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Abstract. The problem of finding a disconnected cut in a graph is NP-hard in general but polynomial-time solvable on planar graphs. The problem of finding a minimal disconnected cut is also NP-hard but its computational complexity was not known for planar graphs. We show that it is polynomial-time solvable on 3-connected planar graphs but NP-hard for 2-connected planar graphs. Our technique for the first result is based on a structural characterization of minimal disconnected cuts in 3-connected $K_{3,3}$-free-minor graphs and on solving a topological minor problem in the dual. In addition we show that the problem of finding a minimal connected cut of size at least 3 is NP-hard for 2-connected apex graphs. Finally, we relax the notion of minimality and prove that the problem of finding a so-called semi-minimal disconnected cut is still polynomial-time solvable on planar graphs.

Keywords. vertex cut, connectivity, planar graph.

1 Introduction

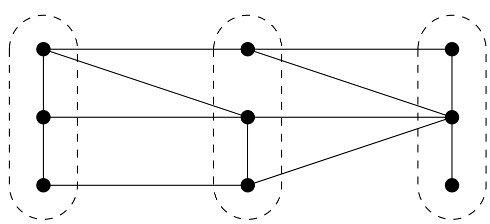
A *cutset* or *cut* in a connected graph is a subset of its vertices whose removal disconnects the graph. The problem Stable Cut is that of testing whether a connected graph has a cut that is an independent set. Le, Mosca, and Müller [17] proved that this problem is NP-complete even for K_4-free planar graphs with maximum degree 5. A connected graph $G = (V, E)$ is *k-connected* for some

⋆ The fist author was supported by Foundation for Polish Science (HOMING PLUS/2011-4/8) and National Science Center (SONATA 2012/07/D/ST6/02432). The second author was supported by EPSRC Grant EP/K025090/1. The research of the last author was co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: ARISTEIA II. An extended abstract of this paper appeared in the proceedings of FCT 2015 [12].

integer k if $|V| \geq k+1$ and every cut of G has size at least k. It is not hard to see that if one can solve Stable Cut for 3-connected planar graphs in polynomial time then one can do so for all planar graphs (in particular the problem is trivial if the graph has a cut-vertex or a cut set of two vertices that are non-adjacent). Hence, the problem is NP-complete for 3-connected planar graphs.

Due to the above it is a natural question whether one can relax the condition on the cut to be an independent set. This leads to the following notion. For a connected graph $G = (V, E)$, a subset $U \subseteq V$ is called a *disconnected* cut if U disconnects the graph and the subgraph induced by U is disconnected as well, that is, has at least two (connected) components. This problem is NP-compete in general [18] but polynomial-time solvable on planar graphs [11]. However, the property of the cut being disconnected can be viewed to be somewhat artificial if one considers the 4-vertex path $P_4 = p_1p_2p_3p_4$, which has two disconnected cuts, namely $\{p_1, p_3\}$ and $\{p_2, p_4\}$. Both these cuts contain a vertex, namely p_1 and p_4, respectively, such that putting this vertex out of the cut and back into the graph keeps the graph disconnected. Therefore, Ito et al. [10] defined the notion of a *minimal disconnected* cut of a connected graph $G = (V, E)$, that is, a disconnected cut U so that $G[(V \backslash U) \cup \{u\}]$ is connected for every $u \in U$ (more generally, we call a cut that satisfies the later condition a *minimal* cut). Here, the graph $G[S]$ denotes the subgraph of G induced by $S \subseteq V(G)$. We note that every vertex of a minimal cut U of a connected graph $G = (V, E)$ is adjacent to every component of $G[V \setminus U]$. See Figure 1 for an example of a planar graph with a minimal disconnected cut. The corresponding decision problem is defined as follows.

Minimal Disconnected Cut
Instance: a connected graph $G = (V, E)$.
Question: does G have a minimal disconnected cut?



S

Fig. 1. An example of a planar graph with a minimal disconnected cut, namely the set S.

Ito et al. [10] showed that Minimal Disconnected Cut is NP-complete. However its computational complexity remained open for planar graphs. As a graph has a stable cut if and only if a graph has a minimal stable cut, the problem of deciding whether a graph has a minimal stable cut is NP-complete for any graph class (and thus for the class of planar graphs) for which Stable Cut is NP-complete. In contrast, the problem of deciding whether a graph has a minimal cut (that may be connected or disconnected) is polynomial-time solvable: given a vertex cut U we can remove vertices from U one by one until the remaining vertices in U form a minimal cut.

Our Results. As a start we observe that Minimal Disconnected Cut is polynomial-time solvable for outerplanar graphs (as these graphs do not contain $K_{2,3}$ as a minor, any minimal cut has size at most 2). In Section 3 we prove that Minimal Disconnected Cut is also polynomial-time solvable on 3-connected planar graphs. The technique used by Ito et al. [11] for solving Disconnected Cut in polynomial time for planar graphs was based on the fact that a planar graph either has its treewidth bounded by some constant or else contains a large grid as a minor. However, grids (which are 3-connected planar graphs) do not have minimal disconnected cuts. Hence, we need to use a different approach, which we describe below.

We first provide a structural characterization of minimal disconnected cuts for the class of 3-connected $K_{3,3}$-minor-free graphs, which contains the class of planar graphs. In particular we show that any minimal disconnected cut of a 3-connected planar graph G has exactly two components and that these components are paths. In order to find such a cut we prove that it suffices to test whether G contains, for some fixed integer r, the biclique $K_{2,r}$ as a contraction. We show that G has such a contraction if and only if its dual contains the multigraph D_r, which is obtained from the r-vertex cycle by replacing each edge by two edges, as a subdivision (see also Figure 2). We then present a characterization of any graph that contains such a subdivision. Next we use this characterization to prove that the corresponding decision problem of finding a multigraph D_r as a subdivision for some $r \geq 2$ is polynomial-time solvable even on general graphs.

In Section 4 we give our second result, namely that, contrary to Disconnected Cut, which is polynomial-time solvable for planar graphs [11], Minimal Disconnected Cut stays NP-complete for the class of 2-connected planar graphs. Our proof is based on a reduction from Stable Cut and as such is different from the NP-hardness proof for general graphs [10], the gadget of which contains large cliques.

In Section 4 we also show that the problem of finding a minimal *connected* cut of size at least 3 is NP-complete for 2-connected *apex graphs* (graphs that can be made planar by deleting one vertex); to the best of our knowledge the computational complexity of this problem has not yet been determined even for general graphs. We note that the problem of finding whether a graph contains a (not necessarily minimal) connected cut of size at most k that separates two given vertices s and t is linear-time FPT when parameterized by k [19].

In Section 5 we consider a generalization of (minimal) disconnected cuts and stable cuts. For a family of graphs $\mathcal{H}$, a connected graph has a *(minimal) $\mathcal{H}$-cut* if it has a (minimal) cut that induces a graph in $\mathcal{H}$. This leads to the corresponding decision problems $\mathcal{H}$-Cut and Minimal $\mathcal{H}$-Cut. For instance, we can describe

(Minimal) Stable Cut as (Minimal) $\{P_1, 2P_1, 3P_1, \ldots\}$-Cut. Moreover, if $\mathcal{H}$ consists of all disconnected graphs, we obtain the (Minimal) Disconnected Cut problem. The problem of finding minimum (that is, smallest) $\mathcal{H}$-cuts that separate two given vertices s and t has been studied from a parameterized point of view for various graph families $\mathcal{H}$ by Heggernes et al. [9]. We show some initial results for Minimal $\mathcal{H}$-Cut, which provide some further insights in our main results.

In Section 6 we relax the notion of minimality for cut sets as follows. If a cut U of a graph $G = (V, E)$ is minimal, each of its vertices is adjacent to every component in $G[V \setminus U]$. What if instead we demand that each vertex $u \in U$ is adjacent to at least two (but maybe not all) components of $G[V \setminus U]$? This leads to the following definition. A disconnected cut U of a connected graph $G = (V, E)$ is *semi-minimal* if $G[(V \setminus U) \cup \{u\}]$ contains fewer components than $G[V \setminus U]$ for every $u \in U$. The corresponding decision problem, which is known to be NP-complete [10], is called Semi-Minimal Disconnected Cut. Note that there exist graphs with a disconnected cut, such as the P_4, that have no semi-minimal disconnected cut. Because for planar graphs Minimal Disconnected Cut is NP-complete and Disconnected Cut is polynomial-time solvable, it is a natural question to determine the complexity of Semi-Minimal Disconnected Cut for planar graphs. We adapt the proof for Disconnected Cut to show that Semi-Minimal Disconnected Cut is also polynomial-time solvable on planar graphs.

We finish our paper with some further observations and open problems in Section 7.

Related Work. Vertex cuts play an important role in graph connectivity. In the literature various kinds of vertex cuts, besides stable cuts, have been studied extensively and we briefly survey a number of results below that have not been mentioned yet.

A cut U of a graph $G = (V, E)$ is a clique cut if $G[U]$ is a clique, a k-clique cut if $G[U]$ has a spanning subgraph consisting of k complete graphs; a strict k-clique cut if $G[U]$ consists of k components that are complete graphs; and a matching cut if $E_{G[U]}$ is a matching. It follows from a classical result of Tarjan [23] that determining whether a graph has a clique cut is polynomial-time solvable. Whitesides [24] and Cameron et al. [4] proved that the problem of testing whether a graph has a k-clique cut is solvable in polynomial time for $k = 1$ and $k = 2$, respectively. Cameron et al. [4] also proved that testing whether a graph has a strict 2-clique cut can be solved in polynomial time. As mentioned the problem of testing whether a graph has a stable cut is NP-complete. This was first shown for general graphs by Chvátal [5]. Also the problem of testing whether a graph has a matching cut is NP-complete. This was shown by Brandstädt et al. [3]. Bonsma [2] proved that this problem is NP-complete even for planar graphs with girth 5 and for planar graphs with maximum degree 4.

The Skew Partition problem is that of testing whether a graph $G = (V, E)$ has a disconnected cut U so that $V \setminus U$ induces a disconnected graph in the complement of G. De Figueiredo, Klein, Kohayakawa and Reed [7] proved that even the list version of this problem, where each vertex has been assigned a list of blocks in which it must be placed, is polynomial-time solvable. Afterwards, Kennedy and Reed [15] gave a faster polynomial-time algorithm for the non-list version.

Finally, for an integer $k \geq 1$, a cut U of a connected graph G is a k-cut of G if $G[U]$ contains exactly k components. For $k \geq 1$ and $\ell \geq 2$, a k-cut U is a (k, ℓ)-cut of a graph G if $G[V \backslash U]$ consists of exactly ℓ components. Ito et al. [11] proved that testing if a graph has a k-cut is solvable in polynomial time for $k = 1$ and NP-complete for every fixed $k \geq 2$. In addition they showed that testing if a graph has a (k, ℓ)-cut is polynomial-time solvable if $k = 1, \ell \geq 2$ and NP-complete otherwise [11]. The same authors showed, by using the approach for solving Disconnected Cut on planar graphs, that both problems are polynomial-time solvable on planar graphs.

2 Preliminaries

Let $G = (V, E)$ be a connected simple graph. A maximal connected subgraph of G is called a *component* of G. Recall that, for a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. A vertex $u \in V \setminus S$ is *adjacent* to a set $S \subseteq V \setminus \{u\}$ if u is adjacent to a vertex in S. We say that two disjoints sets $S \subset V$ and $T \subset V$ are *adjacent* if S contains a vertex adjacent to T, or equivalently, if T contains a vertex adjacent to S.

Let G be a graph. We define the following operations. The *contraction* of an edge uv removes u and v from G, and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to u or v in G. Unless we explicitly say otherwise we remove all self-loops and multiple edges so that the resulting graph stays simple. The *subdivision* of an edge uv replaces uv by a new vertex w with edges uw and vw. Let $u \in V(G)$ be a vertex that has exactly two neighbours v, w, and moreover let v and w be non-adjacent. The *vertex dissolution* of u removes u and adds the edge vw.

A graph G contains a graph H as a *minor* if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. If G does not contains H as a minor, G is *H-minor-free*. We say that G contains H as a *contraction*, denoted by $H \leq_c G$, if H can be obtained from G by a sequence of edge contractions. Finally, G contains H as a *subdivision* if H can be obtained from G by a sequence of vertex deletions, edge deletions and vertex dissolutions, or equivalently, if G contains a subgraph H' that is a *subdivision* of H, that is, H can be obtained from H' after applying zero or more vertex dissolutions. We say that a vertex in H' is a *subdivision vertex* if we need to dissolve it in order to obtain H; otherwise it is called a *branch vertex* (that is, it corresponds to a vertex of H).

For some of our proofs the following global structure is useful. Let G and H be two graphs. An *H-witness structure* $\mathcal{W}$ is a vertex partition of a (not necessarily proper) subgraph of G into $|V(H)|$ nonempty sets $\{W(x)\}_{x \in V(H)}$ called *(H-witness) bags*, such that

(i) each $W(x)$ induces a connected subgraph of G,
(ii) for all $x, y \in V(H)$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G if x and y are adjacent in H.

In addition, we may require the following additional conditions:

(iii) for all $x, y \in V(H)$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G *only* if x and y are adjacent in H,
(iv) every vertex of G belongs to some bag.

By contracting all bags to singletons we observe that H is a minor or contraction of G if and only if G has an H-witness structure such that conditions (i)-(ii) or (i)-(iv) hold, respectively. We note that G may have more than one H-witness structure with respect to the same containment relation.

We denote the complete graph on k vertices by K_k and the complete bipartite graph with bipartition classes of size k and ℓ, respectively, by $K_{k,\ell}$. A graph is *planar* if it can be drawn on the plane in such a way that no two edges cross each other. By Kuratowski's Theorem [16], a graph is planar if and only if it is both K_5-minor-free and $K_{3,3}$-minor-free. Recall that an *apex graph* is a graph that can be made planar by deleting one vertex.

3 The Algorithm

We first present a necessary and sufficient condition for a 3-connected $K_{3,3}$-minor-free graph to have a minimal disconnected cut.

Theorem 1. *A 3-connected $K_{3,3}$-minor-free graph G has a minimal disconnected cut if and only if $K_{2,r} \leq_c G$ for some $r \geq 2$.*

Proof. Let $G = (V, E)$ be a 3-connected graph that has no $K_{3,3}$ as a minor. First suppose that G has a minimal disconnected cut U. Let p and q be the number of components of $G[U]$ and $G[V \setminus U]$, respectively. Because U is a disconnected cut, $p \geq 2$ and $q \geq 2$. By definition, every vertex of every component of $G[U]$ is adjacent to all components in $G[V \setminus U]$. Hence, G contains $K_{p,q}$ as a contraction. Because G has no $K_{3,3}$ as a minor, G has no $K_{3,3}$ as a contraction. This means that $p \leq 2$ or $q \leq 2$. Because $p \geq 2$ and $q \geq 2$ holds as well, we find that $K_{2,r} \leq_c G$ for some $r \geq 2$.

Now suppose that $K_{2,r} \leq_c G$ for some $r \geq 2$. Throughout the remainder of the proof we denote the partition classes of $K_{k,\ell}$ by $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_\ell\}$. We refer to the bags in a $K_{k,\ell}$-witness structure of G corresponding to the vertices in X and Y as x-bags and y-bags, respectively. Because $K_{2,r} \leq_c G$, there exists a $K_{2,r}$-witness structure $\mathcal{W}$ of G that satisfies conditions (i)-(iv). Note that $W(x_1) \cup W(x_2)$ is a disconnected cut. However, it may not be minimal.

Suppose that $W(x_1)$ contains a vertex u that is adjacent to some but not all y-bags, i.e., the number of y-bags to which u is adjacent is h for some

$1 \leq h < r$. Then we move u to a y-bag that contains one of its neighbours unless $W(x_1) \cup W(x_2)$ no longer induce a disconnected graph (which will be the case if u is the only vertex in $W(x_1)$). We observe that $G[W(x_1) \setminus \{u\}]$ may be disconnected, namely when u is a cut vertex in $G[W(x_1)]$. We also observe that u together with its adjacent y-bags induces a connected subgraph of G. Hence, the resulting witness structure $\mathcal{W}'$ is a $K_{q,r'}$-witness structure of G with $q \geq 2$ (as the resulting vertices in $W(x_1) \cup W(x_2)$ still induce a disconnected graph) and $r' = r - (h-1)$. Because $1 \leq h < r$, we find that $2 \leq r' \leq r$. We repeat this rule as long as possible. During this process, $W(x_2)$ does not change, and afterwards, we do the same for $W(x_2)$. Let $\mathcal{W}^*$ denote the resulting witness structure that is a K_{q^*,r^*}-witness structure satisfying conditions (i)-(iv) for some $q^* \geq 2$ and $2 \leq r^* \leq r$.

We will now prove the following claim.

Claim. Every vertex of each x-bag of $\mathcal{W}^*$ is adjacent to all y-bags.

We prove this claim as follows. First suppose that there exists an x-bag of $\mathcal{W}^*$, say $W^*(x_1)$, that contains a vertex u adjacent to some but not to all y-bags of $\mathcal{W}^*$, say u is not adjacent to $W^*(y_1)$. By our procedure we would have moved u to an adjacent y-bag unless that makes the disconnected cut connected. Hence we find that there are exactly two witness bags $W^*(x_1)$ and $W^*(x_2)$ and that $W^*(x_1) = \{u\}$. In our procedure we only moved vertices from x-bags to y-bags. This means that u belonged to an x-bag of the original witness structure $\mathcal{W}$. This x-bag was adjacent to all y-bags of $\mathcal{W}$ (as $\mathcal{W}$ was a $K_{2,r}$-witness structure). As we only moved vertices from x-bags to y-bags, this means that there must still exist a path from u to a vertex in $W^*(y_1)$ that does not use any vertex of $W^*(x_2)$; a contradiction. Hence every x-bag of $\mathcal{W}^*$ only contains vertices that are either adjacent to all y-bags or to none of them.

Now, in order to obtain a contradiction, suppose that an x-bag, say $W^*(x_1)$, contains a vertex u not adjacent to any y-bag. Because G is 3-connected, G contains three vertex-disjoints paths P_1, P_2, P_3 from u to a vertex in $W^*(y_1)$ (by Menger's Theorem). Each P_i contains a vertex v_i in $W^*(x_1)$ whose successor on P_i is outside $W^*(x_1)$ and thus in some y-bag. Hence, by our assumption, v_i has a neighbour in every y-bag (including $W^*(y_1)$). Recall that the number of y-bags is $r^* \geq 2$. We consider the subgraph induced by the vertices from $W^*(y_1)$ and $W^*(y_2)$ together with the vertices on the three paths P_1, P_2, P_3. For $i = 1, \ldots, 3$ we contract all edges on the subpath of P_i from u to v_i to one edge, and we contract both $W^*(y_1)$ and $W^*(y_2)$ to single vertices. These edge contractions modify the subgraph into a graph isomorphic to $K_{3,3}$, which is not possible. Hence, every vertex of each x-bag of $\mathcal{W}^*$ is adjacent to all y-bags. This completes the proof of the claim.

As $q^* \geq 2$ and $r^* \geq 2$, there are at least two x-bags and at least two y-bags in $\mathcal{W}^*$. By combing this observation with the above claim, we find that the x-bags of $\mathcal{W}^*$ form a minimal disconnected cut U of G. This completes the proof of Theorem 1. $\square$

Recall that planar graphs are $K_{3,3}$-minor-free by Kuratowski's Theorem. Hence, by Theorem 1 we may restrict ourselves to finding a $K_{2,r}$-contraction for some $r \geq 2$ in a 3-connected planar graph. Below we state some additional terminology.

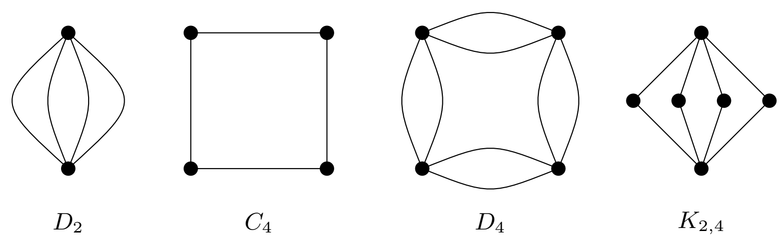


Fig. 2. The graphs $D_2, C_4, D_4, K_{2,4}$. Note that the dual of $C_4 = K_{2,2}$ is D_2, that D_4 is obtained from C_4 by duplicating each edge and that D_4 is the dual of $K_{2,4}$.

Recall that D_n is the multigraph obtained from the cycle on $n \geq 3$ vertices by doubling its edges. We let D_2 be the multigraph that has two vertices with four edges between them. The *dual* graph G_d of a plane graph G has a vertex for each face of G, and there exist k edges between two vertices u and v in G_d if and only if the two corresponding faces share k edges in G. Note that the dual of a graph may be a multigraph. As 3-connected planar graphs have a unique embedding (see e.g. Lemma 2.5.1, p.39 of [21]) we can speak of the dual of a 3-connected planar graph.

Lemma 1. *Let G be a 3-connected planar graph. Then G contains $K_{2,r}$ as a contraction for some $r \geq 2$ if and only if the dual of G contains D_r as a subdivision.*

Proof. We first observe that for all $r \geq 2$, every $K_{2,r}$ has a unique plane embedding, the dual of which is D_r. Then the results follows from a result from [13] that for 3-connected planar graphs comes down to the following statement: a 3-connected planar graph G contains a graph H as a contraction if and only if the dual of G contains the dual of H as a subdivision. $\square$

By Lemma 1 it suffices to check if the dual of the 3-connected planar input graph contains D_r as a subdivision for some $r \geq 2$. We show how to solve this problem in polynomial time for general graphs. In order to do so we need the next lemma which gives a necessary condition for a graph G to be a yes-instance of this problem. In its proof we use the following notation. For a path $P = v_1 v_2 \ldots v_p$, we write $v_i P v_j$ to denote the subpath $v_i v_{i+1} \ldots v_j$ or $v_j P v_i$ if we want to emphasize that the subpath is to be traversed from v_j to v_i.

Lemma 2. *Let v, w be two distinct vertices of a multigraph G such that there exist four edge-disjoint v-w-paths in G. Then G contains a subdivision of D_r for some $r \geq 2$.*

Proof. We prove the lemma by induction on $|V(G)| + |E(G)|$. Then we can assume that G is the union of the four edge-disjoint v-w-paths. Let us call these paths P_1, P_2, P_3, and P_4. If these four paths are vertex-disjoint (apart from v and w) then they form a subdivision of D_2. Hence, we may assume that there exists at least one vertex of G not equal to v or w that belongs to more than one of the four paths.

First suppose that there exists a vertex u that belongs to all four paths P_1, P_2, P_3 and P_4. Let G' be the graph consisting of the vertices and edges of the four subpaths vP_1u, vP_2u, vP_3u and vP_4u. As G' does not contain w, it holds that $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the induction hypothesis, G', and thus G, contains a subdivision of D_r for some $r \geq 2$.

Now suppose that there exists a vertex u that belong to only three of the four paths, say to P_1, P_2, and P_3. Let G' be the graph that consists of the vertices and edges of the four paths uP_1w, uP_2w, uP_3w and uP_1vP_4w. As G' does not contain an edge of vP_2u we find that $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the induction hypothesis, G', and thus G, contains a subdivision of D_r for some $r \geq 2$.

From now on assume that every inner vertex of every path P_i ($i = 1, \dots, 4$) belongs to at most one other path P_j ($j \neq i$). We say that two different paths P_i and P_j *cross* in a vertex u if u is an inner vertex of both P_i and P_j. Suppose P_i and P_j cross in some other vertex u' as well. Then we say that u is *crossed before* u' by P_i and P_j if u is an inner vertex of both vP_iu' and vP_ju'.

We now prove the following claim.

Claim 1. If P_i and P_j ($i \neq j$) cross in both u and u' then we may assume without loss of generality that either u is crossed before u' or u' is crossed before u.

We prove Claim 1 as follows. Suppose that u is not crossed before u' by P_i and P_j and similarly that u' is not crossed before u by P_i and P_j. Then we may assume without loss of generality that u is an inner vertex of vP_iu' and that u' is an inner vertex of vP_ju. See Figure 3 for an example of this situation. However, in that case we can replace P_i and P_j by the paths vP_iuP_jw and $vP_ju'P_iw$. These two paths together with the two unused original paths form a subgraph G' of G with fewer edges than G (as for instance no edge on uP_iu' belongs to G'). We apply the induction hypothesis on G'. This completes the proof of Claim 1.

We need Claim 1 to prove the following claim, which is crucial for our proof.

Claim 2. We may assume without loss of generality that there exists a vertex $u \notin \{v, w\}$ that is on two paths P_i and P_j ($i \neq j$) so that every inner vertex of vP_iu and vP_ju has degree 2 in G.

We prove Claim 2 as follows. By our assumption there exists at least one vertex in G that is on two paths. Let $s \notin \{v, w\}$ be such a vertex, say s belongs to P_1 and P_2. Assume without los of generality that every inner vertex of vP_1s has degree 2. Then, by Claim 1, we find that P_1 and P_2 do not cross in an inner vertex of vP_2s.

If every inner vertex of vP_1s and vP_2s has degree 2 in G then the claim has been proven. Suppose otherwise, namely that there exists an inner vertex s' of

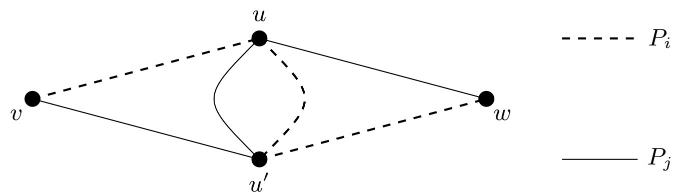


Fig. 3. The paths P_i and P_j where u is not crossed before u' by P_i and P_j and similarly u' is not crossed before u by P_j and P_i. Note that the paths P_i and P_j may have more common vertices, but for clarify this is not been shown.

vP_1s or vP_2s whose degree in G is larger than 2, say s' belongs to vP_2s. As P_1 does not cross vP_2s, we find that s' must belong to P_3 or to P_4. Choose s' in such a way that every inner vertex of vP_2s' has degree 2 in G. Assume without loss of generality that s' belongs to P_3.

If every inner vertex of vP_3s' has degree 2 then the claim has been proven (as every inner vertex of vP_2s' has degree 2 as well). Suppose otherwise, namely that there exists an inner vertex s'' of vP_3s' whose degree in G is larger than 2. Choose s'' in such a way that every inner vertex of vP_3s'' has degree 2 in G. By Claim 1, no inner vertex of vP_3s' belongs to P_2, so s'' does not lie on P_2. This means that s'' belongs either to P_1 or to P_4.

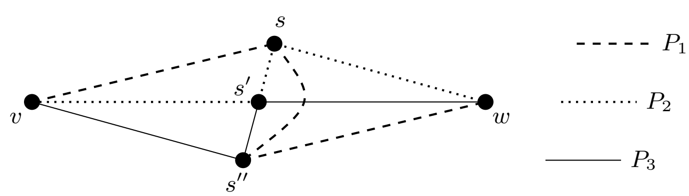


Fig. 4. The paths P_1, P_2 and P_3 where s belongs to P_1 and P_2, s' belongs to vP_2s and P_3 and s'' belongs to vP_3s' and P_1.

Suppose s'' belongs to P_1. See Figure 4 for an example of this situation. As every inner vertex of vP_1s has degree 2, we find that s is an inner vertex of vP_1s''. However, we can now replace P_1, P_2 and P_3 by the three paths $vP_1sP_2w, vP_2s'P_3w$ and $vP_3s''P_1w$. These three paths form, together with P_4, a subgraph of G with fewer edges than G (for instance, no edge of sP_1s'' belongs to G'). We can apply the induction hypothesis on this subgraph. Hence we may assume that s'' does not belong to P_1.

From the above we conclude that s'' belongs to P_4. See Figure 5 for an example of this situation. We consider the paths vP_3s'' and vP_4s''. If every inner vertex of vP_4s'' has degree 2 in G then we have proven Claim 2 (recall that every inner vertex of vP_3s'' has degree 2 in G as well). Suppose otherwise, namely that there exists an inner vertex t of vP_4s'' whose degree in G is larger than 2. Choose t in such a way that every inner vertex of vP_4t has degree 2 in G. By Claim 1 we find that t is not on P_3. If t is on P_2 we can use a similar replacement of three paths by three new paths as before that enables us to apply the induction hypothesis. Hence, we find that t belongs to P_1.

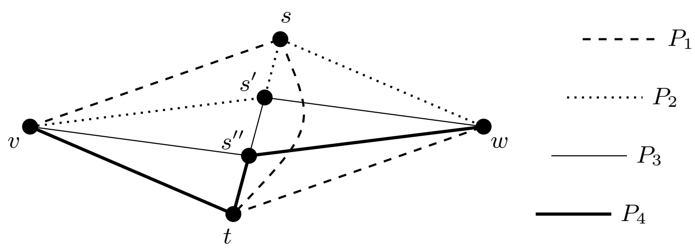


Fig. 5. The paths P_1, P_2, P_3 and P_4 where s belongs to P_1 and P_2, s' belongs to vP_2s and P_3, s'' belongs to vP_3s' and P_4 and t belongs to vP_4s'' and P_1.

As every inner vertex of vP_1s has degree 2 in G we find that s is an inner vertex of vP_1t. Then we take the four paths $vP_1sP_2w, vP_2s'P_3w, vP_3s''P_4w$ and vP_4tP_1w. These four paths form a subgraph G' of G with fewer edges than G (as for instance G' contains no edge from sP_1t). We can apply the induction hypothesis on G'. Hence we may assume that such a vertex t cannot exist. Thus we have found the desired vertex and subpaths, namely s'' with subpaths vP_3s'' and vP_4s''. This completes the proof of Claim 2.

By Claim 2 we may assume without loss of generality that there exists a vertex u that belongs to P_1 and P_2 such that every inner vertex of vP_1u and vP_2u has degree 2. Let G^* be the graph obtained from G by contracting all edges of vP_1u and vP_2u (recall that we remove loops and multiple edges). Let u^* be the new vertex to which all the edges were contracted. Notice that there are four edge-disjoint u^*-w-paths in G^*. Then, by the induction hypothesis, G^* contains a subdivision H of D_r for some $r \geq 2$. If u^* does not belong to H, then G contains H as well and we would have proven the lemma. Assume that u^* belongs to H.

First suppose that u^* is a subdivision vertex of H in G^*. Let u^* have neighbours s_1 and s_2 in H. Take a shortest path Q from s_1 to s_2 in the subgraph of G induced by s_1, s_2 and the vertices of vP_1u and vP_2u. This results in a graph H', which is a subgraph of G and which is a subdivision of D_r as well.

Now suppose that u^* is a branch vertex of H in G^*, say u^* corresponds to $z \in V(D_r)$. Note that any vertex in D_r has one neighbour if $r = 2$ and two neighbours if $r \geq 3$. We let s and t be the two branch vertices of H that correspond to the neighbours of z in D_r (note that $s = t$ if $r = 2$). Let s_1 and s_2 be the neighbours of u^* on the two paths from u^* to s, respectively, in H. Similarly, let t_1 and t_2 be the neighbours of u^* on the two paths from u^* to t, respectively, in H. Note that, as G is a multigraph, it is possible that $s_1 = s_2 = s$ and $t_1 = t_2 = t$.

Recall that every internal vertex on vP_1u and on vP_2u has degree 2 in G. As u is an inner vertex of P_1 and P_2 but not of P_3 and P_4, it has degree 4 in G. As G is the union of P_1, P_2, P_3 and P_4, we find that v has degree 4 as well. Then, after uncontracting u^*, we have without loss of generality one of the following two situations in G. First, u is adjacent to s_1 and s_2 and v is adjacent to t_1 and t_2. In that case u and v become branch vertices of a subdivision of D_{r+1} in G (to which the internal vertices on the paths uP_1v and uP_2v belong as well, namely as subdivision vertices). Second, u is adjacent to s_1 and t_1, whereas v is adjacent to s_2 and t_2. Then u and v become subdivision vertices of a subdivision of D_r in G (and we do not use the internal vertices on the paths uP_1v and uP_2v). This completes the proof of the lemma. □

Lemma 2 gives us the following result.

Theorem 2. *It is possible to find in $O(mn^2)$ time whether a graph G with n vertices and m edges contains D_r as a subdivision for some $r \geq 2$.*

Proof. Let G be a graph with m edges. We check for every pair of vertices s and t whether G contains four edge-disjoint paths between them. We can do this via a standard reduction to the maximum flow problem. Replace each edge uv by the arcs (u, v) and (v, u). Give each arc capacity 1. Introduce a new vertex s' and an arc (s', s) with capacity 4. Also introduce a new vertex t' and an arc (t, t') with capacity 4. Check if there exists an (s', t')-flow of value 4 by using the Ford-Fulkerson algorithm. As the maximum value of an (s', t')-flow is at most 4, this costs $O(m)$ time per pair, so $O(mn^2)$ time in total.

If there exists a pair s, t in G with four edge-disjoint paths between them then G has a subdivision of D_r, for some $r \geq 2$, by Lemma 2. If not then we find that G has no subdivision of any D_r ($r \geq 2$) as any subdivision of D_r immediately yields four edge-disjoint paths between two vertices and our algorithm would have detected this. □

We are now ready to state our main result.

Theorem 3. Minimal Disconnected Cut *can be solved in $O(n^3)$ time on 3-connected planar graphs with n vertices.*

Proof. Let G be a 3-connected planar graph with n vertices. By Theorem 1 it suffices to check whether $K_{2,r} \leq_c G$ for some $r \geq 2$. By Lemma 1, the latter is equivalent to checking whether the dual of G, which we denote by G^*, contains D_r as a subdivision for some $r \geq 2$. To find G^* we first embed G in the plane using the linear-time algorithm from Mohar [20]. As the number of edges in a planar graph is linear in the number of vertices, G^* has $O(n)$ vertices and $O(n)$ edges and can be constructed in $O(n)$ time. We are left to apply Theorem 2. □

4 Hardness

We prove the following result, which shows that Theorem 3 can be viewed as best possible.

Theorem 4. Minimal Disconnected Cut *is* NP*-complete for the class of* 2*-connected planar graphs.*

Proof. As we can check in polynomial time whether a given subset of vertices in a graph is a minimal disconnected cut, the problem belongs to NP. To show NP-hardness we reduce from Stable Cut. Recall that this problem is to test whether a graph has a cut that is an independent set and that it is an NP-complete problem for planar graphs [17] even if they are 2-connected (as the answer is trivially yes if the input graph contains a cut vertex[1]).

Let G be a 2-connected planar graph with n vertices and m edges. We construct in polynomial time a graph G' by adding for each edge $e = uv$ in G a new vertex x_e that we make adjacent (only) to u and v. Note that G' is a planar graph with $m + n$ vertices and $3m$ edges. Moreover, G' is 2-connected. Hence, it suffices to prove that G has a stable cut if and only if G' has a minimal disconnected cut.

First suppose that G has a stable cut S. As long as S contains a vertex u so that the subgraph of G induced by $(V(G) \setminus S) \cup \{u\}$ is disconnected we move u from S to $V(G) \setminus S$. Because G is 2-connected, the resulting set $S^* \subseteq S$ is a stable cut of size at least 2. By our procedure, S^* is a minimal disconnected cut of G as well. Because S^* is an independent set, at least one vertex of every pair of adjacent vertices u, v in G does not belong to S^*, say u does not belong to S. Let F be the component of $G[V \setminus S^*]$ that contains u. Then either v belongs to F as well or v belongs to S^*. In both cases we place x_{uv} in F (so we neither create any new components in $G^* - S$ nor do we reduce the number of components). After doing this for each pair of adjacent vertices in G we find that S^* is also a minimal disconnected cut of G^*.

Now suppose that G' has a minimal disconnected cut S'. Consider an edge uv of G' that belongs to G as well. The vertex x_{uv} has degree 2 and both its neighbours u and v are adjacent. Hence, x_{uv} cannot belong to S' (as otherwise x_{uv} would have neighbours in at most one component of $G' - S'$). Moreover, at most one of u and v can belong to S' as otherwise, due to their adjacency, they would belong to the same component of $G'[S']$, meaning that any other component of $G'[S']$ is not adjacent to the 1-vertex component of $G' - S'$ that contains x_{uv}. Hence, S' is a stable cut of G' that only contains vertices of G. Because at least one vertex of any pair of adjacent vertices u, v belongs to the same component of $G' - S'$ that contains the vertex x_{uv}, we find that $G - S'$ has just as many (and thus at least two) components as $G' - S'$. We conclude that S' is a stable cut of G as well. □

[1] We recall from Section 1 that Stable Cut is NP-complete even for 3-connected planar graphs, but we do not need 3-connectivity in our proof.

A cut S in a graph G is a *minimal connected* cut if $G[S]$ is connected and for all $u \in S$ we have that $G[(V \setminus S) \cup \{u\}]$ is connected. We call the problem of testing whether a graph has a minimal connected cut of size at least k the Minimal Connected Cut(k) problem. By modifying the proof of Theorem 4 we obtain the following result.

Theorem 5. Minimal Connected Cut(3) *is* NP*-complete even for the class of* 2*-connected apex graphs.*

Proof. We can check in polynomial time whether a given subset of vertices in a graph is a minimal connected cut. Hence the problem belongs to NP. As mentioned, we are following the line of the proof of Theorem 4, so we reduce from the Stable Cut problem restricted to 2-connected planar graphs.

Let G be a 2-connected planar graph with n vertices and m edges. We construct in polynomial time a graph G'' by adding for each edge $e = uv$ in G a new vertex x_e that we make adjacent (only) to u and v. We say that these newly added vertices are of *x-type*. Afterward we add a new vertex y that we make adjacent to all vertices of G (so not to the x-type vertices). Note that G'' is an apex graph with $m + n + 1$ vertices and $3m + n$ edges. Moreover, G'' is 2-connected. We claim that G has a stable cut if and only if G'' has a minimal connected cut of size at least 3.

First suppose that G has a stable cut S. Following the same arguments as in the proof of Theorem 4 we find that S contains a subset S^* that is a minimal disconnected cut of $G'' - \{y\}$. Adding y to S^* yields a minimal connected cut of G''. Because G is 2-connected, $S^* \cup \{y\}$ has size at least 3.

Now suppose that G'' has a minimal disconnected cut S'' of size at least 3. Consider an edge uv of G'' that belongs to G as well. The vertex x_{uv} has degree 2 and its two neighbours u, v are adjacent. Hence, x_{uv} cannot belong to S'' (as otherwise x_{uv} would have neighbours in at most one component of $G'' - S''$). Moreover, at most one of u and v can belong to S''. This can be seen as follows. For contradiction, assume that u and v both belong to S''. Because S'' has size at least 3, we find that S'' contains some vertex $w \notin \{u, v\}$. This is not possible, as w is not adjacent to the 1-vertex component of $G'' - S''$ that contains x_{uv}.

Let $T = S'' \setminus \{y\}$ if $y \in S''$ and let $T = S''$ otherwise. Because at most one of every pair of adjacent vertices in G and no x-type vertices belong to S'', we find that T is a stable cut of $G'' - \{y\}$ that only contains vertices of G. Because at least one vertex of any pair of adjacent vertices u, v belongs to the same component of $G'' - S''$ that contains the vertex x_{uv}, we find that $G - T$ has just as many (and thus at least two) components as $G'' - S''$. Hence $G - T$ has at least two components. We conclude that T is a stable cut of G. □

Note that we cannot use the reduction in the proof of Theorem 5 to get NP-hardness for Minimal Connected Cut(1), the reason being that the gadget graph constructed contains many minimal disconnected cuts of size 2.

5 A Generalization

Let $\mathcal{H}$ be a graph class. Recall that a given connected graph has a minimal $\mathcal{H}$-cut if it has a (minimal) cut that induces a graph in $\mathcal{H}$ and that the corresponding decision problems are called $\mathcal{H}$-Cut and Minimal $\mathcal{H}$-Cut.

From the proof of Theorem 4 we find that the Minimal Stable Cut problem, that is, the problem of determining whether a graph has a minimal stable cut is NP-complete even for 2-connected planar graphs. The argument in this proof to move any cut vertex not adjacent to all components outside the stable cut until a minimal stable cut is obtained can be generalized to $\mathcal{H}$-cuts if an extra condition is added.

Observation 1 *Let $\mathcal{H}$ be a graph class of graphs closed under vertex deletion. Then a connected graph has a minimal $\mathcal{H}$-cut if and only if it has a $\mathcal{H}$-cut.*

Due to Observation 1, the problems $\mathcal{H}$-Cut and Minimal $\mathcal{H}$-Cut are polynomially equivalent if $\mathcal{H}$ is closed under vertex deletion. Recall that if we let $\mathcal{H}$ be the class of disconnected graphs, we obtain the (Minimal) Disconnected Cut problem. However, we cannot combine Observation 1 with results for the Disconnected Cut problem to obtain corresponding results for the Minimal Disconnected Cut problem, because the class of disconnected graphs is not closed under vertex deletion. This is also clear from the fact that Disconnected Cut is polynomial-time solvable for planar graphs [11], whereas we showed in Section 4 that Minimal Disconnected Cut is NP-complete even for 2-connected planar graphs.

Also if for instance $\mathcal{H}$ consists of all linear forests on at least two components (disjoint unions of two or more paths) we cannot use Observation 1, but in that case we can determine the complexity of Minimal $\mathcal{H}$-Cut by first giving the following description of minimal disconnected cuts in planar graphs (the second statement is a structural observation which is not needed for the proof of this result).

Theorem 6. *Let G be a $K_{3,3}$-minor-free graph. Let U be any minimal cut of G. Then every component of $G[U]$ is a path or a cycle, or in case G is planar and U is disconnected, every component of $G[U]$ is a path. Moreover, G has a minimal disconnected cut of size 2 or for every minimal disconnected cut U of G it holds that $G[V \setminus U]$ has exactly two components.*

Proof. Let V_1 and V_2 be the vertex sets of any two components of $G[V \setminus U]$. Suppose that U contains a vertex s of degree 3 in $G[U]$. Then s has neighbours t_1, t_2, t_3 in U. As every vertex of U is adjacent to both V_1 and V_2, the vertices s, t_1, t_2, t_3 form, together with V_1 and V_2, a $K_{3,3}$-minor of G, a contradiction. Hence, every component of $G[U]$ has maximum degree at most 2, so is either a path or a cycle. Suppose that G is planar and that U is disconnected. For contradiction, assume that $G[U]$ contains a component with vertex set U_1 that is a cycle. As every vertex of U is adjacent to both V_1 and V_2 we find that U_1

and a vertex of another component of $G[U]$ form, together with V_1 and V_2, a K_5-minor of G, which is not possible as G is planar.

Now suppose that G has at least one minimal disconnected cut but not one of size 2. Let U be a minimal disconnected cut of G. Then $G[V \setminus U]$ must have exactly two components; otherwise three vertices from U and three components of $G[V \setminus U]$ form a $K_{3,3}$-minor of G, as every vertex of U is adjacent to every component of $G[V \setminus U]$ by definition. $\square$

Let $\mathcal{P}$ consist of all disjoint unions of two or more paths. Theorems 3, 4 and 6 have the following consequence.

Corollary 1. Minimal $\mathcal{P}$-Cut *is polynomial-time solvable for k-connected planar graphs if $k \geq 3$ and* NP*-complete if $k \leq 2$.*

6 Semi-Minimality

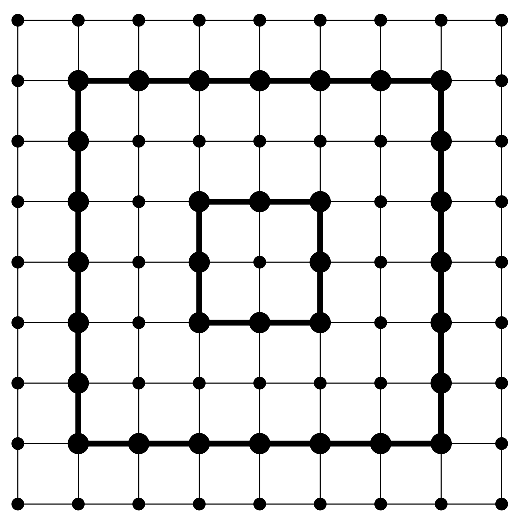


Fig. 6. The grid M_9, where the two thick cycles correspond to cycles C_b and C_d in the proof of Theorem 7.

The $m \times m$ *grid* M_m has all pairs (i, j) for $i, j = 0, 1, \ldots, m-1$ as the vertex set, and two vertices (i, j) and (i', j') are joined by an edge if and only if $|i - i'| + |j - j'| = 1$. See Figure 6 for an example. We need the following result due to Robertson, Seymour and Thomas (see e.g. [6] for the definition of treewidth).

Lemma 3 ([22]). *For every integer m, every planar graph of treewidth at least $6m - 4$ contains M_m as a minor.*

We also use the well-known result of Courcelle [6] that states that on any class of graphs of bounded treewidth, every problem definable in monadic second-order logic can be solved in time linear in the number of vertices of the graph (we refer to [6] for more details on monadic second-order logic).

Lemma 4. *The* Semi-Minimal Disconnected Cut *problem can be defined in monadic second order logic.*

Proof. We can express the property that $G = (V, E)$ has a semi-minimal disconnected cut in monadic second order logic as follows. We first note that a graph G has a disconnected cut if and only if V can be partitioned into four sets U_1, U_2, V_1, V_2 such that the following three conditions hold:

1. every vertex of V belongs to exactly one set of $\{U_1, U_2, V_1, V_2\}$;
2. sets U_1, U_2, V_1, V_2 are all nonempty;

3a. sets U_1 and U_2 are nonadjacent;
3b. sets V_1 and V_2 are nonadjacent.

It is readily seen that these conditions can be expressed in monadic second order logic:

- $\phi_1 = \forall u(U_1(u) \vee U_2(u) \vee V_1(u) \vee V_2(u))$;
- $\phi_2 = \exists u U_1(u) \wedge \exists u U_2(u) \wedge \exists u V_1(u) \wedge \exists u V_2(u)$;
- $\phi_{3a} = \forall u \forall v((U_1(u) \wedge U_2(v)) \rightarrow \neg E(u, v))$;
- $\phi_{3b} = \forall u \forall v((V_1(u) \wedge V_2(v)) \rightarrow \neg E(u, v))$.

We are left to express the semi-minimality in monadic second order logic. This condition is equivalent to demanding that for all $u \in U_1 \cup U_2$ there exists a set $Z_u \subseteq V$ (so Z_u may be different for different vertices u of $U_1 \cup U_2$) such that the following three conditions hold:

4a. $Z_u \cap (V_1 \cup V_2)$ contains a neighbour s of u;
4b. $(V_1 \cup V_2) \setminus Z_u$ contains a neighbour t of u;
4c. there is no edge between any vertex of $Z_u \cap (V_1 \cup V_2)$ and any vertex of $(V_1 \cup V_2) \setminus Z_u$.

Also these conditions can be easily formulated in monadic second order logic:

- $\phi_{4a} = \exists s(E(s, u) \wedge Z_u(s) \wedge (V_1(s) \vee V_2(s))$;
- $\phi_{4b} = \exists t(E(t, u) \wedge (V_1(t) \vee V_2(t)) \wedge \neg Z_u(t))$;
- $\phi_{4c} = \forall s \forall t((Z_u(s) \wedge (V_1(s) \vee V_2(s)) \wedge (V_1(t) \vee V_2(t)) \wedge \neg Z_u(t)) \rightarrow \neg E(s, t))$.

Then G has a semi-minimal disconnected cut if and only if the following monadic second order logic sentence is true:

$$\exists U_1 \exists U_2 \exists V_1 \exists V_2 (\phi_1 \wedge \phi_2 \wedge \phi_{3a} \wedge \phi_{3b} \wedge \forall u((U_1(u) \vee U_2(u)) \rightarrow \exists Z_u(\phi_{4a} \wedge \phi_{4b} \wedge \phi_{4c}))).$$

This completes the proof of Lemma 4. □

We also need the following lemma.

Lemma 5. *Let U be a disconnected cut of a connected graph $G = (V, E)$. If every component of $G[U]$ is adjacent to at least two components of $G[V \setminus U]$, then G has a semi-minimal disconnected cut $U' \subseteq U$.*

Proof. Let $D_1, \ldots, D_p$ be the components of $G[U]$. For each D_i we do as follows. As long as there exists a vertex $u \in D_i$ that is adjacent to at most one component of $G[V \setminus U]$ we move u from D_i to $G[V \setminus U]$. Because D_i is adjacent to at least two components of $G[V \setminus U]$, this process stops before D_i becomes empty. Afterwards, D_i only contains vertices adjacent to none or at least two components of $G[V \setminus U]$. Note that D_i contains at least one vertex adjacent to at least two components of $G[V \setminus U]$. We move all vertices not adjacent to any components of $G[V \setminus U]$ from D_i to $G[V \setminus U]$. Afterward, D_i is still nonempty. Hence, after doing this for each D_i, we have obtained a set $U' \subseteq U$ that is a semi-minimal disconnected cut of G. $\square$

We are now ready to prove the main result of this section.

Theorem 7. *The* Semi-Minimal Disconnected Cut *problem can be solved in linear time for planar graphs.*

Proof. Let G be a planar graph. We use Bodlaender's algorithm [1] to test in linear time whether the treewidth of G is at most $6 \cdot 9 - 5$. If so, then by Lemma 4 and the aforementioned theorem of Courcelle [6], we can test in linear time whether G has a semi-minimal disconnected cut. If not, then G contains M_9 as a minor by Lemma 3. Let $\mathcal{W}$ be a corresponding witness structure. We notice that the vertices of M_9 can be partitioned into 5 nested cycles C_a, C_b, C_c, C_d and C_e of length 1 (with slight abuse of terminology), 8, 16, 24 and 32, respectively; see also Figure 6. We let U be the union of vertices in the sets $W(x)$ for all $x \in V(C_b) \cup V(C_d)$. Then, as G is planar, U is a disconnected cut of G. Moreover, as $V(C_b) \cup V(C_d)$ satisfies the condition of Lemma 5, U satisfies this condition as well. Hence, we can apply this lemma to obtain a semi-minimal disconnected cut $U' \subseteq U$ of G. $\square$

7 Conclusions

Our main results are that Minimal Disconnected Cut is NP-complete for 2-connected planar graphs and polynomial-time solve for planar graphs that are 3-connected. Our proof technique for the latter result was based on translating the problem to a dual problem, namely the existence of a subdivision of D_r for *some* r, for which we obtained a polynomial-time algorithm even for general graphs. One can also solve the problem of determining whether a graph contains D_r as a subdivision for some *fixed* integer r by using the algorithm of Grohe, Kawarabayashi, Marx, and Wollan [8] which tests in cubic time, for any fixed graph H, whether a graph contains H as a subdivision. However, when r is part of the input we observe the following.

Theorem 8. *The problem of deciding whether a graph contains the graph D_r as a subdivision is* NP*-complete if r is part of the input.*

Proof. We reduce from the problem Hamilton Cycle, which is well known to be NP-complete [14]. Let G be a graph with n vertices and m vertices. We replace each edge $e = uv$ in G by two paths us_ev and ut_ev where s_e and t_e are two new vertices (so we add $2m$ new vertices in total). The resulting graph contains D_n as a subdivision if and only if G has a hamilton cycle. $\square$

We finish our paper with some open problems. Recall that our construction in Theorem 5 does not work for proving NP-hardness of Minimal Connected Cut(1), which is the problem of deciding whether a graph has a minimal connected cut. The computational complexity of this problem is not known. In fact we do not know this even for (3-connected) planar graphs, and we pose these questions as open problems. Note that these problems fall under a more general study into minimal $\mathcal{H}$-cuts that we introduced in Section 5.

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