 2550030.
- [LM01] J. M. Landsberg and L. Manivel, *The projective geometry of Freudenthal’s magic square*, J. Algebra **239** (2001), no. 2, 477–512.
- [LM02] ———, *Triality, exceptional Lie algebras and Deligne dimension formulas*, Adv. Math. **171** (2002), no. 1, 59–85.
- [LM25] Y. Liu and L. Manivel, *On linear sections of the spinor tenfold II*, arXiv e-print math/2504.21056, 2025.
- [Mac72] I. G. Macdonald, *Some irreducible representations of Weyl groups*, Bull. London Math. Soc. **4** (1972), 148–150.
- [Man06] L. Manivel, *Configurations of lines and models of Lie algebras*, J. Algebra **304** (2006), no. 1, 457–486.
- [Mat26] R. M. Mathews, *Cubic curves and desmic surfaces*, Trans. Amer. Math. Soc. **28** (1926), no. 3, 502–522.
- [Osh06] T. Oshima, *A classification of subsystems of a root system*, arXiv e-print math/0611904, 2006.
- [PY08] J. V. Pereira and S. Yuzvinsky, *Completely reducible hypersurfaces in a pencil*, Adv. Math. **219** (2008), no. 2, 672–688.

- [Seg18] C. Segre, *Sui complessi lineari di piani nello spazio a cinque dimensioni*, Annali di Mat. Pura. App. **27** (1918), 75–123.
- [ST54] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.
- [Ste79] C. Stephanos, *Sur les systèmes desmiques de trois tétraèdres*, Bull. Sc. Math. **3** (1879), 424–456.
- [Tho13] J. A. Thorne, *Vinberg’s representations and arithmetic invariant theory*, Algebra Number Theory **7** (2013), no. 9, 2331–2368.
- [Vin76] E. B. Vinberg, *The Weyl group of a graded Lie algebra*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 3, 488–526.
- [Yuz09] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of curves*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1641–1648.

TOULOUSE MATHEMATICS INSTITUTE, CNRS/TOULOUSE UNIVERSITY, FRANCE
Email address: `manivel@math.cnrs.fr`